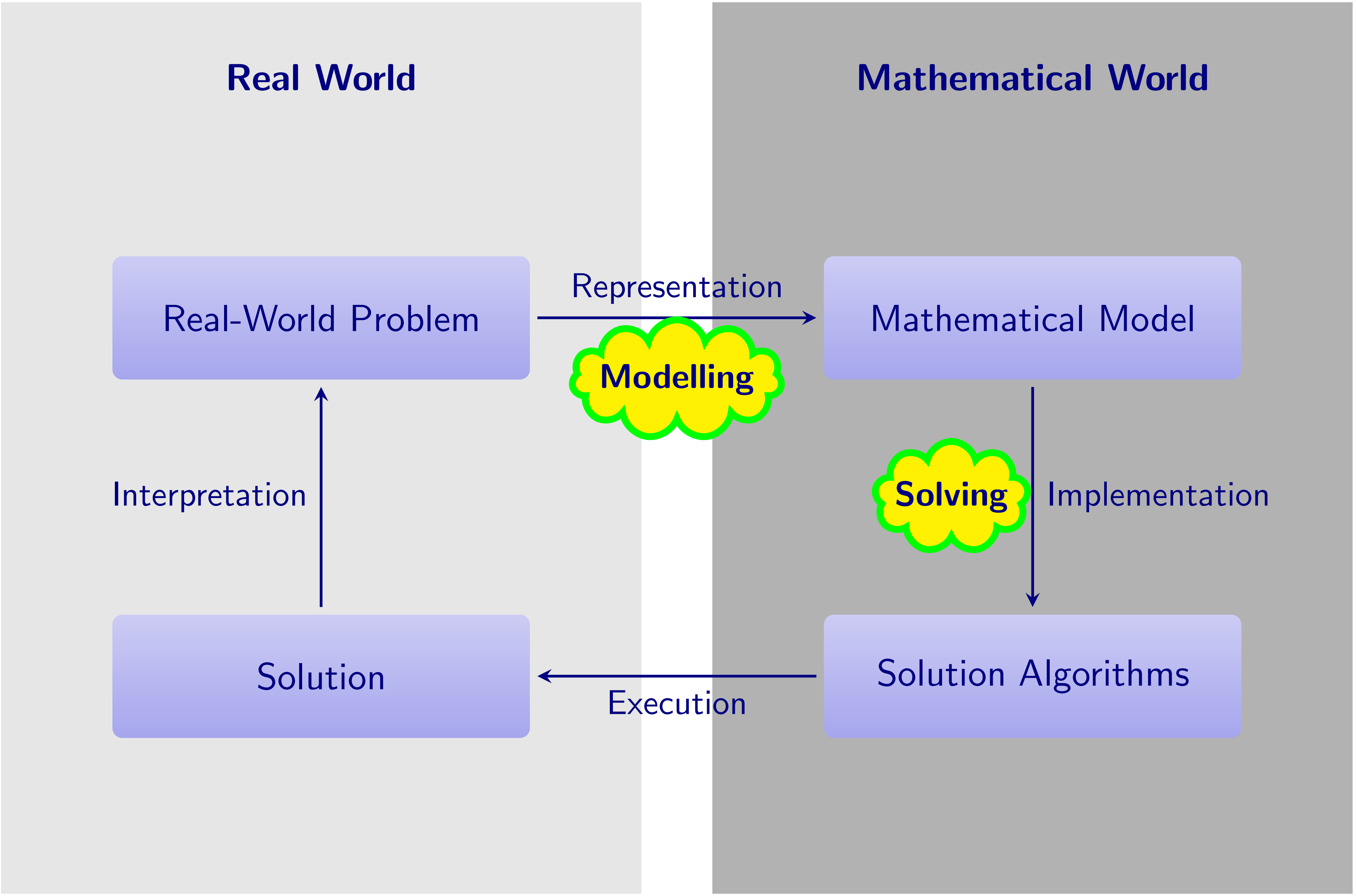
Mathematical Tools for Engineering and Management

Revision

08 Feb 2012

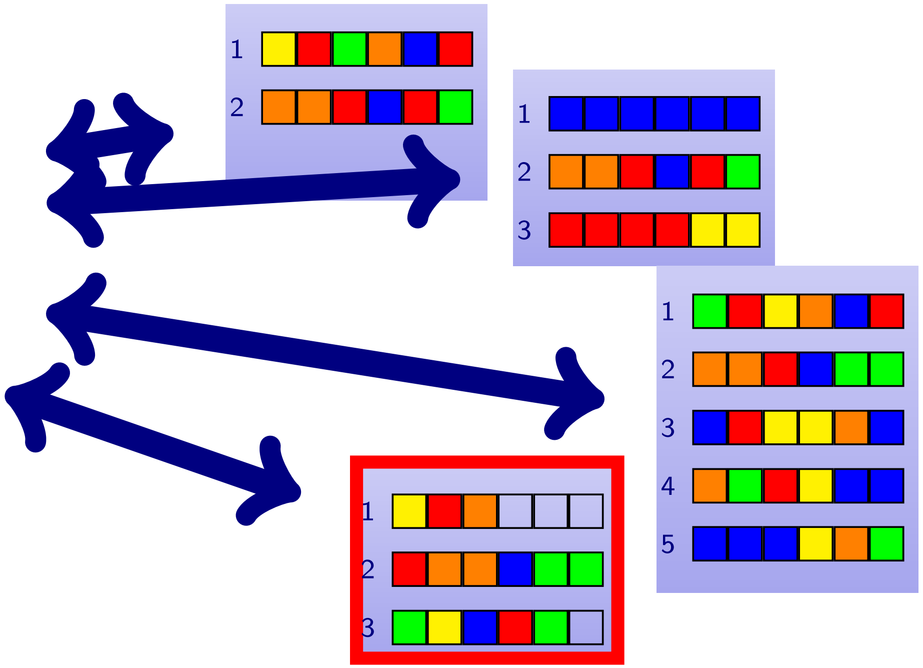
- Models, Data and Instances
- Linear Optimization
  - Modelling as a linear program
  - Solving a linear program (graphically, and in princple by the simplex algorithm)
  - Sensitivity analysis
- (Mixed) Integer Programming
  - Modelling as a (mixed) integer program
  - How to solve a (mixed) integer program (in principle)
- Combinatorial Optimization
  - Exemplary problems, algorithms, and runtimes
- Nonlinear Optimization
  - Local and global optima, convex optimization
- Scheduling
- Lot Sizing
- Multicriteria Optimization


Real World
Mathematical World
Real-World Problem
Representation
Mathematical Model
Modelling
Interpretation
Solving
Implementation
Solution
Execution
Solution Algorithms

Given n sequences of colours $c_{i,j}$ $(1 \leq i \leq n,\ j \in \mathbb{N})$ find a mapping $f : \mathbb{N} \to \{0, \ldots, n\}$ such that $\#\{k \mid c_{f(k),j(k)} \neq c_{f(k+1),j(k+1)}\}$ is minimized, where $j(k) :=$

$$\begin{cases} 1 & \text{if } k{=}1 \text{ or } f(k'){\neq}f(k)\ \forall k'{<}k \\ j(k'){+}1 & \text{otherwise, where } k' \text{ is maximal with } k'{<}k,\ f(k'){=}f(k) \end{cases}$$

Model



Data Sets

- ▷ A model is a mathematical formulation of the problem, independent of any concrete data (as possible input)
- ▷ An instance is a mathematical model, together with one associated data set

Mathematical optimization

objective to maximize/minimize — constraints to respect — solution: variable assignment

Mixed integer programming

linear objective

linear constraints

both continuous and integer variables

Linear programming

only continuous variables

Integer programming

only integer variables

Nonlinear optimization

non-linear objective allowed

non-linear constraints allowed

continuous and/or integer variables

S

Sets of relevant elements
(for example: products, cities, machines, types of raw material, ...)

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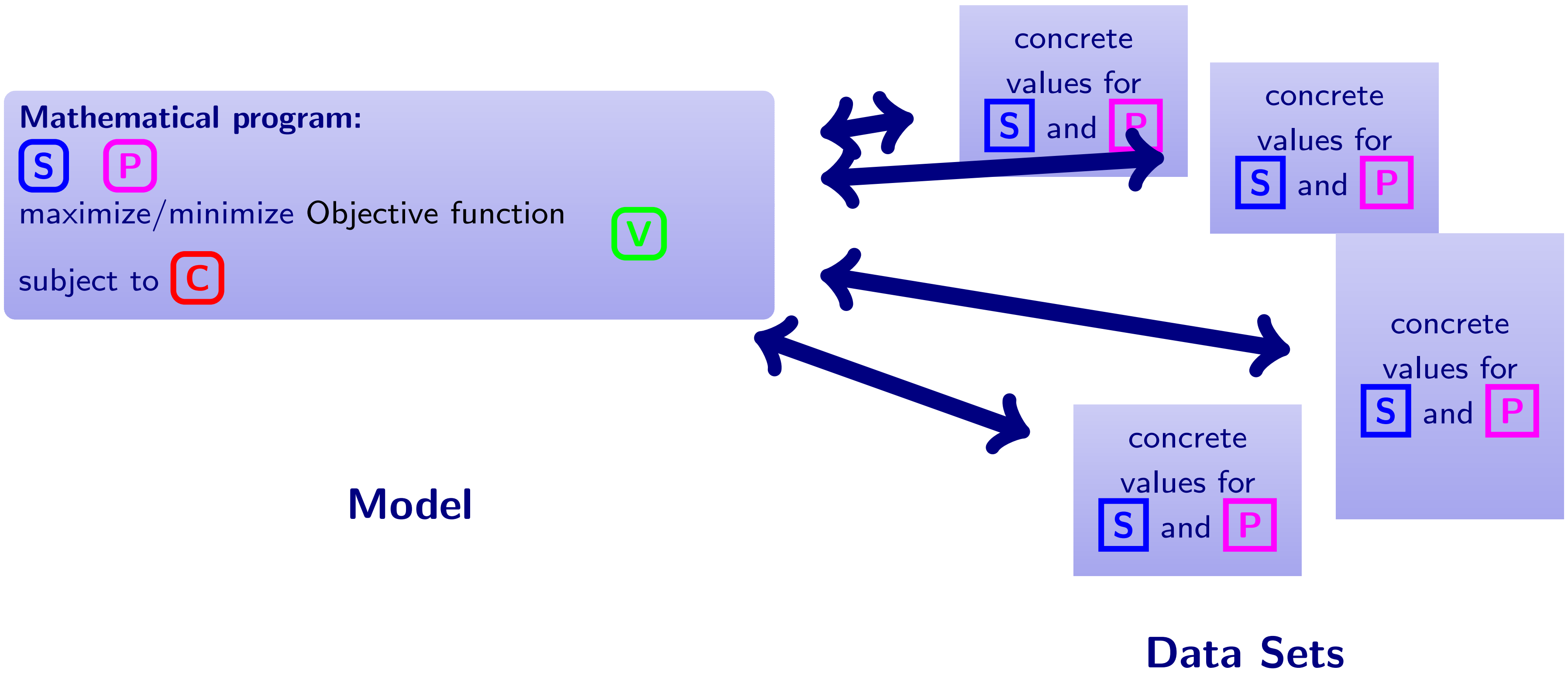
▷ Mathematical Program: Collection of constraints and variables together with an Objective function to be maximized/minimized



- Input: One data set $\widehat{=}$ Values for sets and parameters.
- Output: Variable assignment such that the objective function value is maximal/minimal and the constraints are respected

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   ➡ Which decisions have to be made?

   ➡ In which numbers are they best represented?

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Fruit	**Bananas**	**Pineapples**
Revenue	\$10000	\$20000
Land use	5a	3a
Time use	4h	7h
Water consumpt.	400l	400l

Available capacities and water resources:

- Land: 50a
- Working time: 70h
- Water supply: 4500l

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maximize	(total revenue)	$10x_b + 20x_p$			**Objective**
subject to	(total land usage)	$5x_b + 3x_p$	$\leq$	50	C
	(total working time)	$4x_b + 7x_p$	$\leq$	70	
	(total water consumption)	$4x_b + 4x_p$	$\leq$	45	
	(non-negativity)	x_b, x_p	$\geq$	0	V

	x_b	x_p	revenue	total land use	total working time	total water cons.
feasible	5	6	170	43	62	44
feasible	4	7	180	41	65	44
infeasible	2	9	200	37	**71**	44
optimal	0	10	200	30	70	40
			available:	50	70	45

$$
\begin{array}{llrrcl}
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\text{subject to} & & \text{(total land usage)} & 5x_b + 3x_p & \leq & 50 \\
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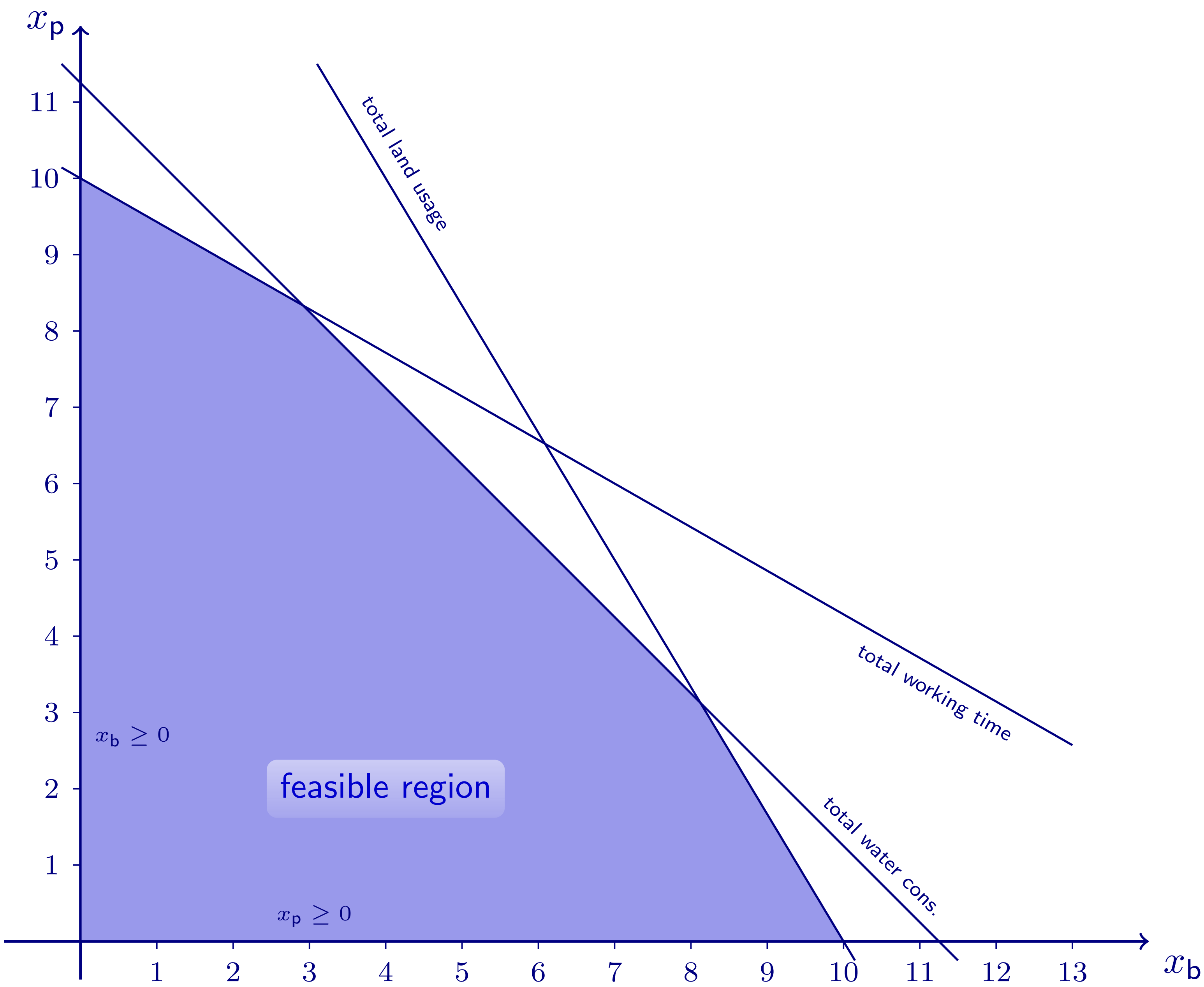
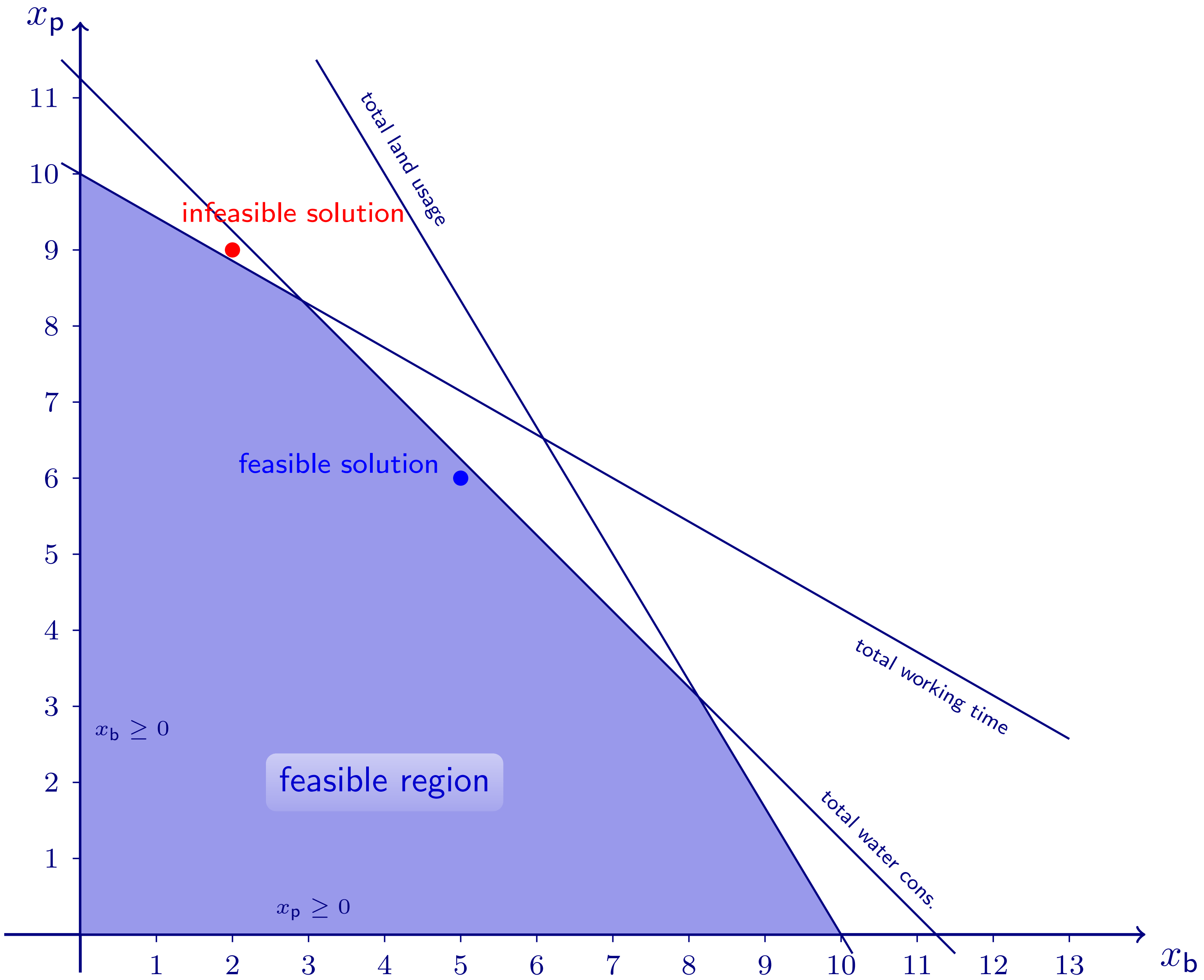


x_p
x_b
11
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10 11 12 13
total land usage
total working time
total water cons.
$x_b \geq 0$
$x_p \geq 0$
feasible region



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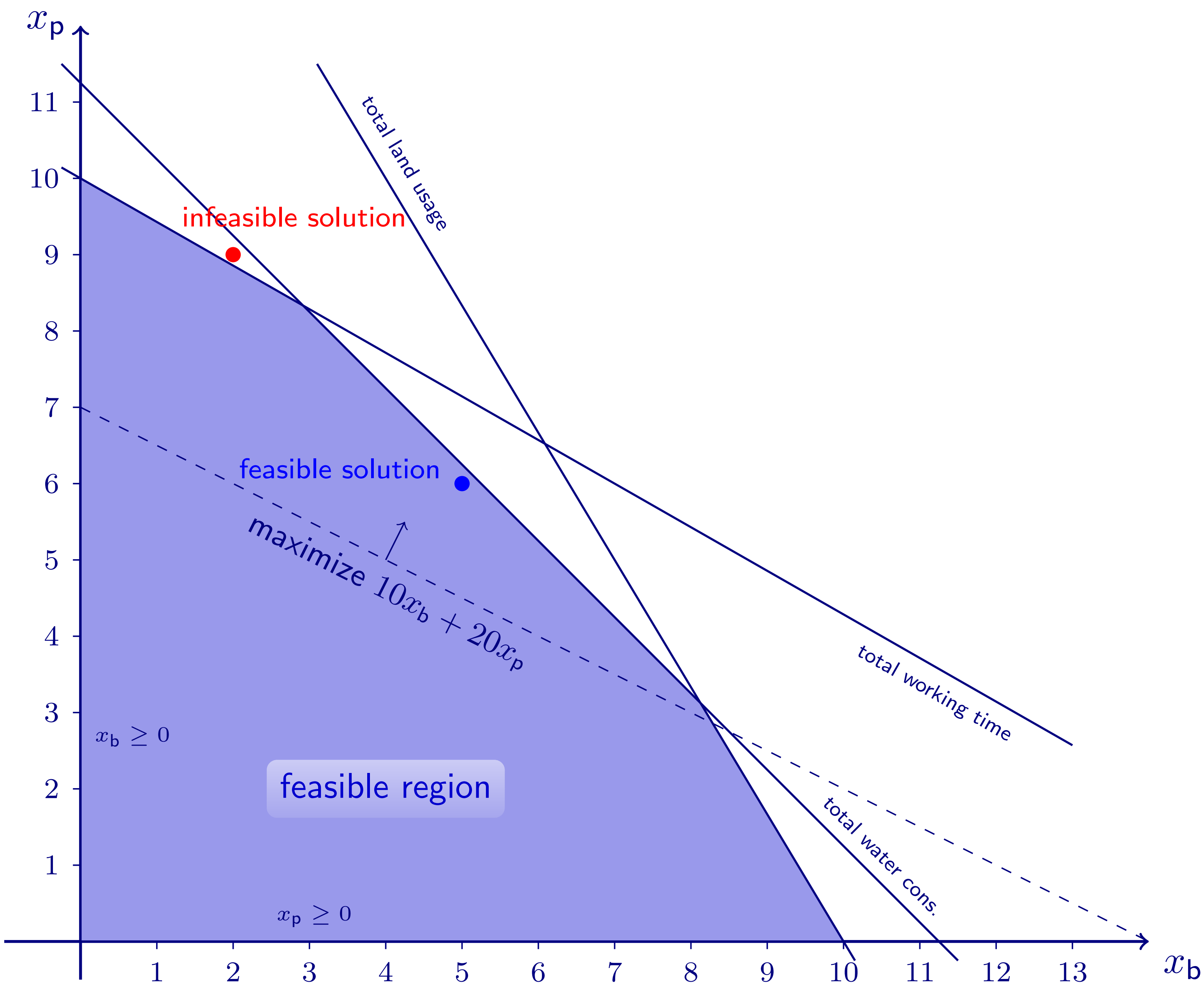

x_p
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total land usage
infeasible solution
feasible solution
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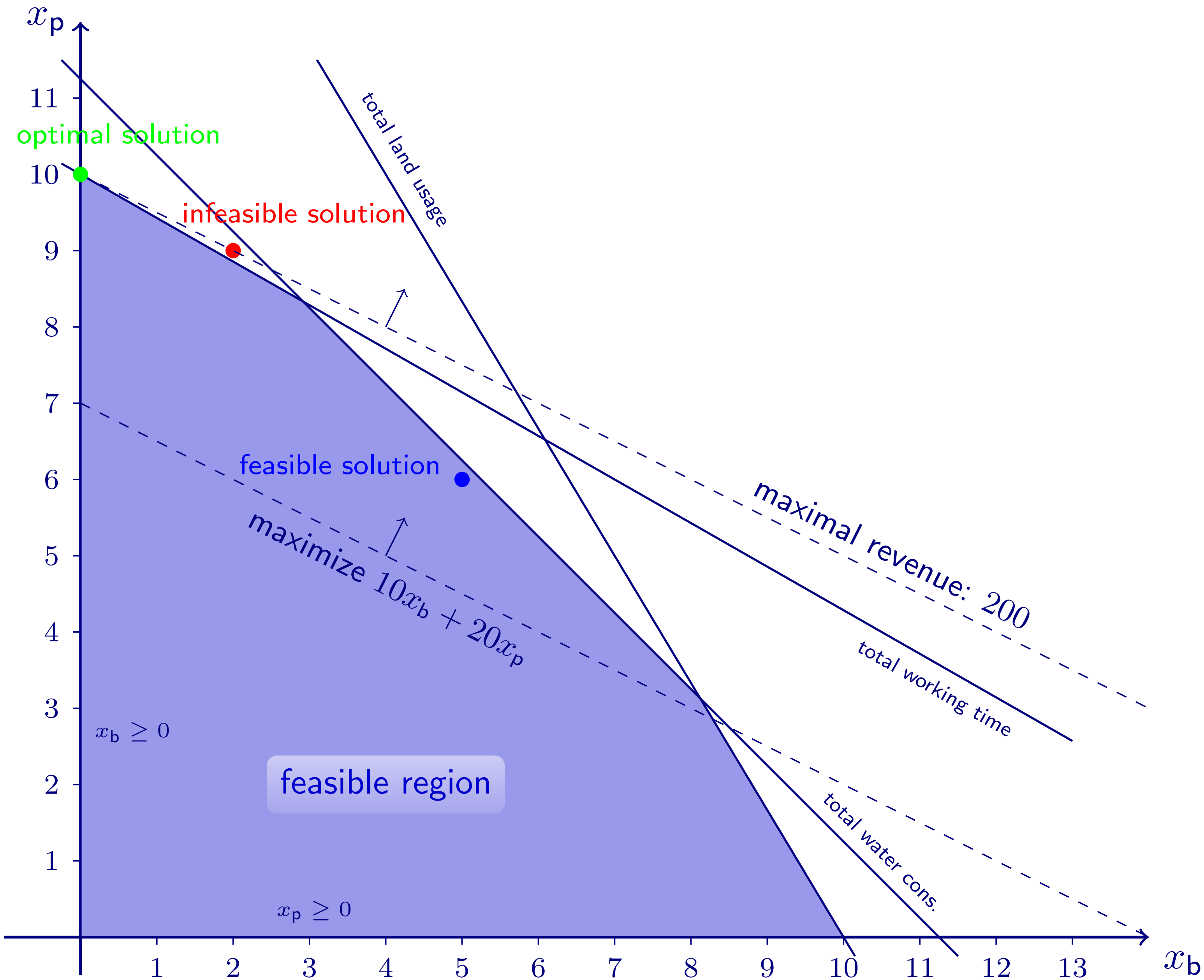
total working time
$x_b \geq 0$
feasible region
total water cons.
$x_p \geq 0$



x_p
optimal solution
infeasible solution
total land usage
feasible solution
maximize $10x_b + 20x_p$
maximal revenue: 200
total working time
$x_b \geq 0$
feasible region
total water cons.
$x_p \geq 0$
x_b
1 2 3 4 5 6 7 8 9 10 11
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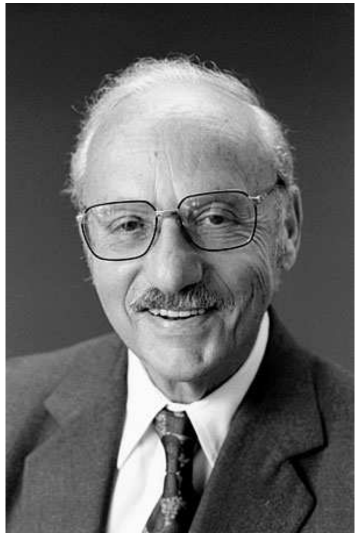
  - Otherwise: stop – an optimal solution is reached!
- ▷ Special cases:
  - No starting vertex can be found ➡ Problem is infeasible
  - No neighbouring vertex in some objective-increasing direction ➡ Problem is unbounded

LP algorithm zoo

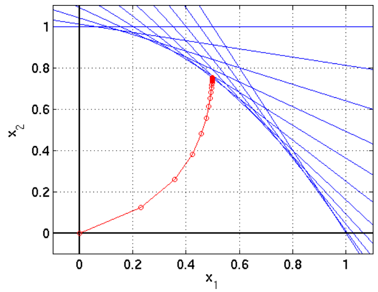
- Simplex Algorithm
  - ➡ developed by George B. Dantzig in 1947
  - ➡ Variants:
    - Dual Simplex Algorithm
    - Network Simplex

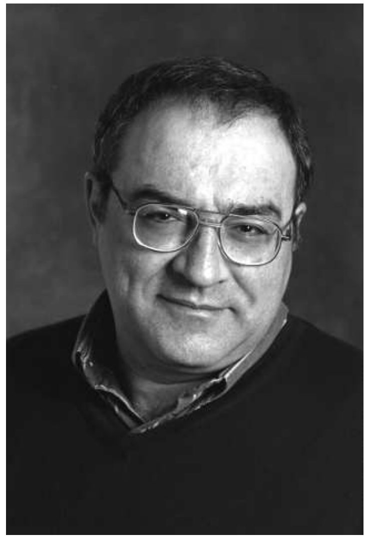


George Bernard Dantzig
(1914–2005)

- Ellipsoid Method
  - ➡ developed by L.G. Khachiyan in 1979
  - ➡ theoretically fast (polynomial), but practically useless

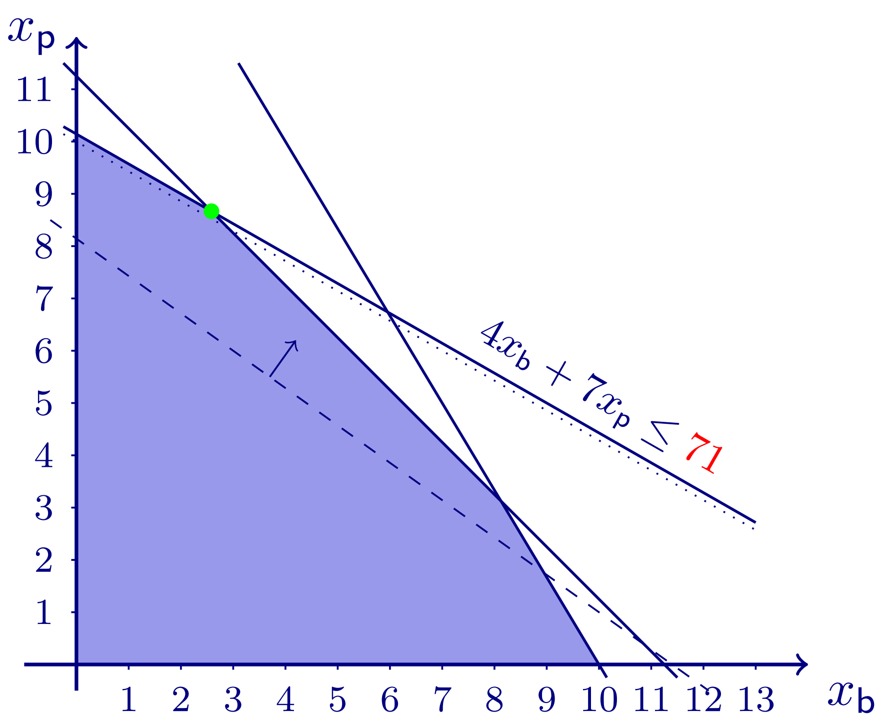
- Interior Point Methods
  - ➡ Barrier Method (Karmarkar, 1984)
  - ➡ theoretically and practically fast
  - ➡ used for large-scale LPs





Leonid Genrikhovich Khachiyan
(1952–2005)

The shadow price of a constraint is the rate of change in the objective function per unit increase of the constraint's right-hand side



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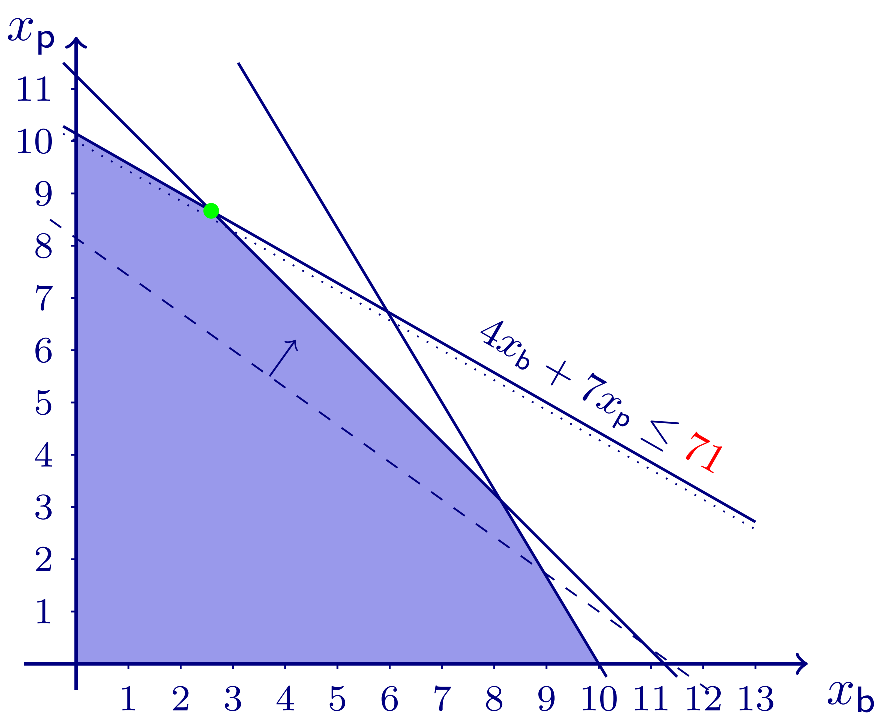

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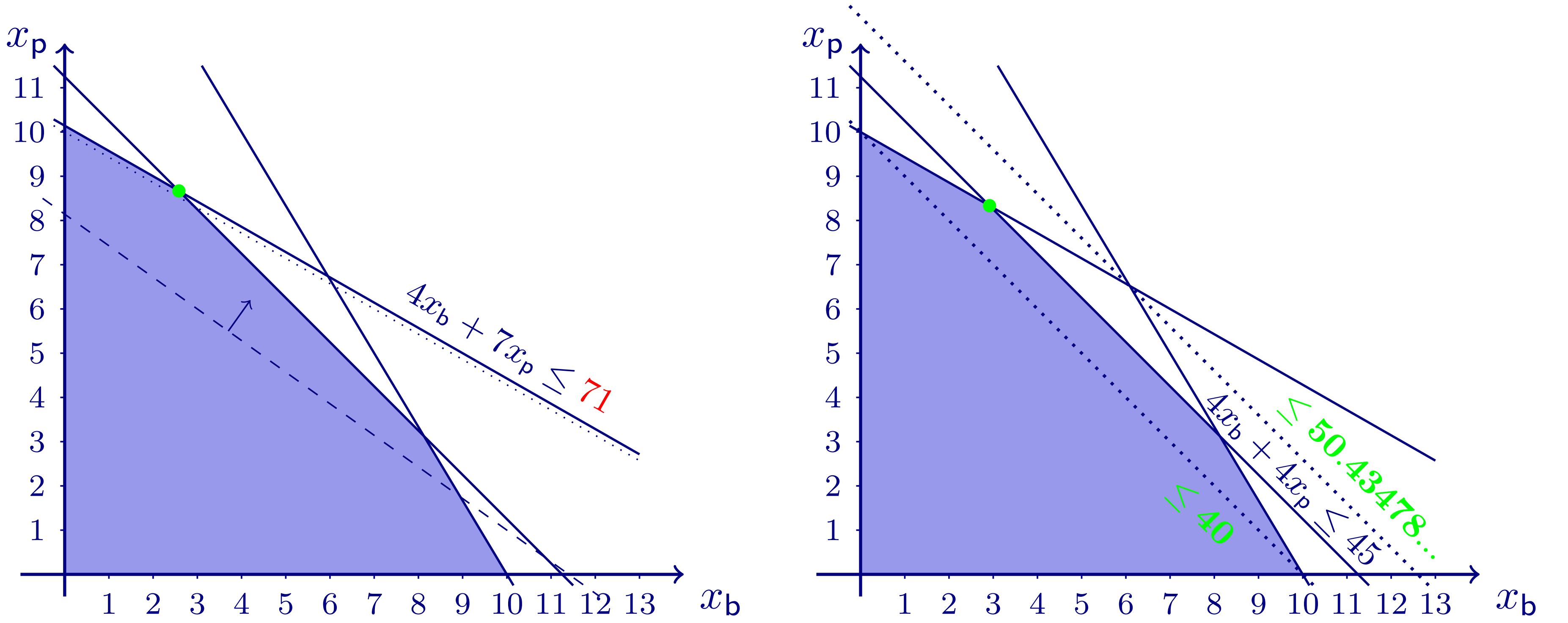
- ▷ Shadow prices for non-binding constraints are always 0
- ▷ Shadow price for a constraint is only valid if the RHS is in a certain range



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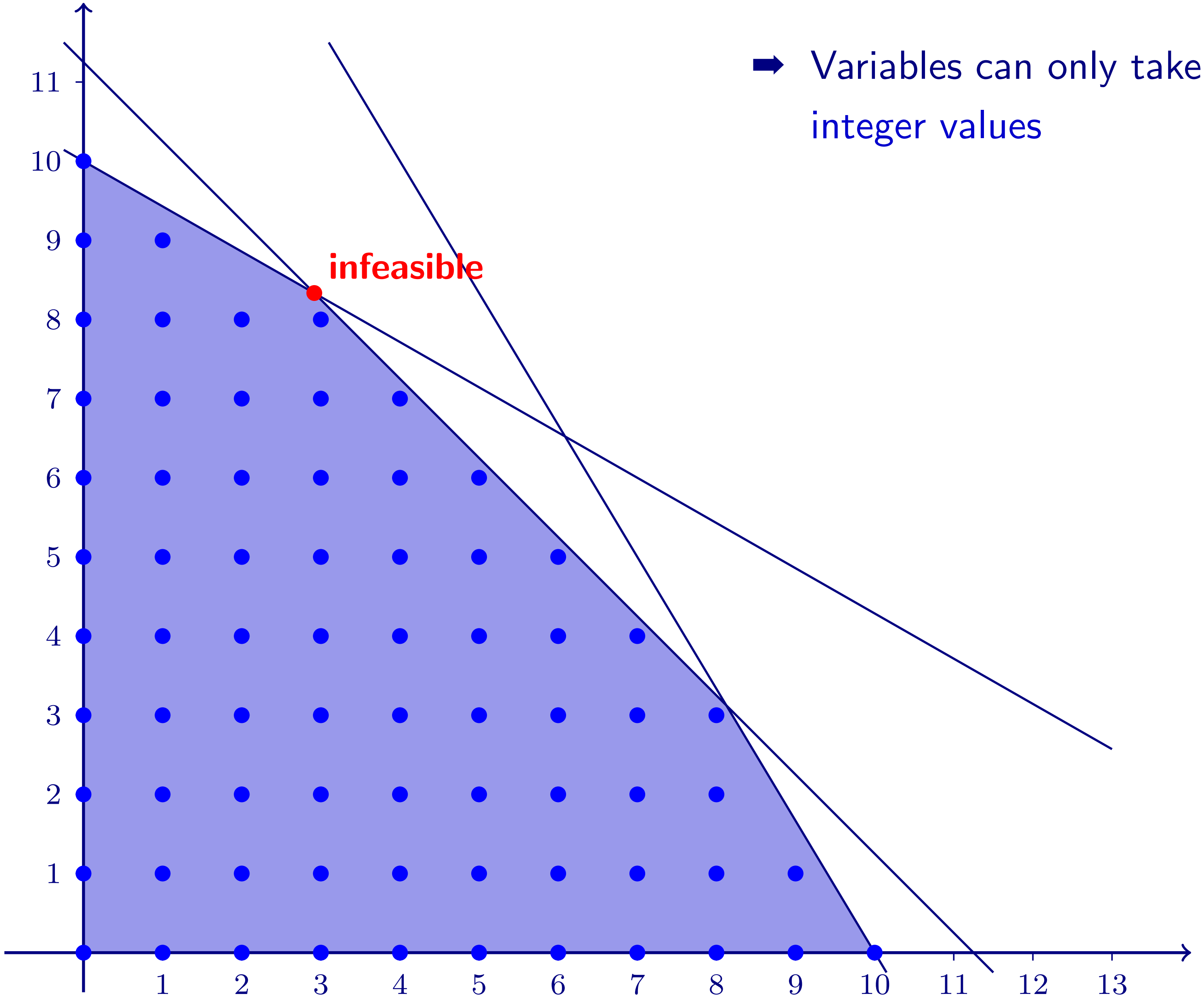
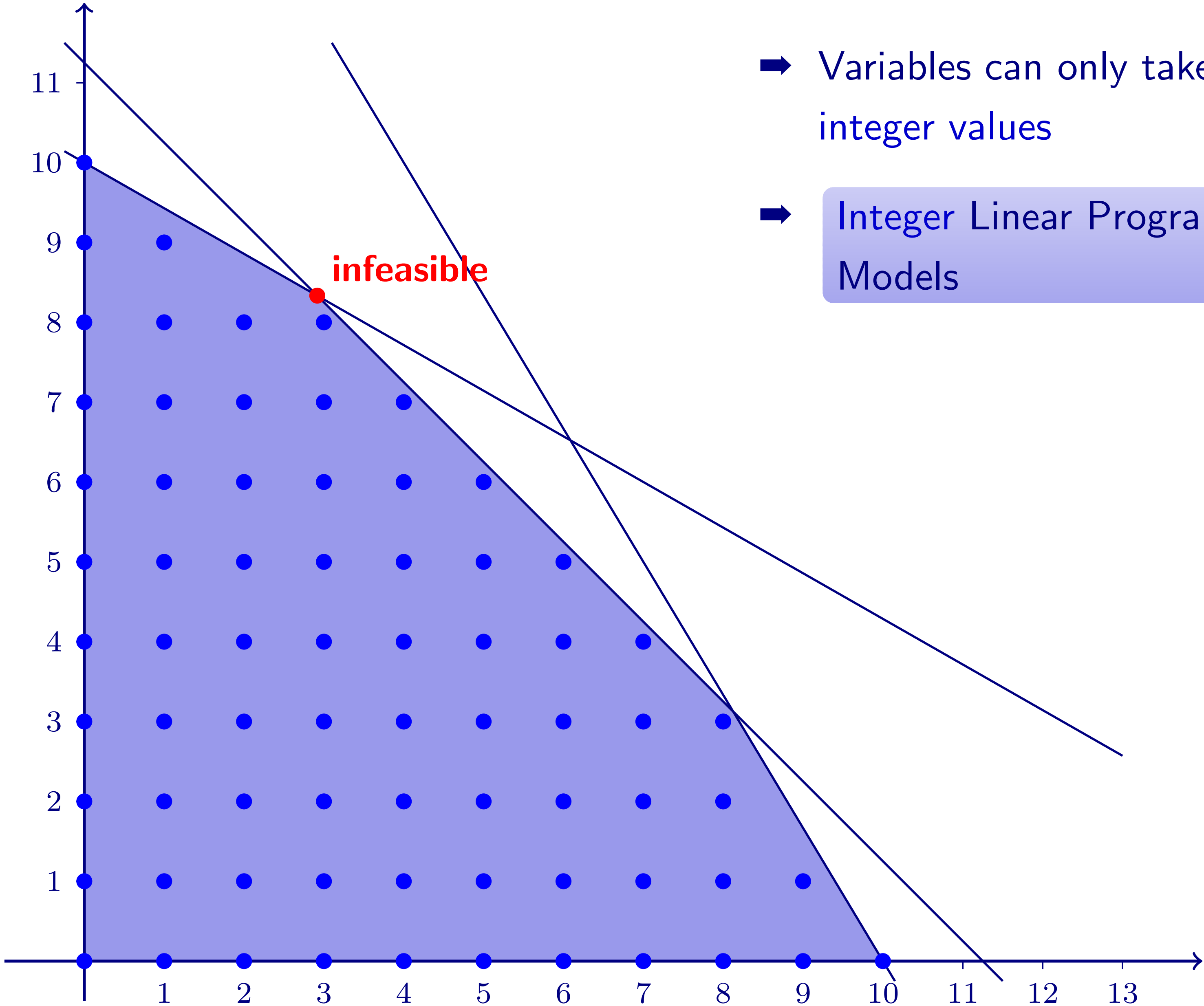


➡ Variables can only take integer values



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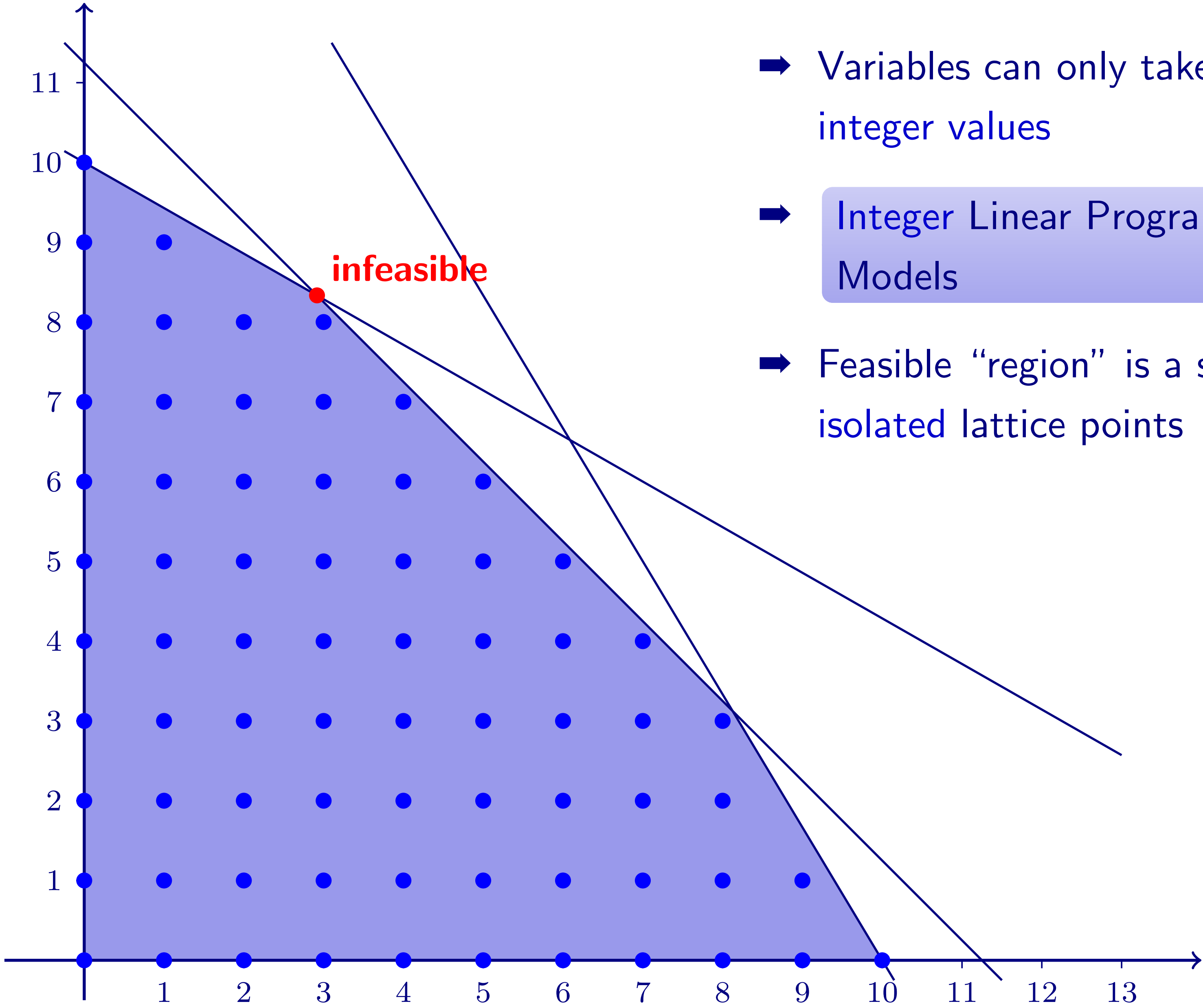

- ➡ Variables can only take integer values
- ➡ Integer Linear Programming Models
- ➡ Feasible "region" is a set of isolated lattice points

LP-relaxation

maximize/minimize $\sum_{j=1}^{n} c_j x_j$ **Objective function**

subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for all $i = 1, \ldots, m$ **C**

$\ell_j \leq x_j \leq u_j$ for all $j = 1, \ldots, n$ **V**

➡ Integer Program: x_j integer for all $j = 1, \ldots, n$

➡ Mixed Integer Program: x_j integer for all $j = 1, \ldots, \ell$ $(\ell < n)$

➡ Binary variables: $x_j \in \{0, 1\}$ for all $j = 1, \ldots, \ell$

- ▷ Trigger a yes/no-decision if some quantity reaches some value V
  - ➡ binary variable $y \in \{0, 1\}$, meaning: $y = 1 \Leftrightarrow$ “yes”
  - ➡ (...linear expression for quantity...) $\leq V + M \cdot y$ (M: large number)

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- ▷ Set packing constraints:
  - ➡ choose at most/at least/exactly one of the binary variables $y_1, \ldots, y_n$
  - ➡ $y_1 + y_2 + \ldots + y_n \leq 1$ / ≥ 1 / $= 1$

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- ▷ Lots of other types...

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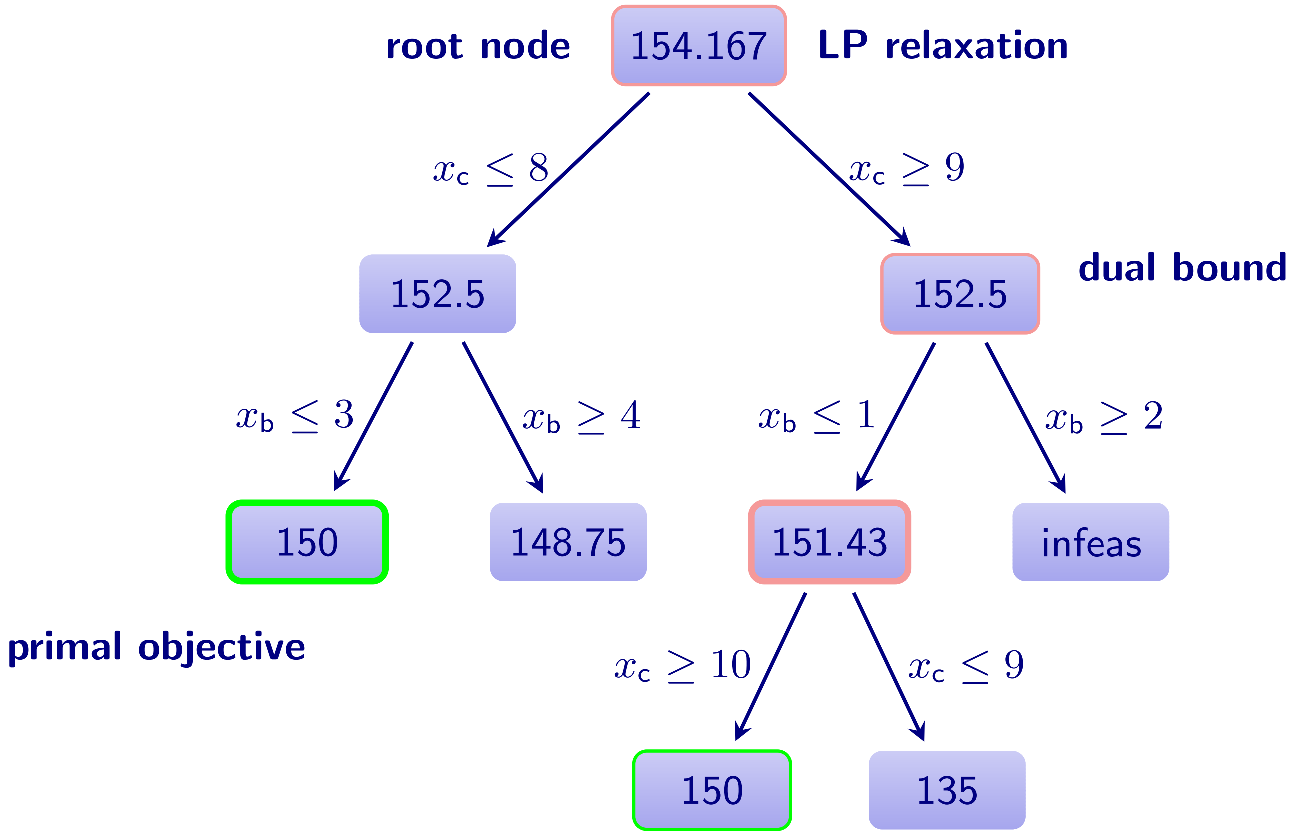
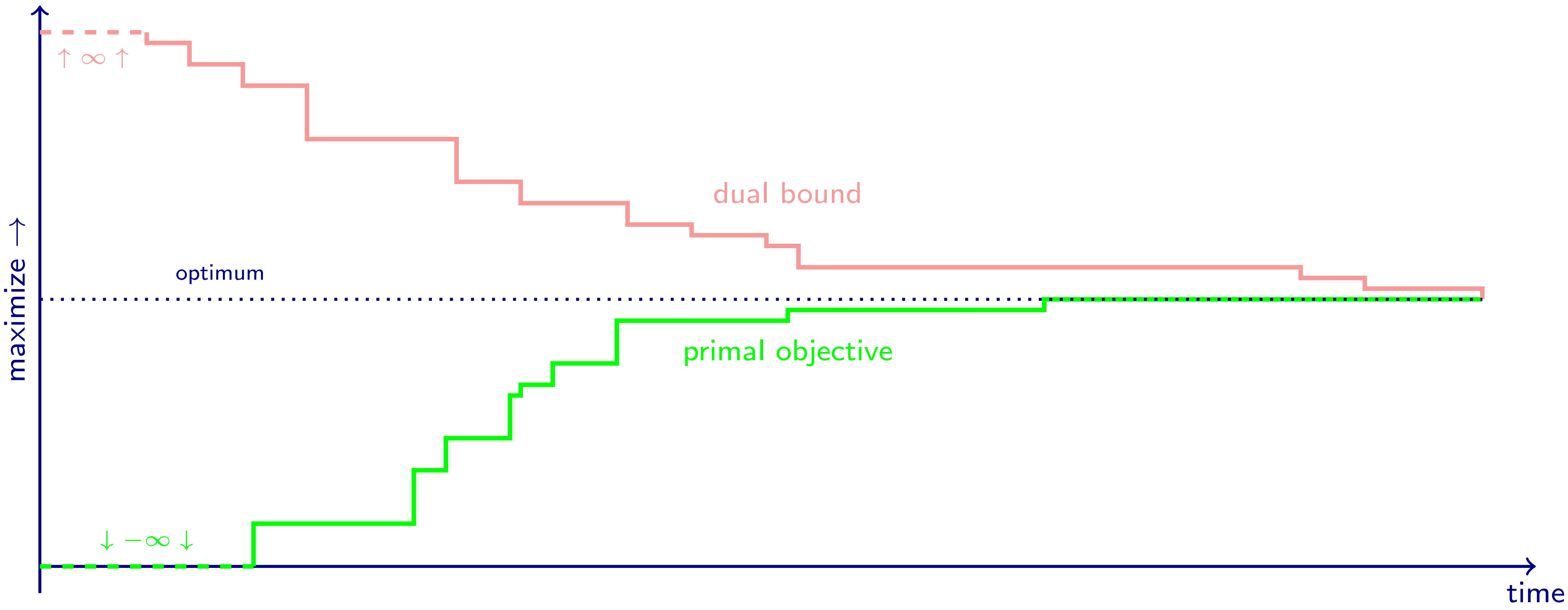
- ▷ If the LP-optimum is not integer: split the problem into two subproblems and iterate
- ➡ Branch-and-bound tree (example for maximization problem):



▷ The (absolute) gap during branch and bound: gap := bestdual − bestprimal



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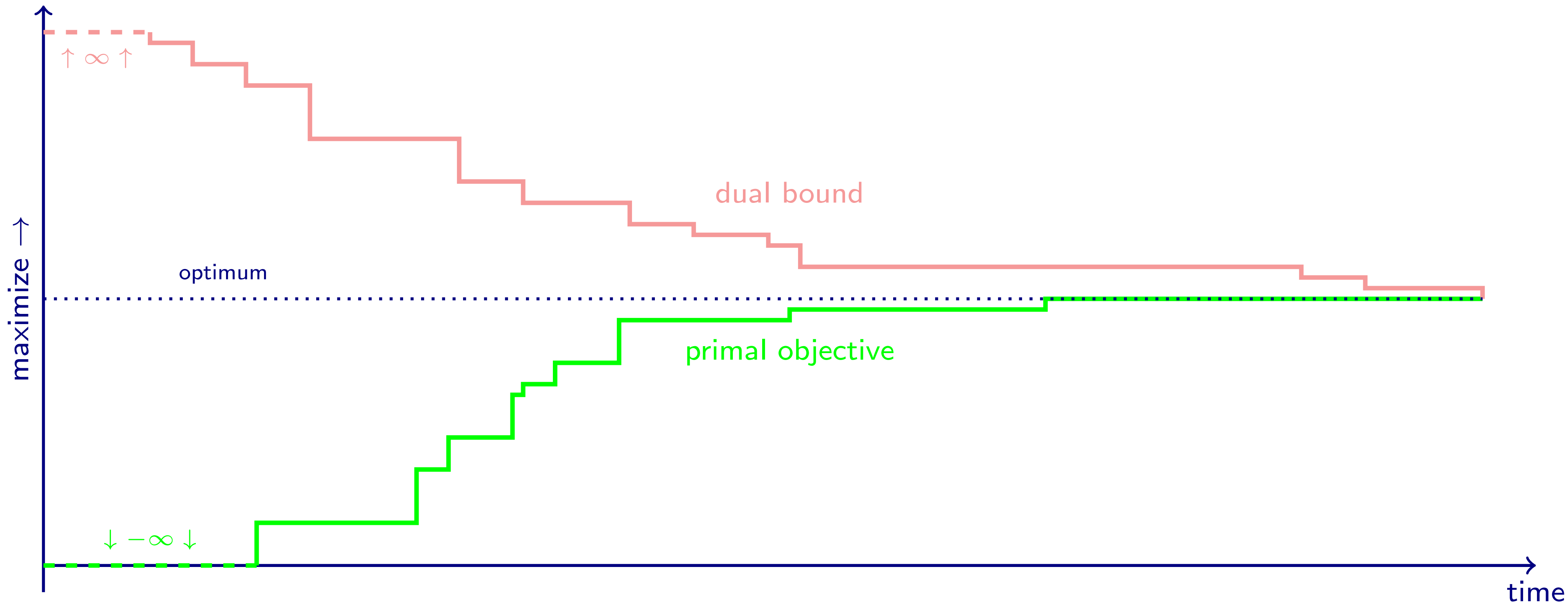

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Gap

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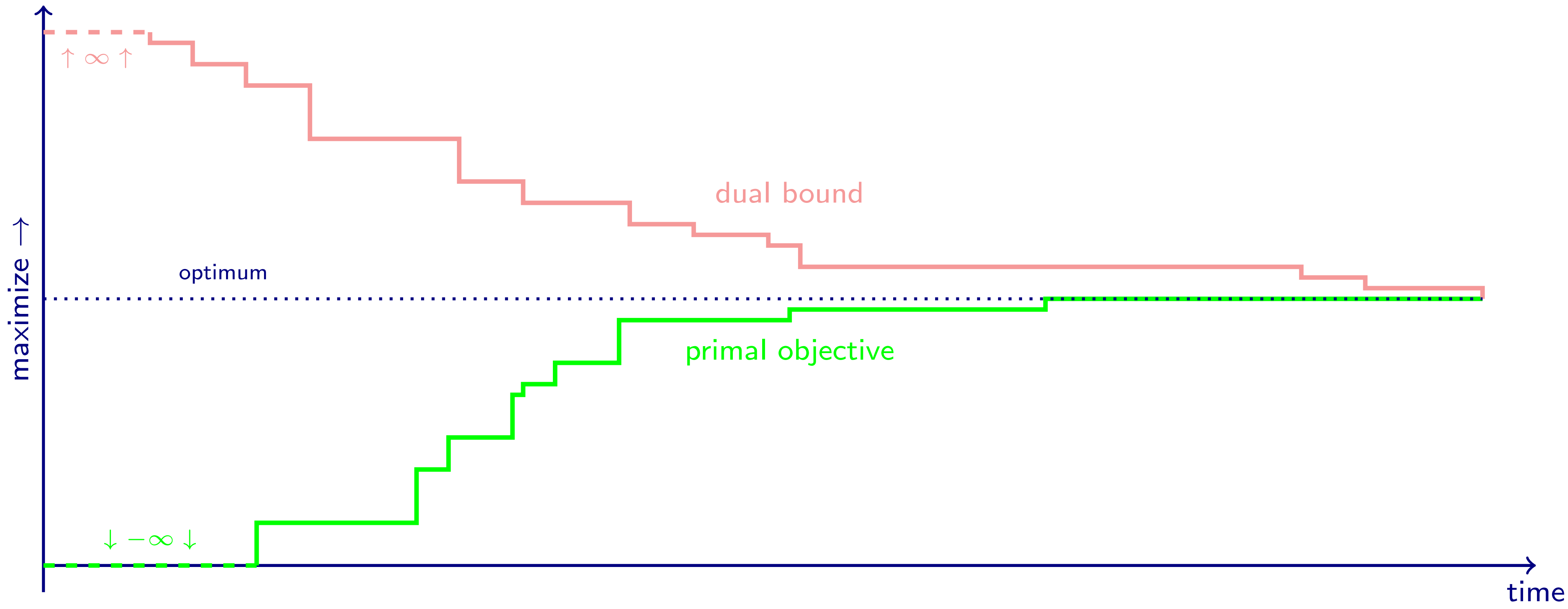

- Stop traversing the tree, if the gap is 0, i.e. the value of the best primal solution and the dual bound coincide
- In practise (and for large-scale MIPs): stop traversing the tree already if the relative gap $\dfrac{\text{gap}}{|\,\text{bestdual}\,|}$ is below a certain target (e.g. 5%, 1%, 0.5%, ...)

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- Examples:
  - Travelling Salesman Problem (TSP)
  - Minimum Spanning Tree (MST)
  - Shortest Path Problem (SPP)
  - Network Flow, Knapsack Problem, Bin Packing, Stable Set Problem, ...

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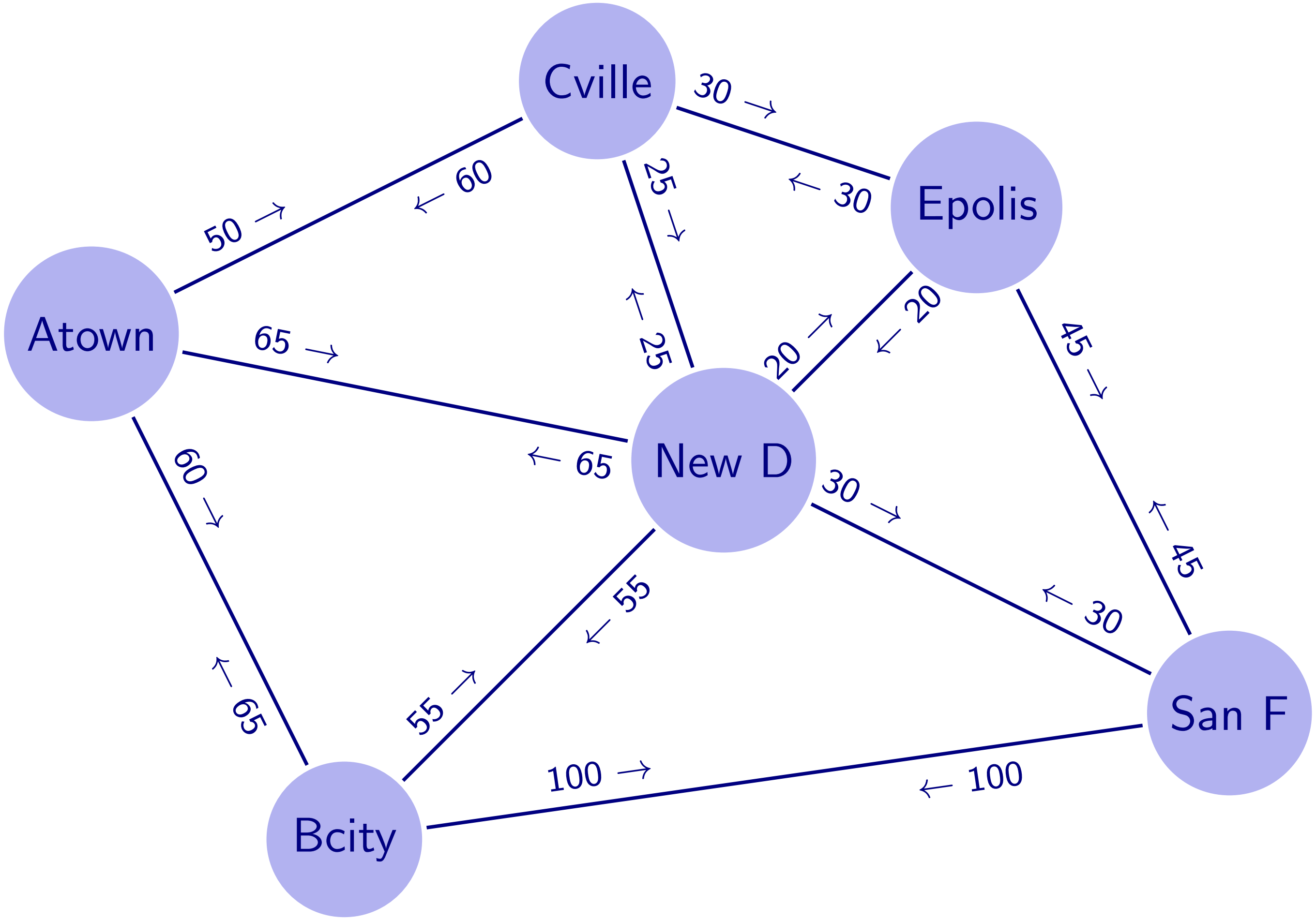

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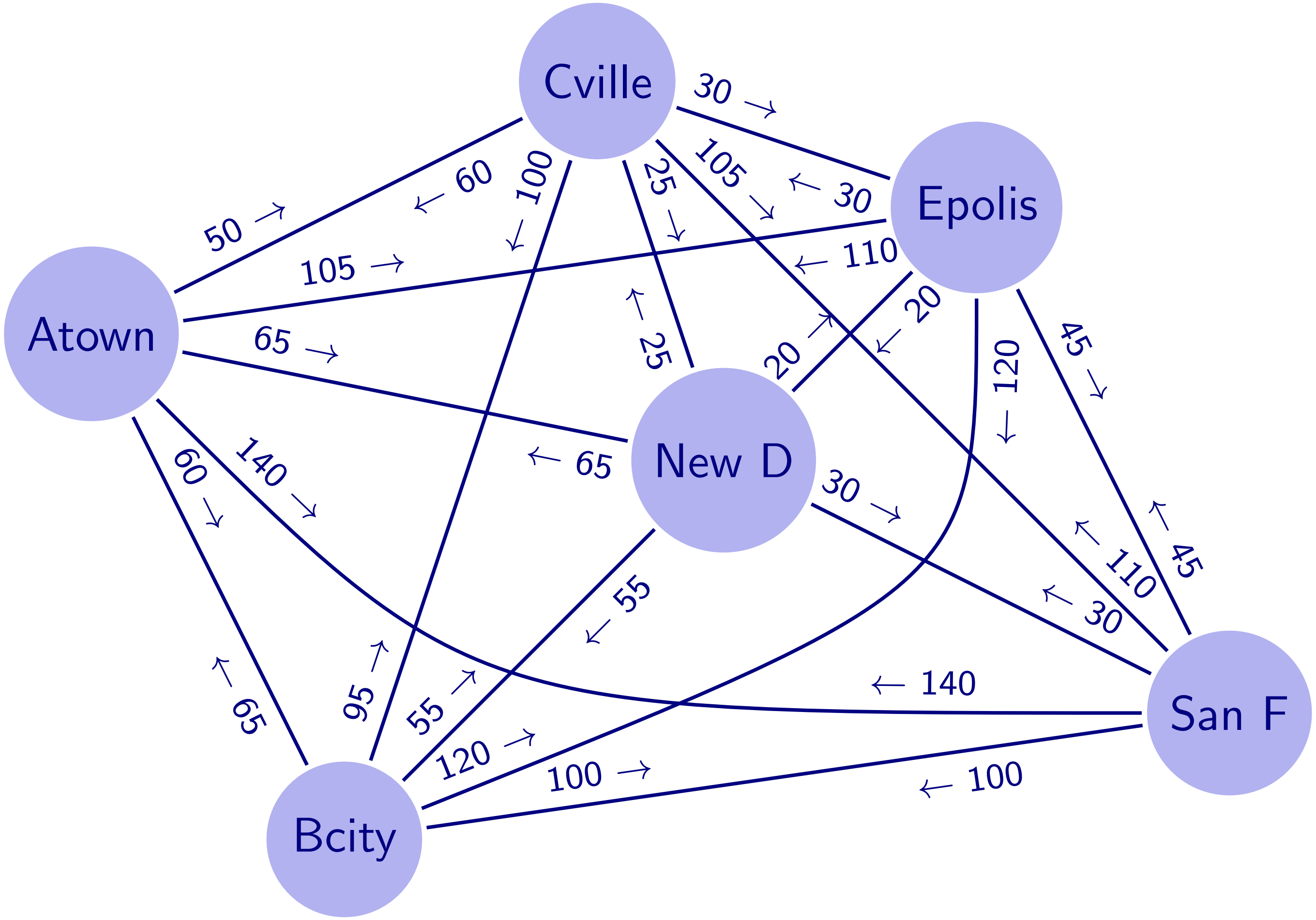
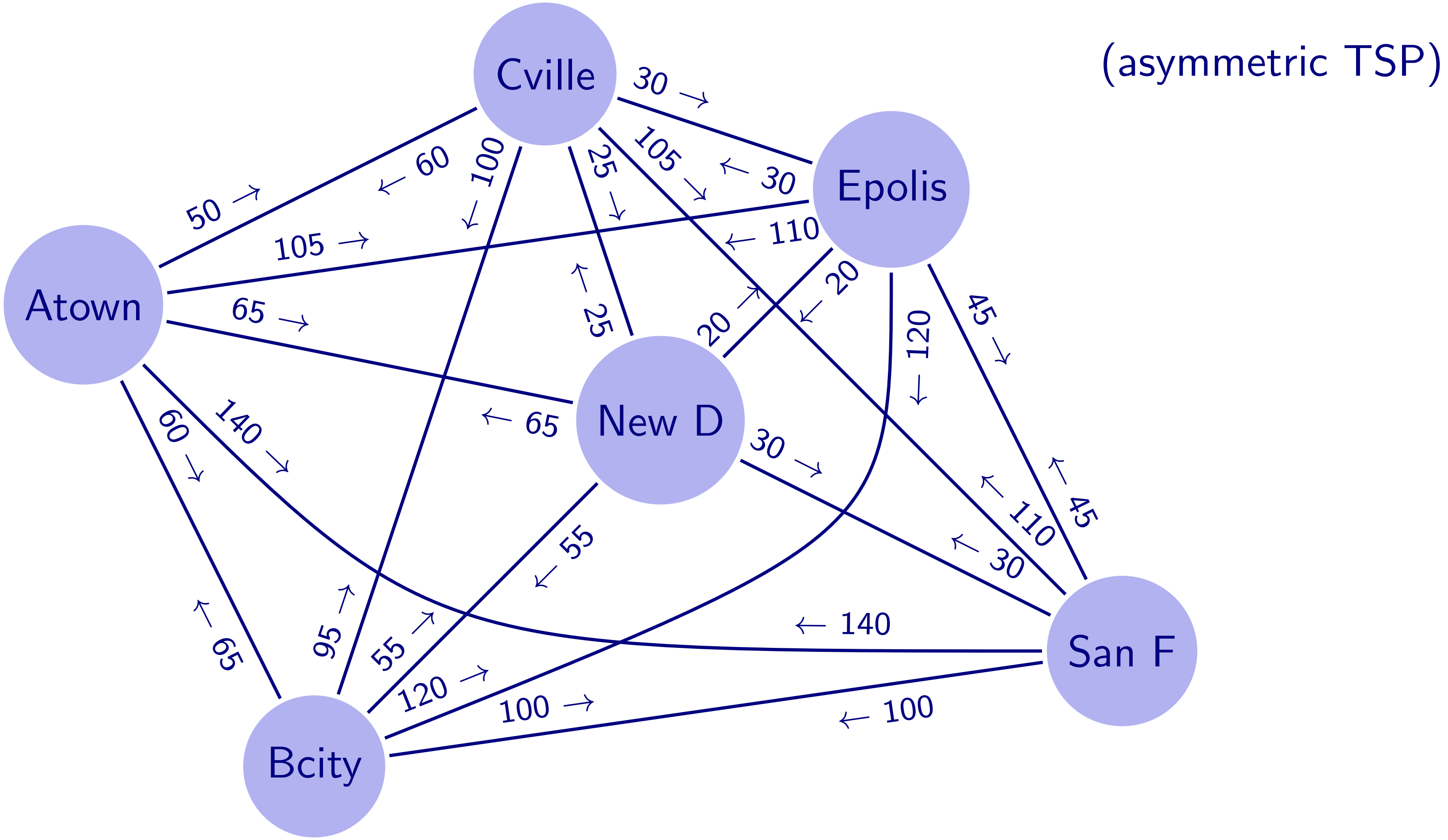


Problem formulation: Given a set of cities together with travel times to travel from every city to every other, find a tour leading through every city such that the total travel time is minimized.



▷ Combinatorial explosion:

# cities	# possible tours	time to try out all tours
5	24	0.012 milliseconds
10	362,880	0.18 seconds
15	87,178,291,200	12 hours
20	121,645,100,408,832,000	1927 years

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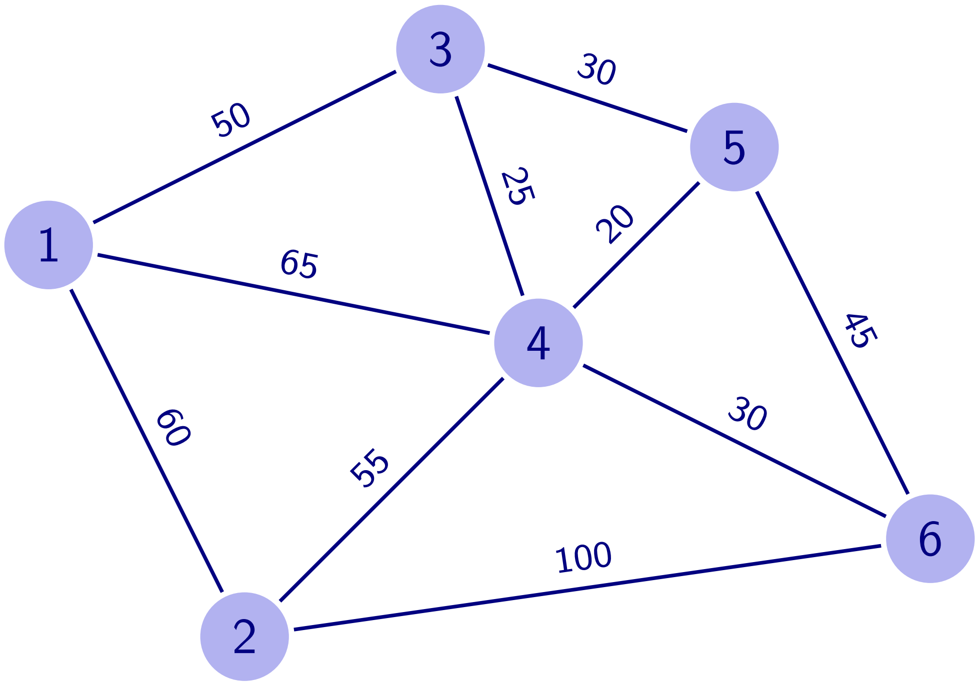
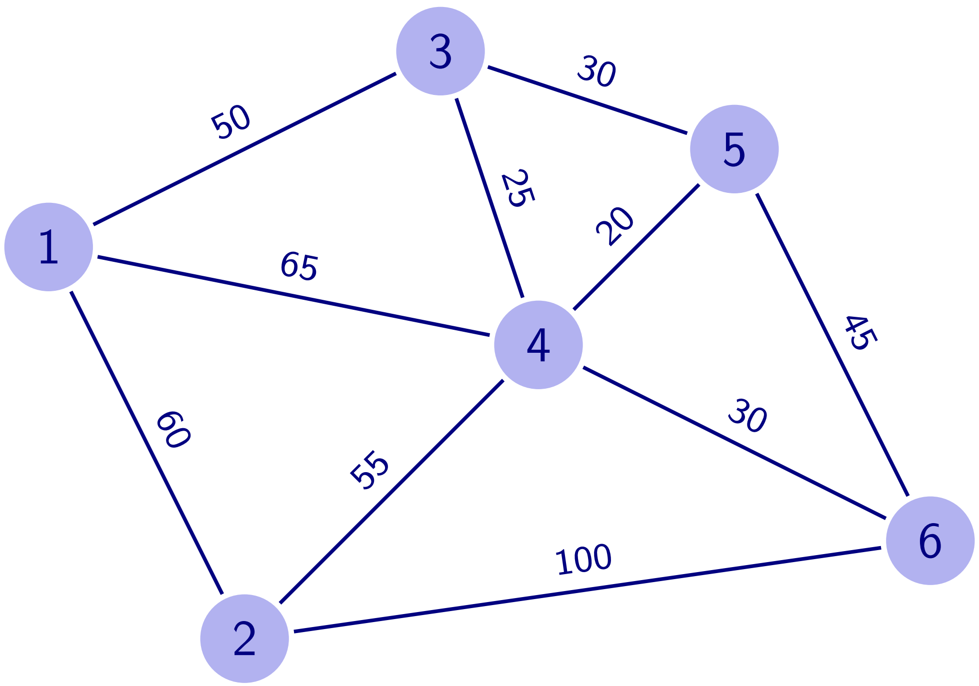
- ▷ TSP can be formulated as a binary integer program
  - ➡ Exponentially many (subtour elimination) constraints
  - ➡ Dual method to provide upper bounds for the solution
- ▷ Approximation algorithms and heuristics
  - ➡ Graph algorithms
  - ➡ Primal methods to find (good) feasible solutions

▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...



▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...

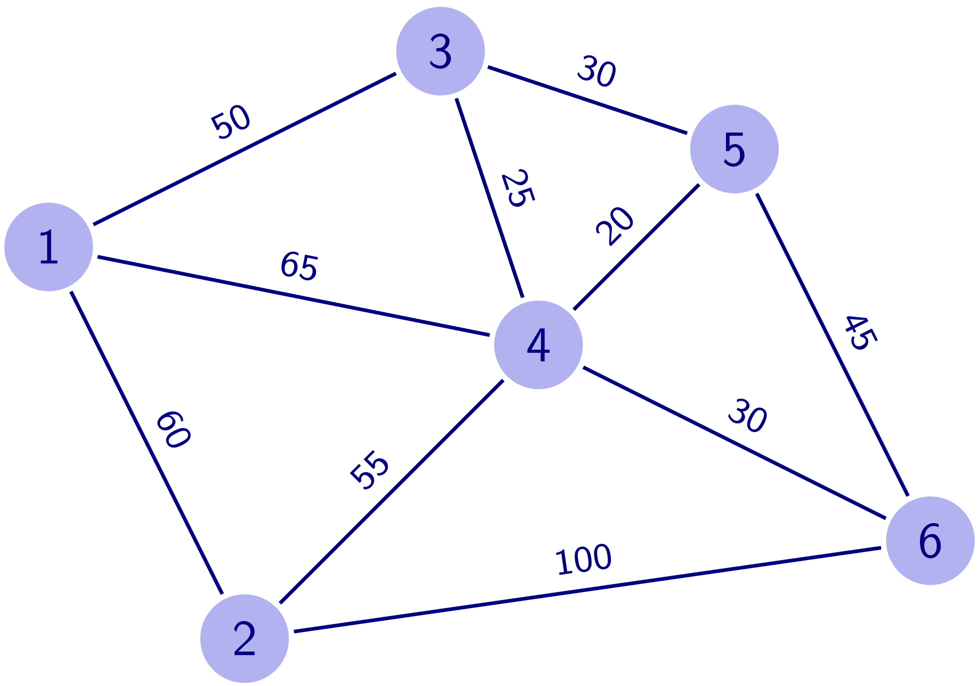
...find a minimum spanning tree for G



▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...

...find a minimum spanning tree for G, that is: a subset E' of the edges such that

- the edges in E' form a tree (connected and no cycles)
- all vertices of G are in the tree
- the total weight of the tree edges is minimal



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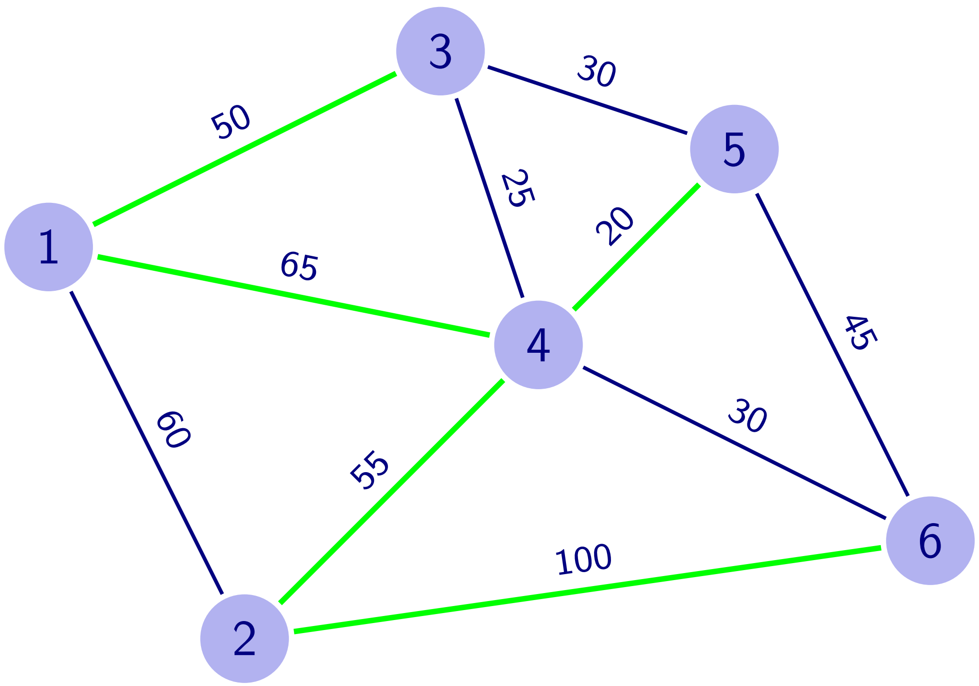

➡ total weight: 290

▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...

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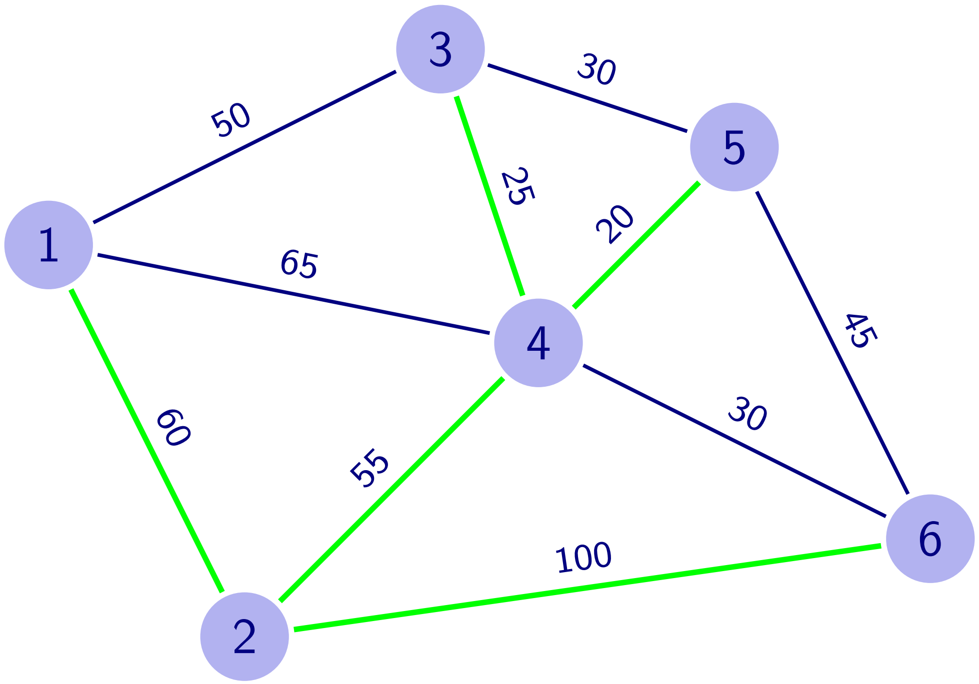

➡ total weight: 260

▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...

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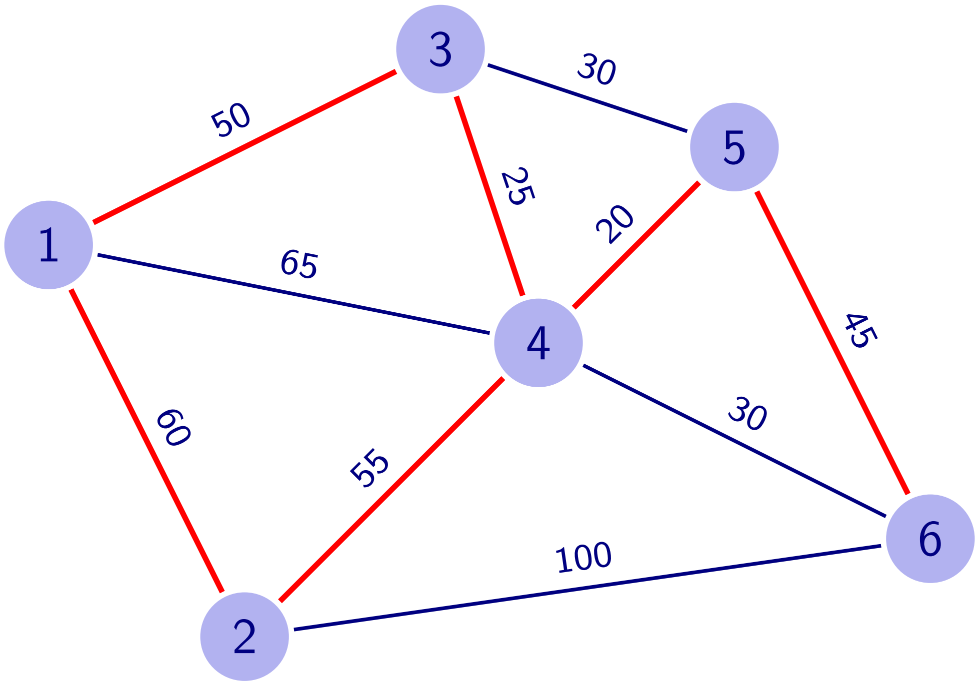

➡ not allowed: not a tree!

▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...

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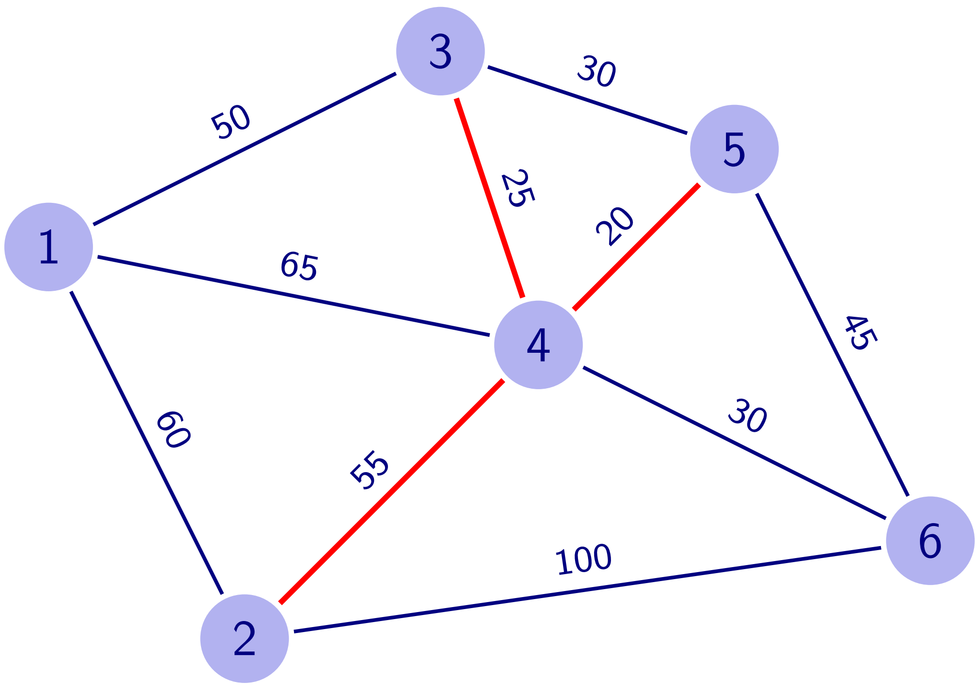

➡ not allowed: misses vertices!

▷ Given a graph $G = (V, E)$ with non-negative edge-weights w_e for all $e \in E$...

...find a minimum spanning tree for G, that is: a subset E' of the edges such that

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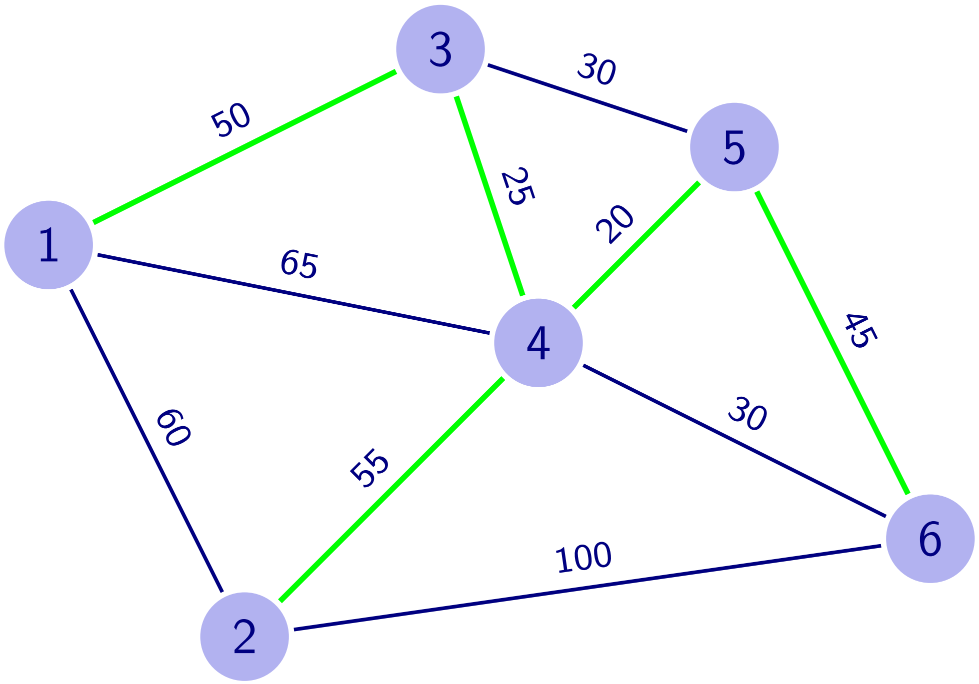
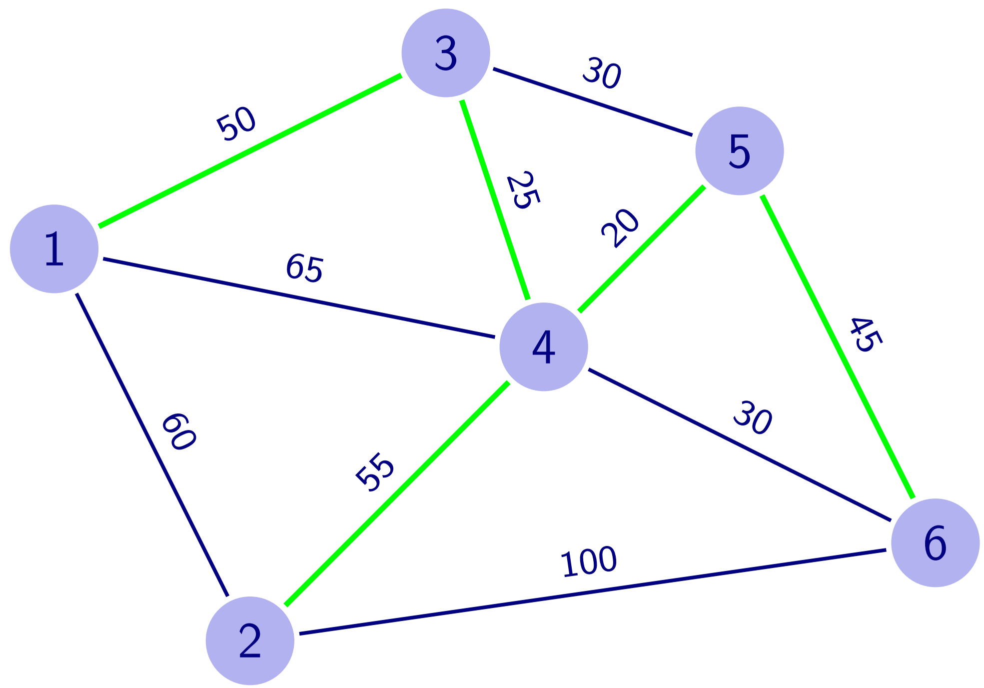


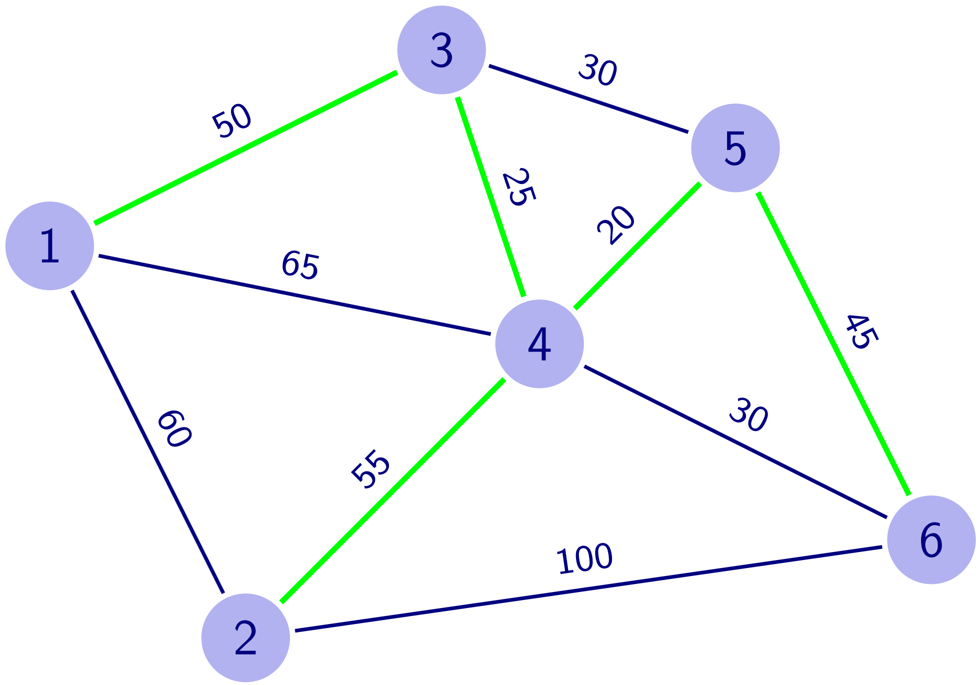
➡ total weight: 195

▷ Idea: at every step select the next cheap edge, as long as it doesn't result in a cycle



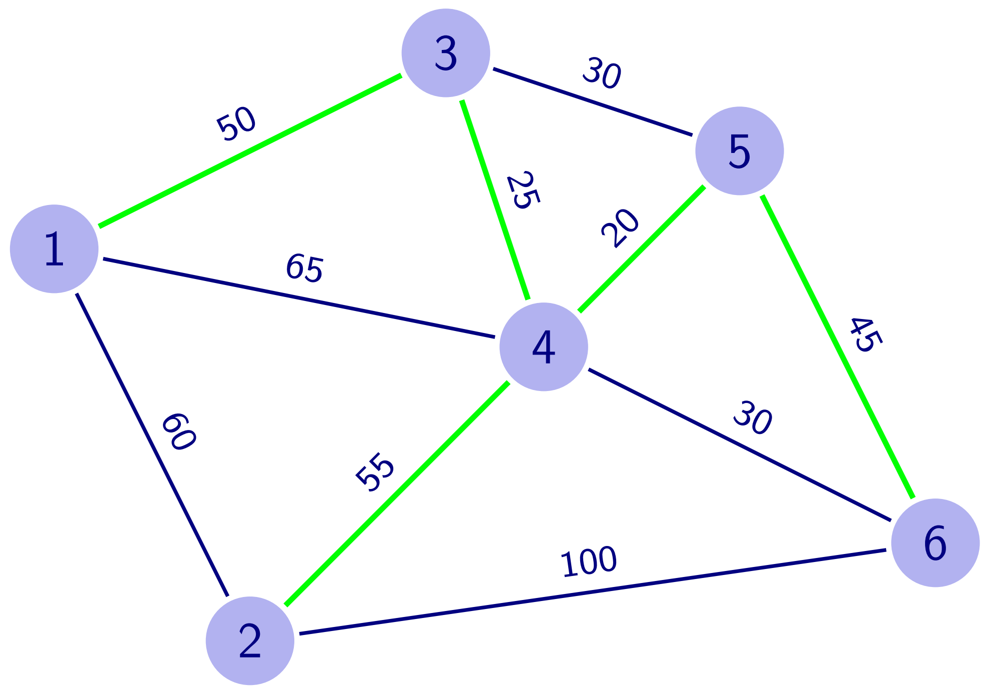
- Idea: at every step select the next cheap edge, as long as it doesn't result in a cycle

  ➡ Greedy algorithm

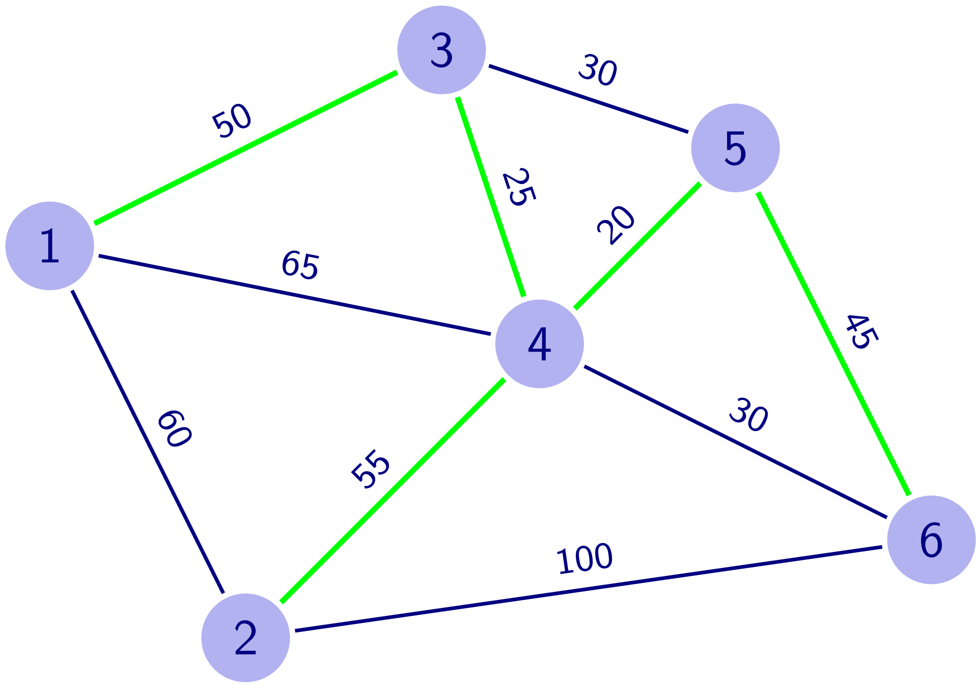


▷ Idea: at every step select the next cheap edge, as long as it doesn't result in a cycle

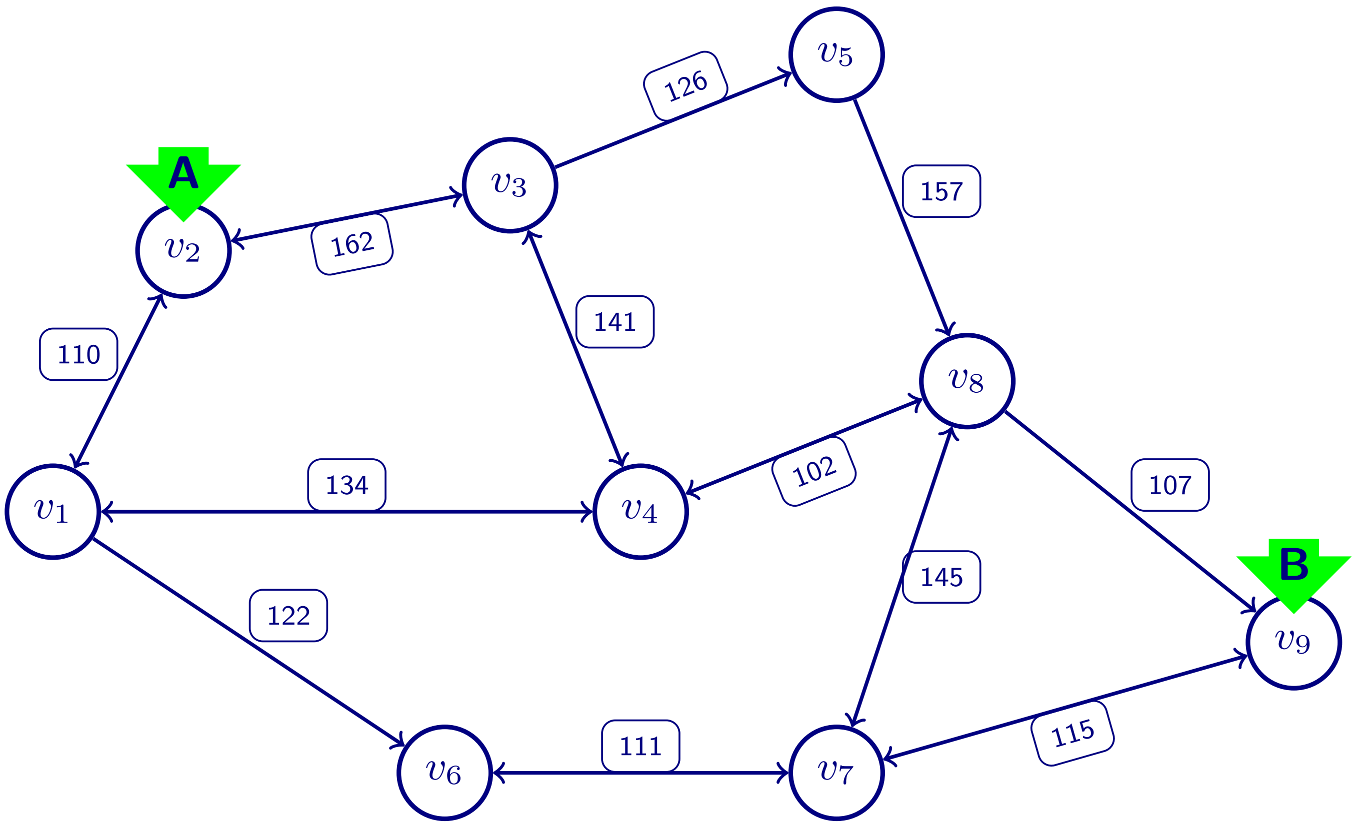
- ➡ Greedy algorithm
- ➡ Polynomial runtime
  - ➡ efficient algorithm



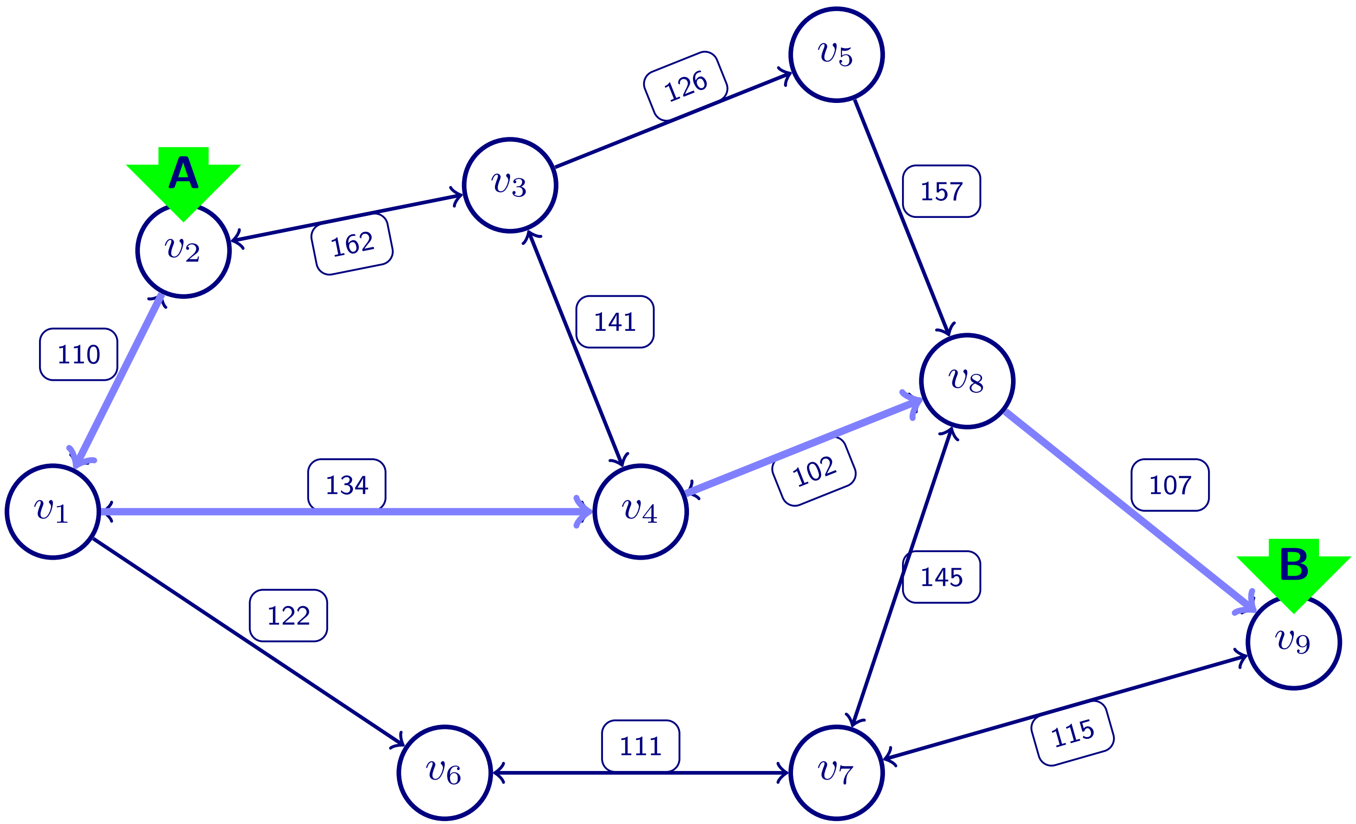
- Idea: at every step select the next cheap edge, as long as it doesn't result in a cycle
  - ➡ Greedy algorithm
  - ➡ Polynomial runtime
    - ➡ efficient algorithm
  - ➡ Yields an optimal solution for every input graph (proof!)
    - ➡ exact algorithm



▷ Given a network – i.e. a directed graph – with a length for each arc, a start node A and a destination B...



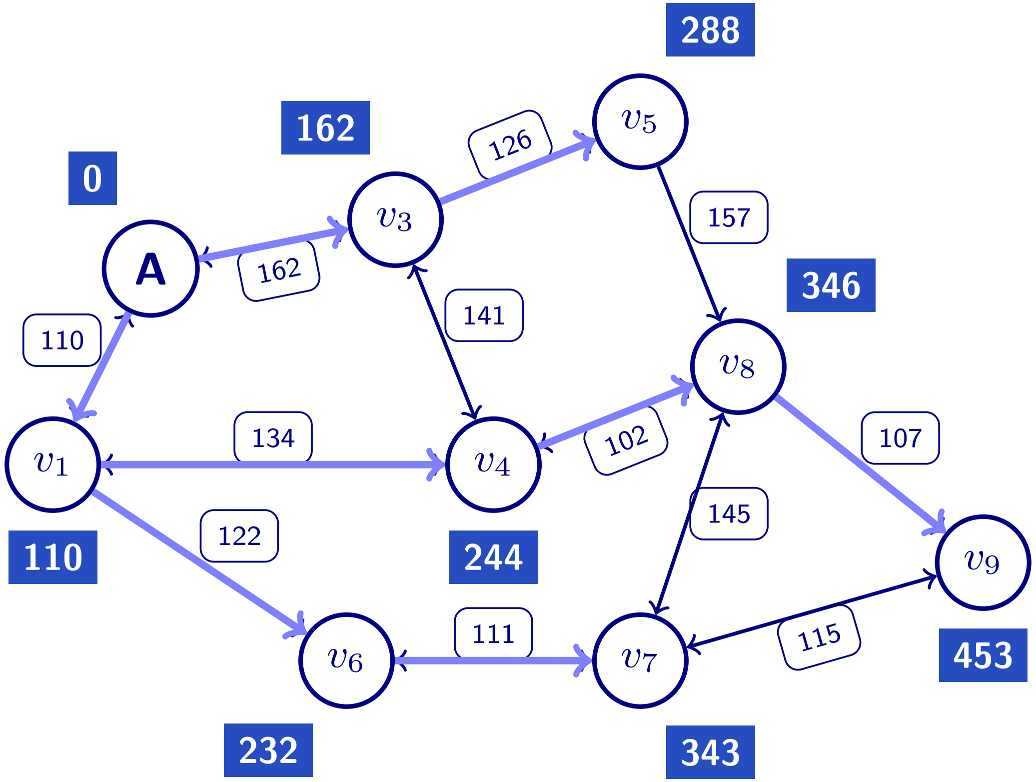
- Given a network – i.e. a directed graph – with a length for each arc, a start node A and a destination B...
- ...compute a shortest path through the network from A to B



- Computes a complete shortest path tree from start node A to all other nodes

Edsger Wybe Dijkstra
(1930–2002)

▷ Computes a complete shortest path tree from start node A to all other nodes



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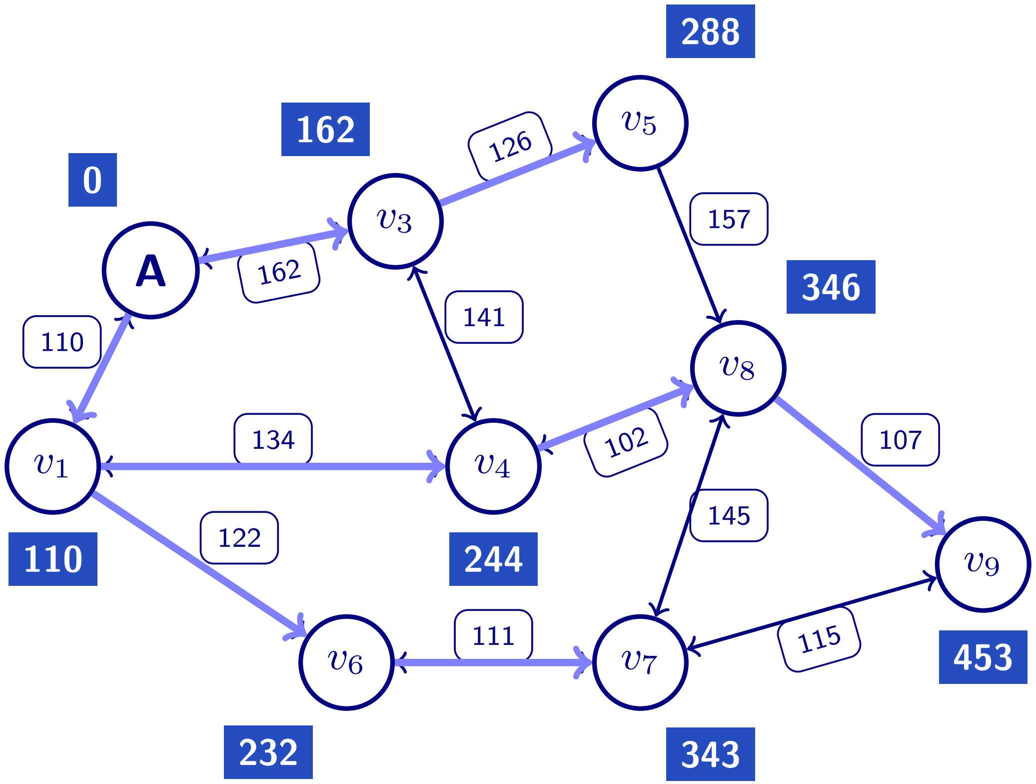
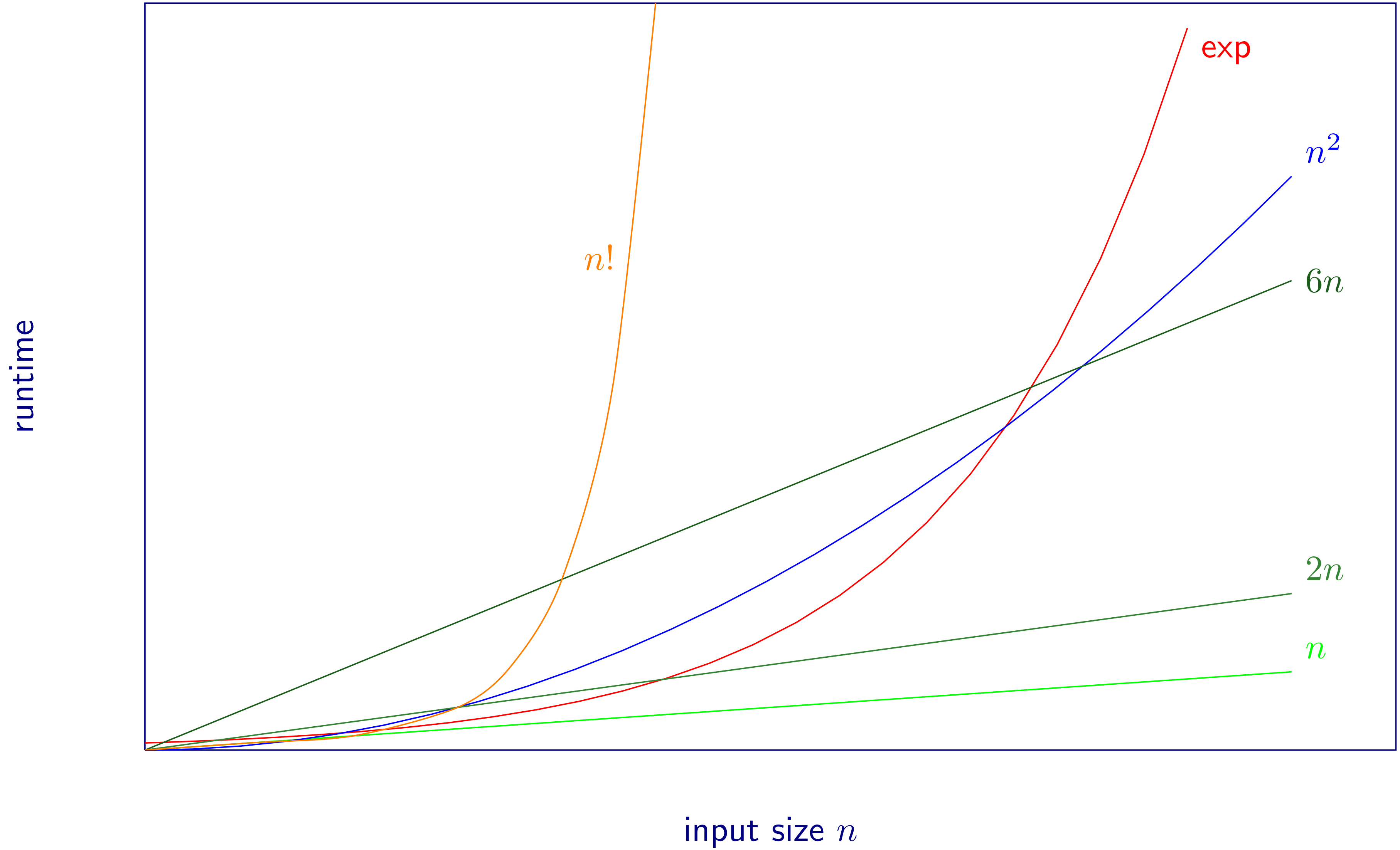


Edsger Wybe Dijkstra
(1930–2002)

▷ Polynomial runtime ➡ efficient algorithm

▷ Always yields an optimal solution ➡ exact algorithm



linear — polynomial — exponential

An algorithm is called efficient if it has polynomial runtime (i.e. its runtime can be bounded by a polynomial in the input size).

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An algorithm is called exact if it guarantees to return an optimal solution (i.e. it can be proved that it always (i.e. for every input!) returns a solution with best possible objective).

	efficient	not efficient
exact	Dijkstra's algorithm Kruskal's algorithm Ellipsoid method	Simplex algorithm (?) Branch & bound Complete enumeration
not exact	TSP heuristic using MST approximation algorithms	

- Models, Data and Instances
- Linear Optimization
  - Modelling as a linear program
  - Solving a linear program (graphically, and in princple by the simplex algorithm)
  - Sensitivity analysis
- (Mixed) Integer Programming
  - Modelling as a (mixed) integer program
  - How to solve a (mixed) integer program (in principle)
- Combinatorial Optimization
  - Exemplary problems, algorithms, and runtimes
- Nonlinear Optimization
  - Local and global optima, convex optimization
- Scheduling
- Lot Sizing
- Multicriteria Optimization

▷ Model (non-linear program):

Objective maximize $10x_{\mathsf{b}} + 150\sqrt{x_{\mathsf{p}}} - 20x_{\mathsf{p}}$

C subject to

$$\begin{aligned} 5x_{\mathsf{b}} + 3x_{\mathsf{p}} &\leq 50 && \text{(total land usage)} \\ 4x_{\mathsf{b}} + 7x_{\mathsf{p}} &\leq 70 && \text{(total working time)} \\ 4x_{\mathsf{b}} + 4x_{\mathsf{p}} &\leq 45 && \text{(total water consumption)} \end{aligned}$$

V

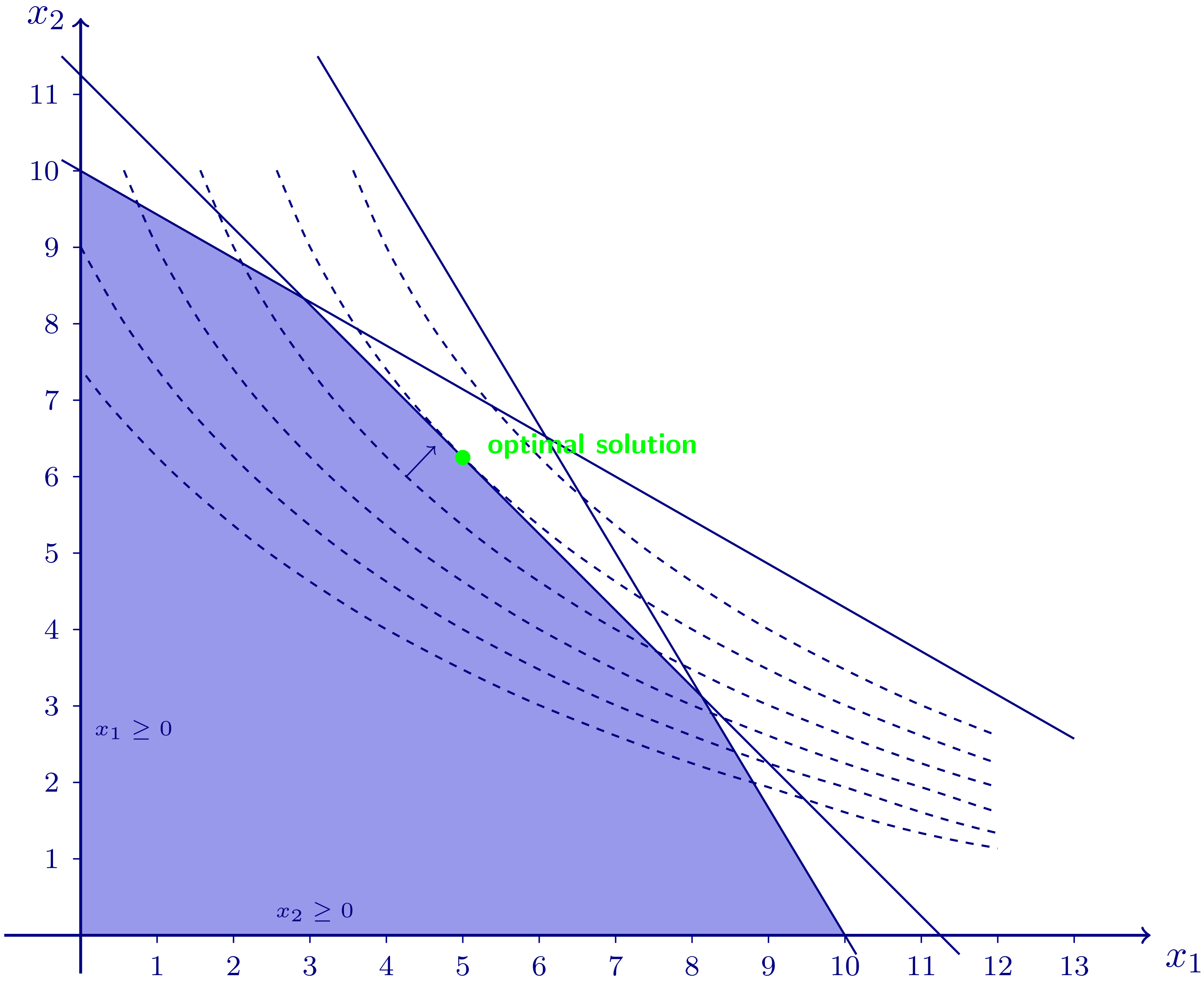
$$x_{\mathsf{b}}, x_{\mathsf{p}} \geq 0 \quad \text{(non-negativity)}$$

▷ Model (non-linear program):

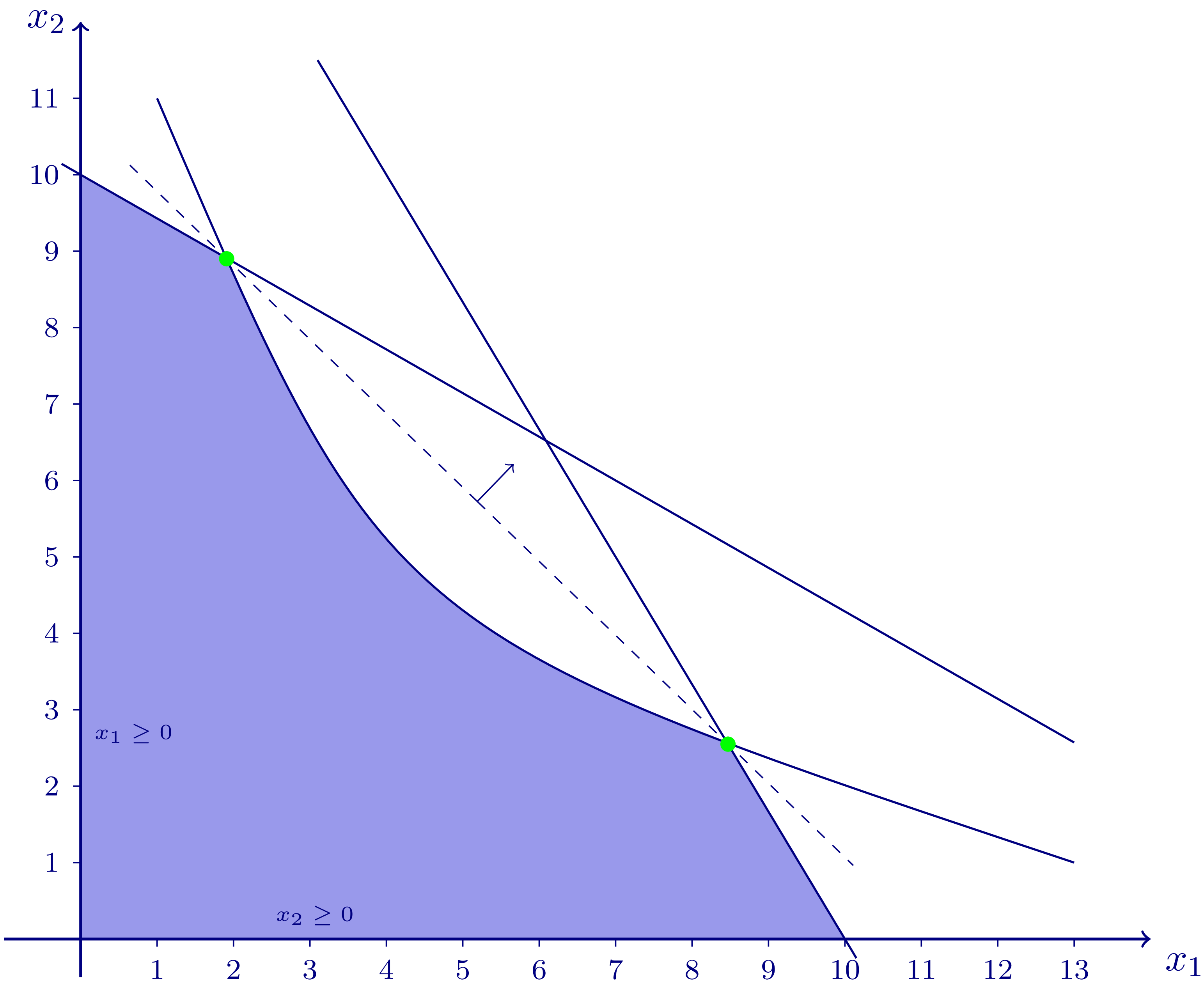
Objective	maximize	$10x_{\mathsf{b}} + 150\sqrt{x_{\mathsf{p}}} - 20x_{\mathsf{p}}$			
	subject to	$5x_{\mathsf{b}} + 3x_{\mathsf{p}}$	$\leq$	50	(total land usage)
C		$4x_{\mathsf{b}} + 7x_{\mathsf{p}}$	$\leq$	70	(total working time)
		$4x_{\mathsf{b}} + 4x_{\mathsf{p}}$	$\leq$	45	(total water consumption)
V		$x_{\mathsf{b}}, x_{\mathsf{p}}$	$\geq$	0	(non-negativity)

▷ Examples of non-linear terms:

- Products of variables: $x_i \cdot x_j$ } quadratic expressions
- Squares of variables: x_i^2 } quadratic expressions
- Higher-order terms of variables: $x_i \cdot x_j \cdot x_k$, $x_j^5 \cdot x_j$
- Absolute values or maxima/minima: $|x_i|$, $\max x_j$
- Terms including elementary functions: $\sin x_i$, $2^{x_i \cdot x_j}$, $\frac{1}{\sqrt{x_i}}$, $\log(x_i + x_j^{x_k})$



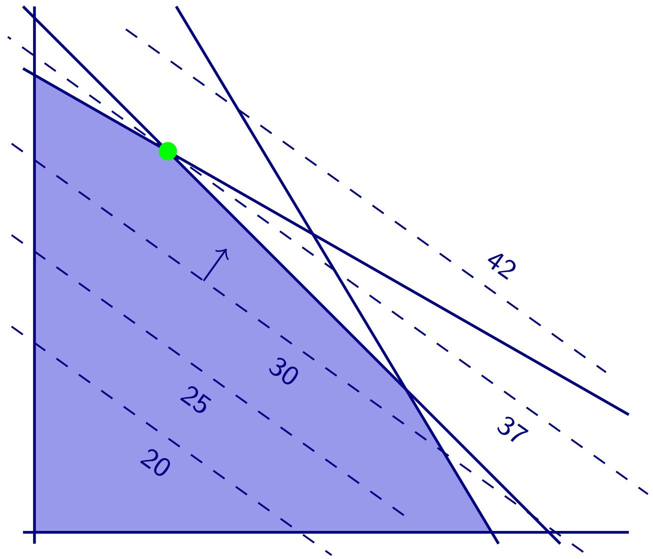
x_2
x_1
11
10
9
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
9
10
11
12
13
optimal solution
$x_1 \geq 0$
$x_2 \geq 0$



x_2
x_1
11
10
9
8
7
6
5
4
3
2
1
1 2 3 4 5 6 7 8 9 10 11 12 13
$x_1 \geq 0$
$x_2 \geq 0$

▷ **Linear models**

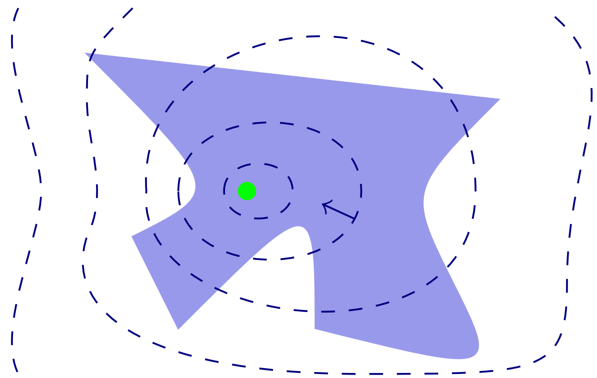
- Linear objective
  - ➡ Level sets are straight lines (in higher dimension: hyperplanes)
- Linear constraints
  - ➡ Feasible region is a polygon (in higher dimension: polyhedron)



➡ Optimal solutions can always be found in vertices

▷ **Non-linear models**

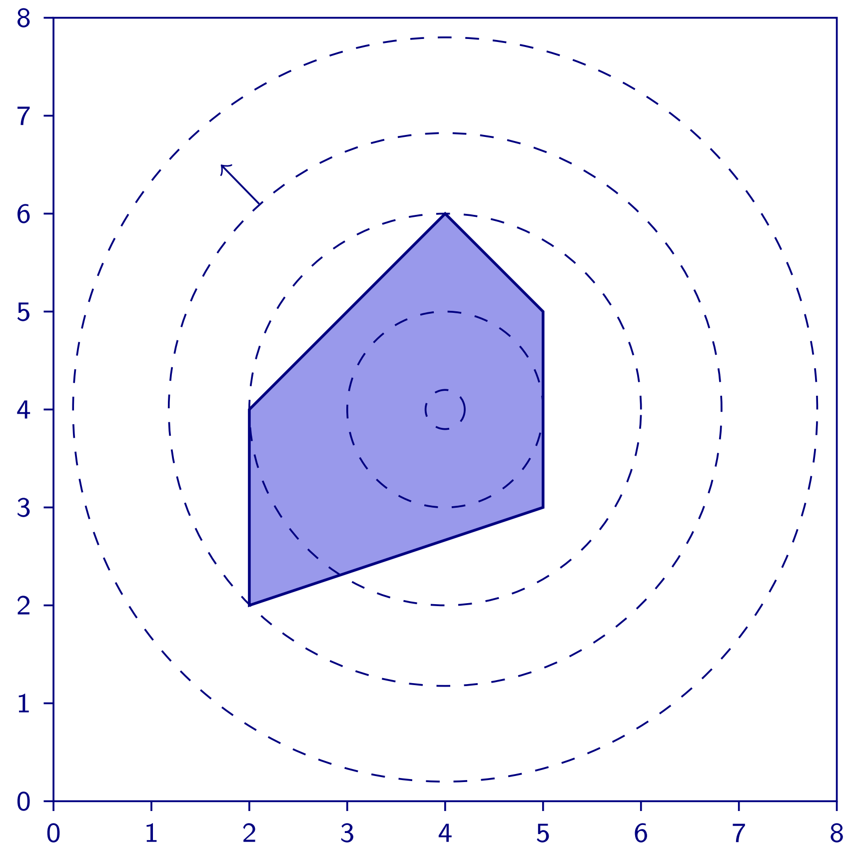
- Non-linear objective
  - ➡ Level sets can be complicated curves
- Non-linear constraints
  - ➡ Feasible region can be complicated



➡ Finding optimal solution can be difficult

- Example:

$$
\begin{aligned}
\max \quad & \sqrt{(x-4)^2+(y-4)^2} \\
\text{s.t.} \quad x &\geq 2 \\
x &\leq 5 \\
-x+y &\leq 2 \\
x+y &\leq 10 \\
x-3y &\leq -4
\end{aligned}
$$



Local and global optima

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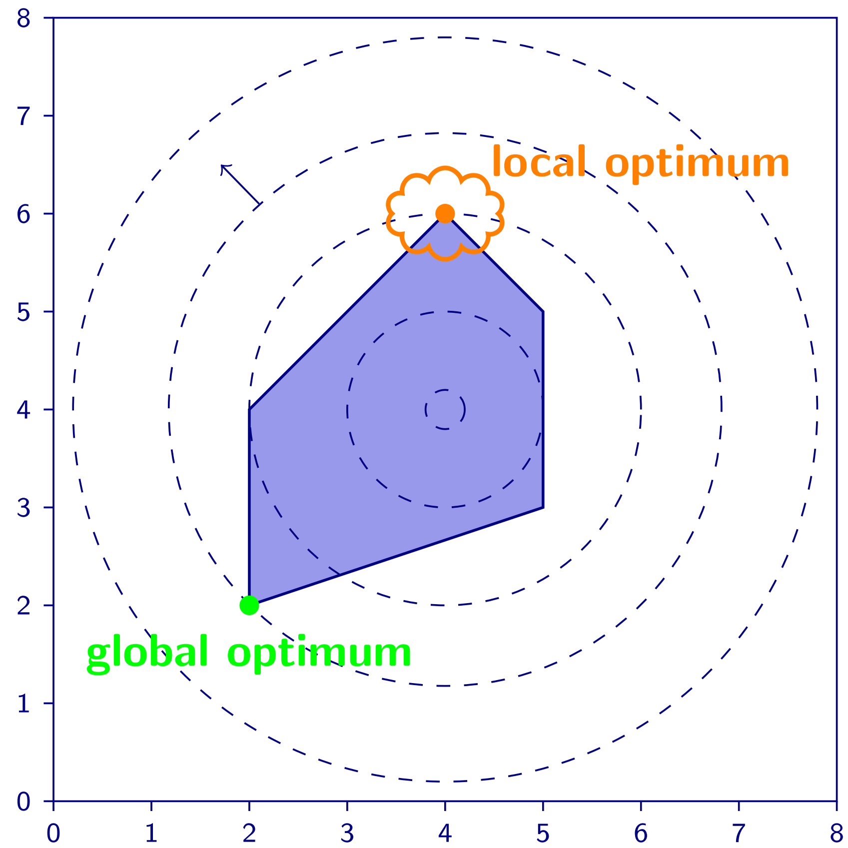

➡ (4,6) is a local (but not a global) optimum

➡ (2,2) is a (local and) global optimum

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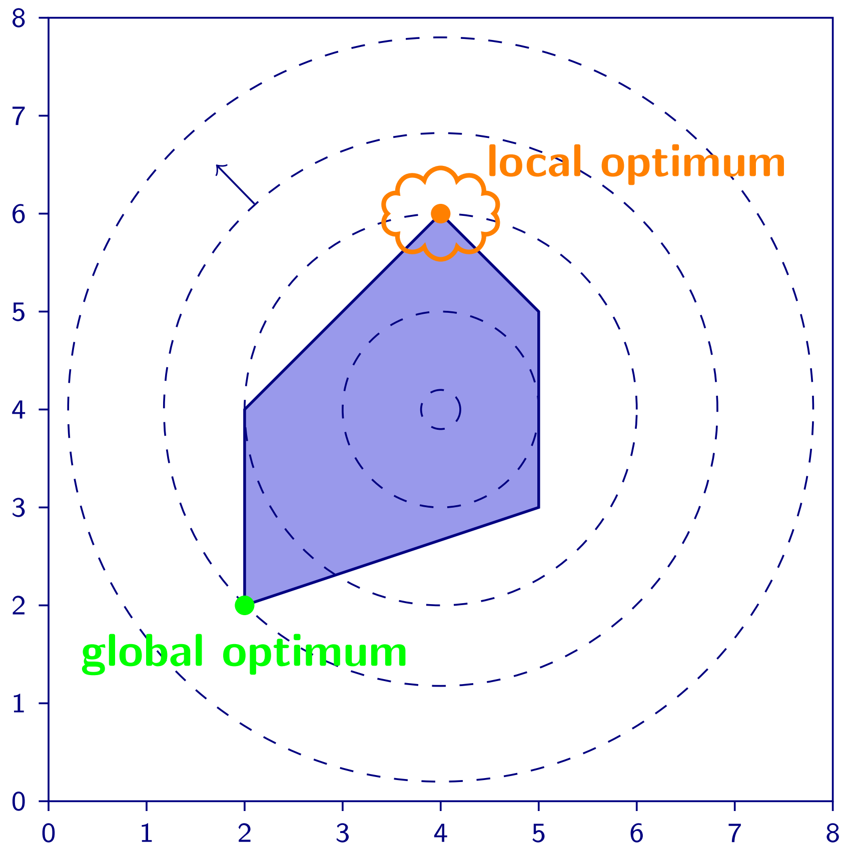
➡ (2,2) is a (local and) global optimum



A feasible solution is called locally optimal if there is no nearby feasible solution with a better objective function value

A feasible solution is called globally optimal if there is no feasible solution at all with a better objective function value

▷ Usual strategy of solvers for non-linear models:

- Find a point somewhere in the feasible region
- Follow steps to find a local optimum

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- If a concave function is maximized over a convex feasible set
- If a convex function is minimized over a convex feasible set
- If the problem is linear

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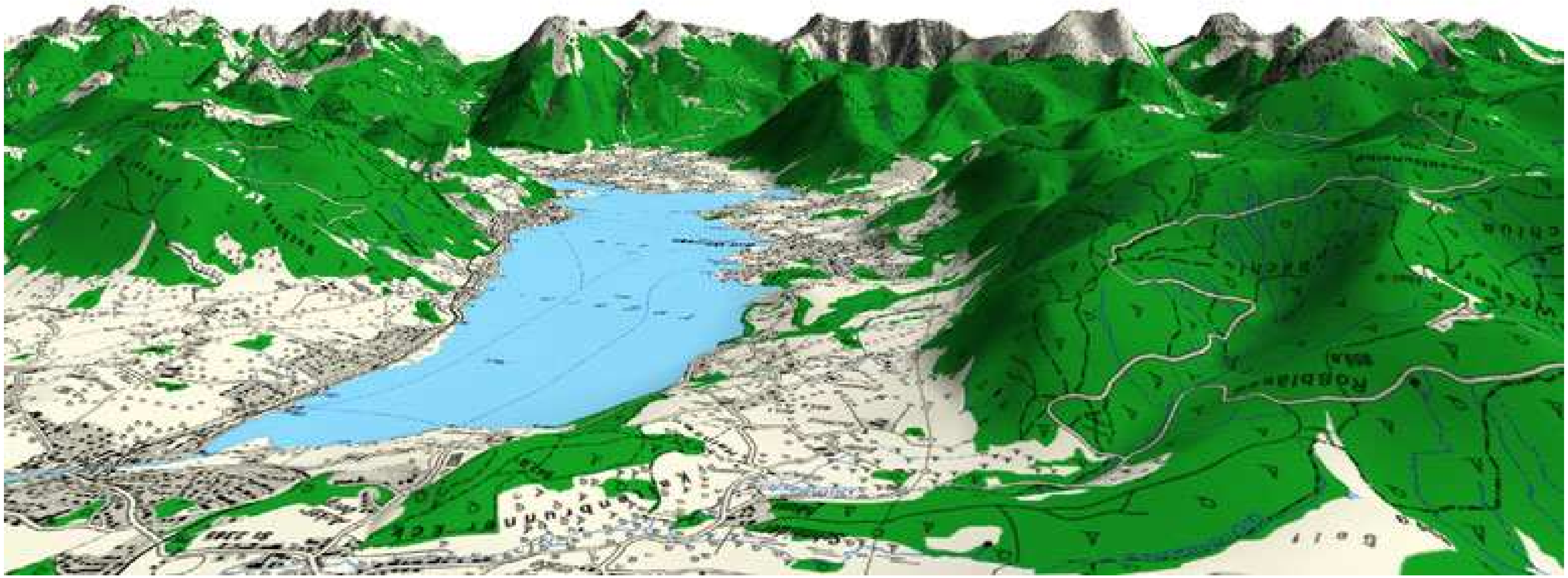
- In special cases, this works nonetheless:
  - If a concave function is maximized over a convex feasible set
  - If a convex function is minimized over a convex feasible set
  - If the problem is linear
- Possibilities otherwise:
  - Reformulate or approximate as a linear model
  - Rely on heuristic strategies and luck...

- Non-linear optimization is like mountain-climbing in the fog



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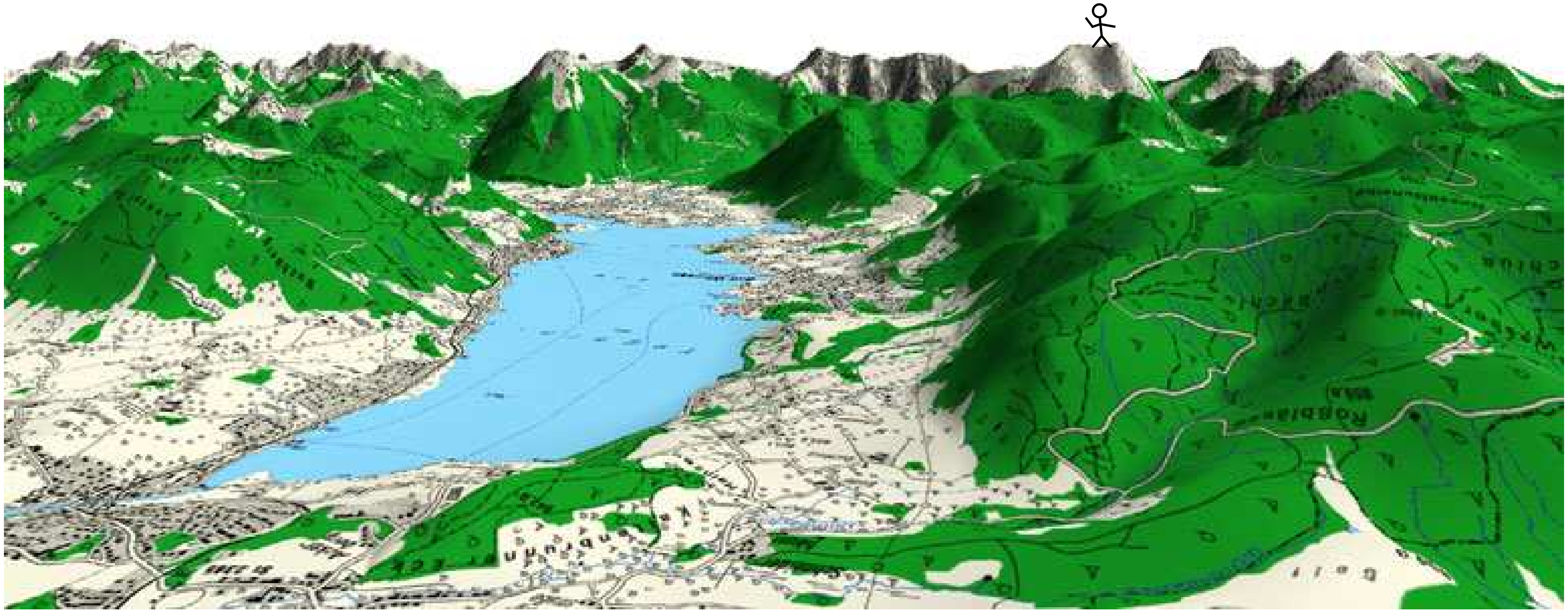

- ▷ How do you know that you're on the highest mountain if you can't see the other peaks?

- ▷ Models, Data and Instances
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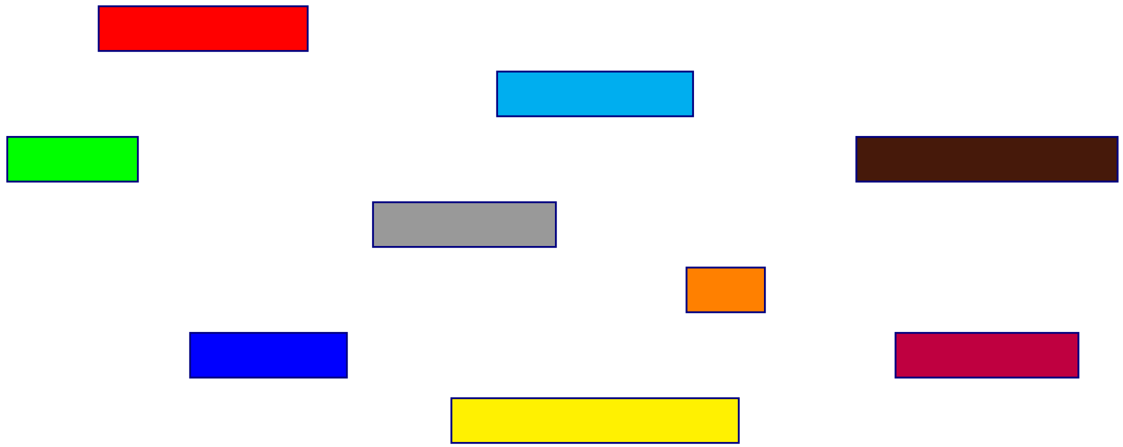
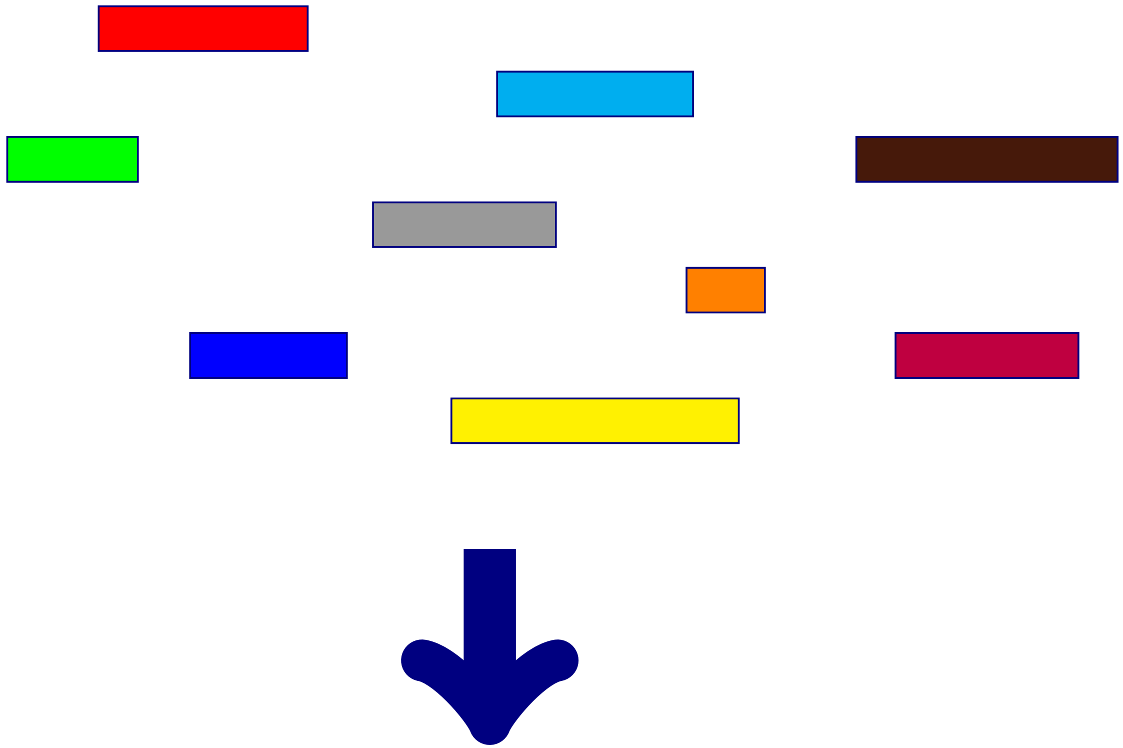
- Jobs:

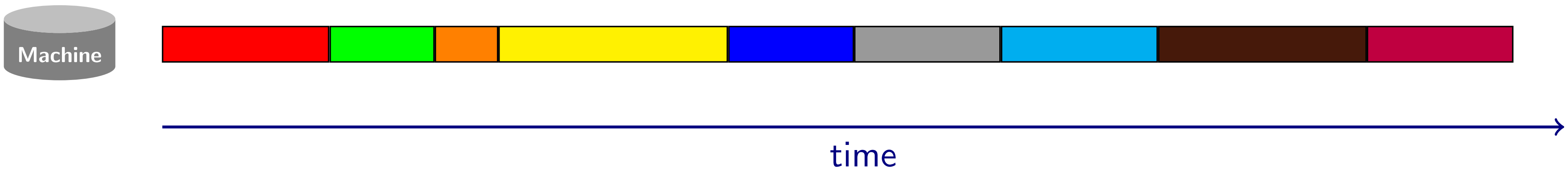


General principle

- Jobs:



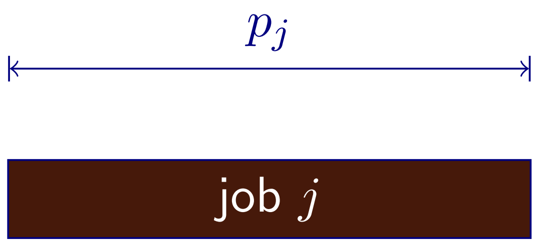
- Schedule (Gantt chart): ➡ optimal with respect to an objective to specify!



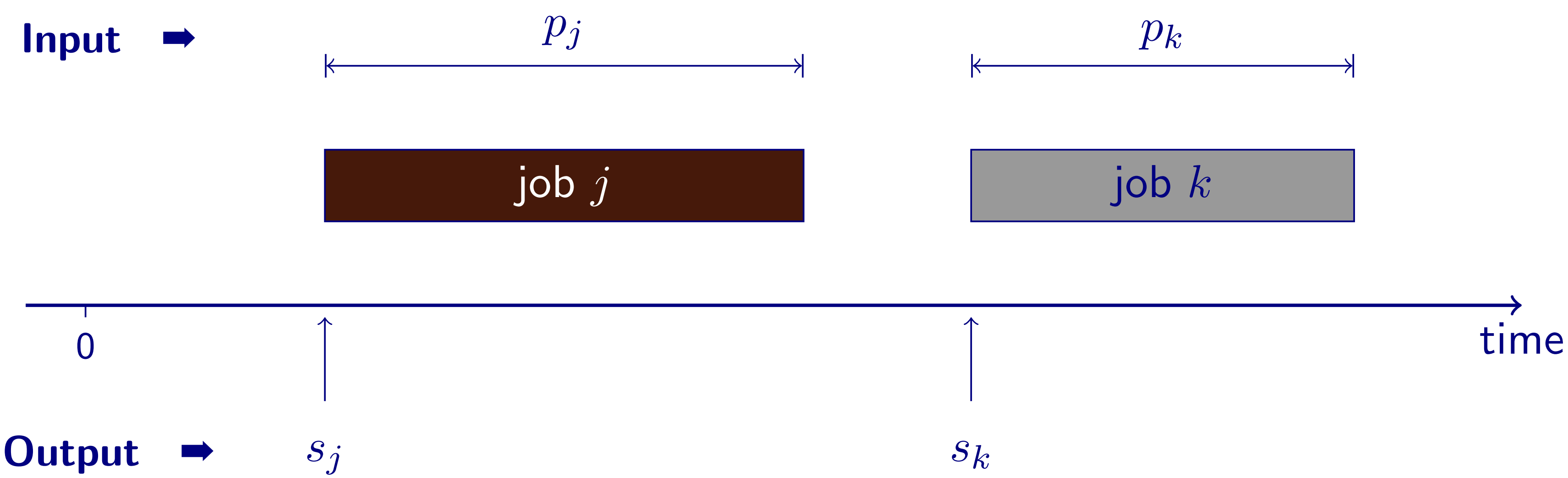
Basic terminology

- Jobs usually have: a processing time p_j

Input ➡



- Jobs usually have: a **processing time** p_j
- A schedule has to provide: a **start time** s_j, such that different jobs do not overlap



- ▷ Jobs usually have: a processing time p_j
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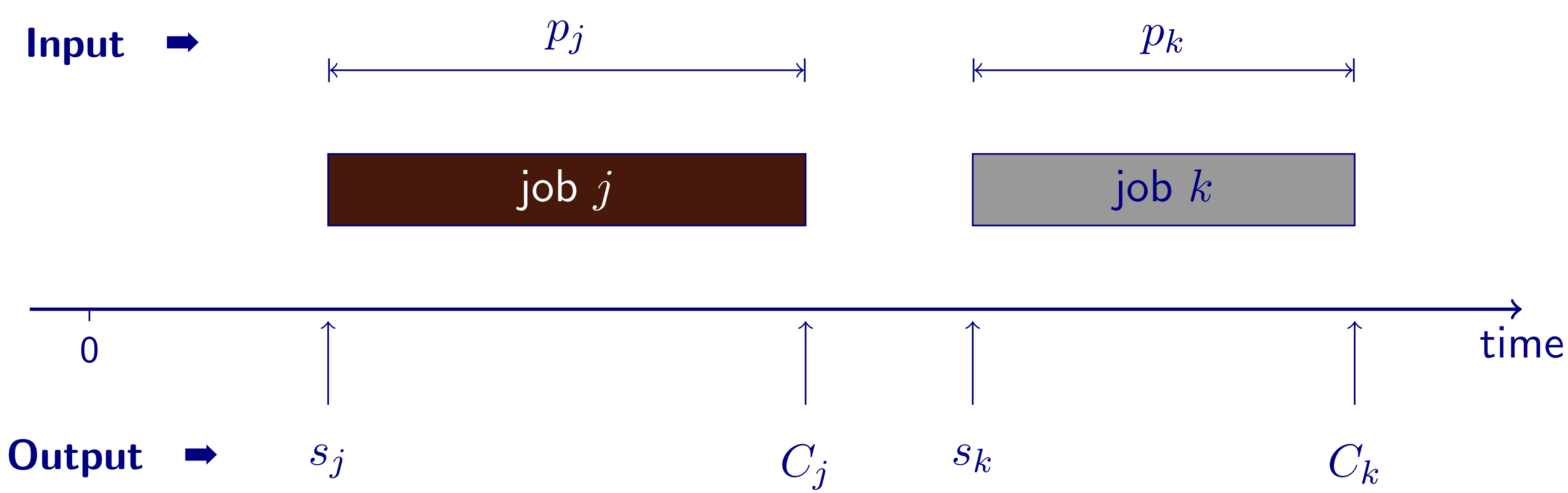
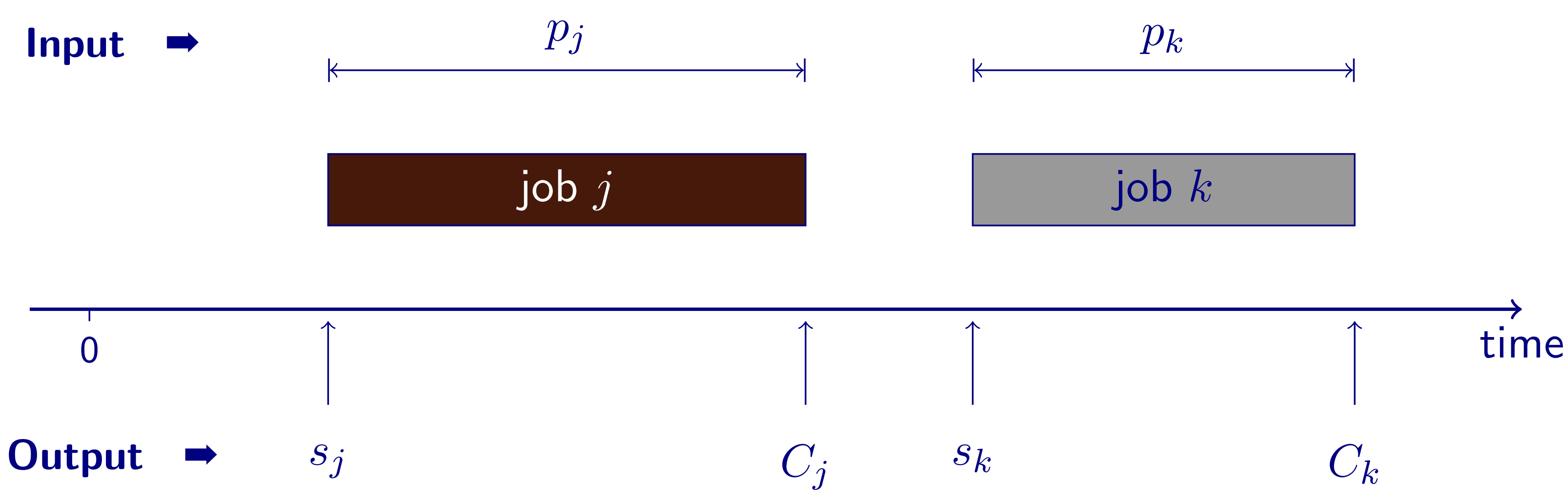
➡ Completion time $C_j := s_j + p_j$



- Jobs usually have: a **processing time** p_j
- A schedule has to provide: a **start time** s_j, such that different jobs do not overlap

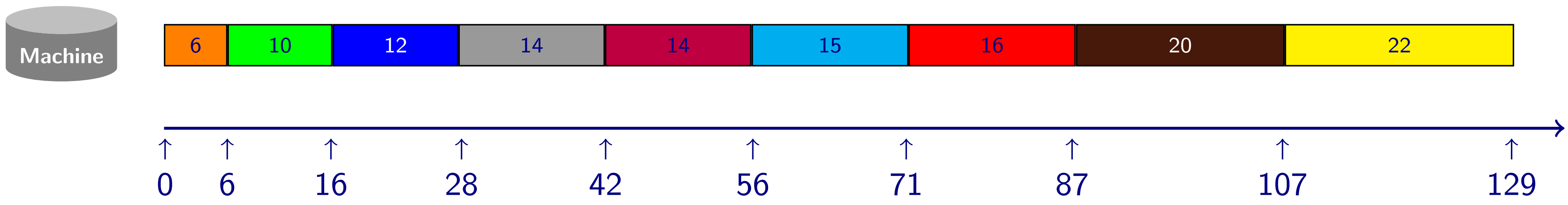
➡ **Completion time** $C_j := s_j + p_j$

- Possible objective functions: (to minimize)
  - ➡ **Sum of completion times** $\sum_{j=1}^{n} C_j$
  - ➡ **Makespan** $\max_{j=1,\ldots,n} C_j$



▷ Single Machine, minimize sum of completion times

➡ easy (greedy algorithm)



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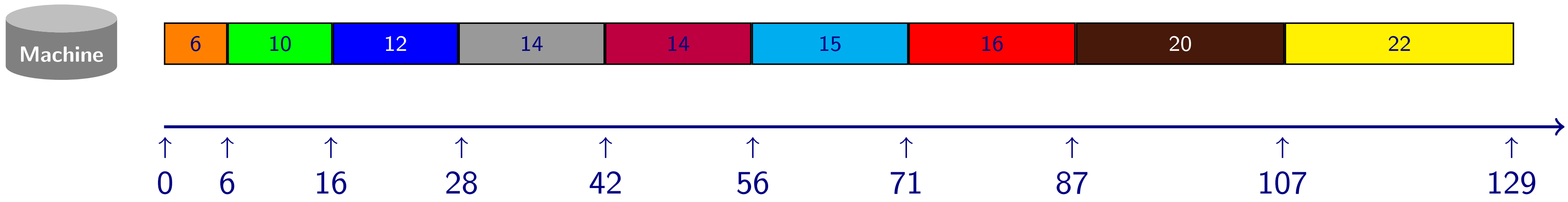
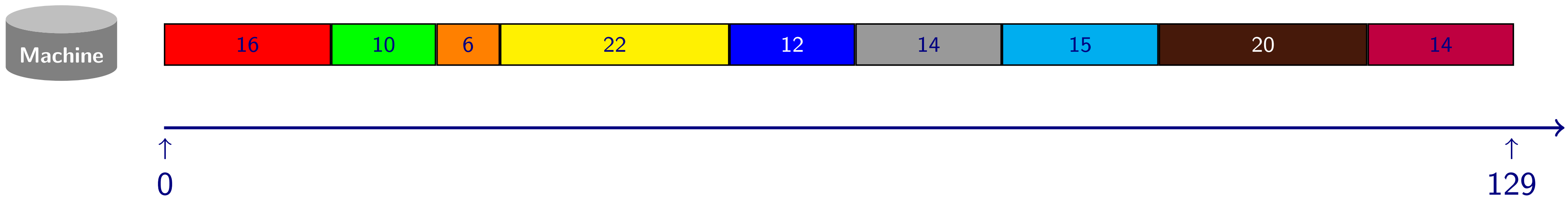


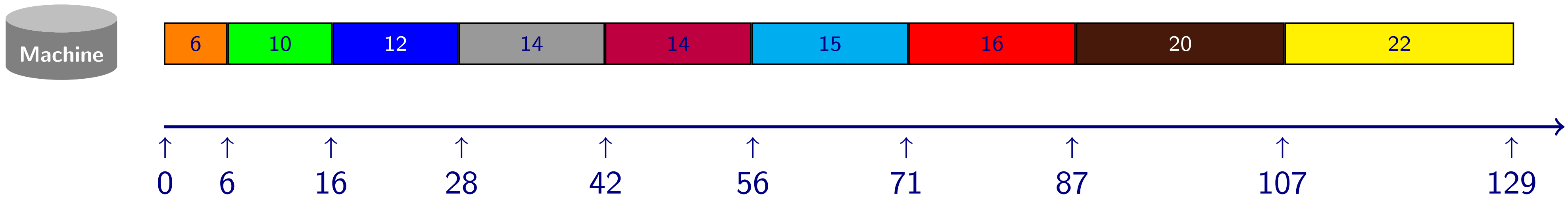
▷ Single Machine, minimize makespan

➡ trivial (always the same)

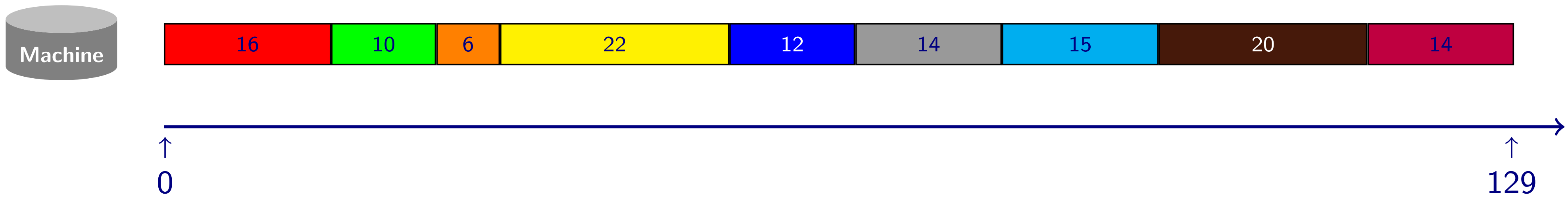


Scheduling – executive summary I

- ▷ Single Machine, minimize sum of completion times
  - ➡ easy (greedy algorithm)



- ▷ Single Machine, minimize makespan
  - ➡ trivial (always the same)



- ▷ Single Machine, jobs with release dates, minimize sum of completion times
  - ➡ similarly easy (greedy algorithm)

▷ Jobs with precedence constraints (project scheduling)

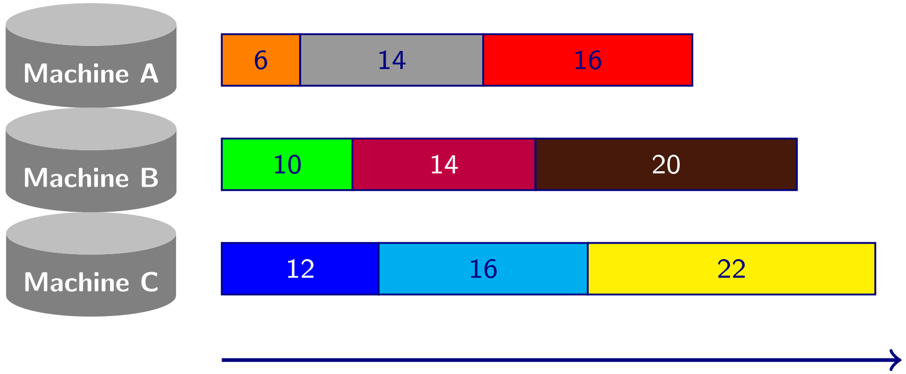
- Single machine ➡ easy (greedy)
- Multiple machines ➡ hard
- Unlimited number of machines ➡ easy again (critical path method)

▷ Jobs with precedence constraints (project scheduling)

- Single machine ➡ easy (greedy)
- Multiple machines ➡ hard
- Unlimited number of machines ➡ easy again (critical path method)

▷ Multiple machines, minimize sum of completion times

➡ easy (greedy)

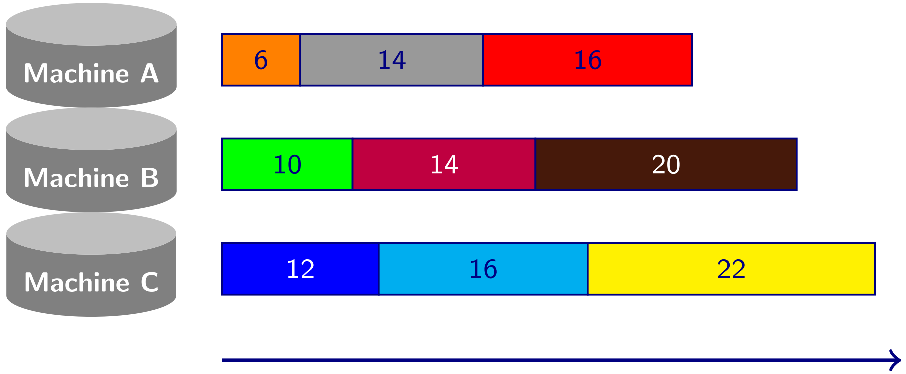


▷ Jobs with precedence constraints (project scheduling)

- Single machine ➡ easy (greedy)
- Multiple machines ➡ hard
- Unlimited number of machines ➡ easy again (critical path method)

▷ Multiple machines, minimize sum of completion times

➡ easy (greedy)



▷ Multiple machines, minimize makespan

➡ hard (partitioning problem)

- ▷ Models, Data and Instances
- ▷ Linear Optimization
  - ➡ Modelling as a linear program
  - ➡ Solving a linear program (graphically, and in princple by the simplex algorithm)
  - ➡ Sensitivity analysis
- ▷ (Mixed) Integer Programming
  - ➡ Modelling as a (mixed) integer program
  - ➡ How to solve a (mixed) integer program (in principle)
- ▷ Combinatorial Optimization
  - ➡ Exemplary problems, algorithms, and runtimes
- ▷ Nonlinear Optimization
  - ➡ Local and global optima, convex optimization
- ▷ Scheduling
- ▷ Lot Sizing
- ▷ Multicriteria Optimization

Data:

- d_t Demand in period t
- f_t fixed (start-up) costs in period t
- c_t unit production costs in period t
- h_t unit holding costs in period t
- C_t available capacity in period t

Variables:

- x_t production in period t
- y_t installation of capacity in period t
- s_t inventory at the end of period t

$$\min \sum_t c_t x_t + f_t y_t + h_t s_t$$

$$\begin{aligned}
s_{t-1} + x_t &= d_t + s_t && t = 1, \ldots, n \\
x_t &\leq C_t y_t && t = 1, \ldots, n \\
x_t, s_t &\geq 0 && t = 1, \ldots, n \\
s_0 &= 0 && \\
y_t &\in \{0, 1\} && t = 1, \ldots, n
\end{aligned}$$

- NP-hard problem
- poly-solvable cases:
  - ▷ Wagner-Whitin: $C_t = \infty$ for all periods t. In practice, $C_t = M$ with M very large value.
  - ▷ constant capacity: $C_t = C$ for all periods t.
  - ▷ Discrete lot sizing: constant capacity $C_t = C$ and $x_t = C_t y_t$ for all periods t
  - ▷ capacity in each period an integer multiple of constant batch size: $C_t = C y_t$ with $y_t \in \mathbb{Z}_+$ for all periods t.

- ▷ Models, Data and Instances
- ▷ Linear Optimization
  - ➡ Modelling as a linear program
  - ➡ Solving a linear program (graphically, and in princple by the simplex algorithm)
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Multicriteria MIP model

$$\max \sum_{j=1}^{n} c_j^1 x_j, \sum_{j=1}^{n} c_j^2 x_j, \ldots, \sum_{j=1}^{n} c_j^q x_j$$

$$\begin{aligned} \text{s.t.} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i & \qquad i = 1, \ldots, m \\ \ell_j \le x_j \le u_j & \qquad j = 1, \ldots, n \\ x_j \in \mathbb{Z}_+ & \qquad j = 1, \ldots, k \end{aligned}$$

Multicriteria MIP model

$$
\begin{aligned}
\max \sum_{j=1}^{n} c_j^1 x_j, \sum_{j=1}^{n} c_j^2 x_j, \ldots, \sum_{j=1}^{n} c_j^q x_j & \\
\text{s.t.} \sum_{j=1}^{n} a_{ij} x_j \le b_i & \qquad i = 1, \ldots, m \\
\ell_j \le x_j \le u_j & \qquad j = 1, \ldots, n \\
x_j \in \mathbb{Z}_+ & \qquad j = 1, \ldots, k
\end{aligned}
$$

Ideas:

- find efficient (non-dominated) solutions (A solution is efficient or non-dominated if no objective value can be improved without reducing the other objective values)
- combine objective functions to weighted linear combination
- maximize one objective subject to bounds on all other objectives
- goal programming: solver get's numerical requirements $\tilde{c}_j$ that have to be achieved as much as possible

Oral Exam takes place on Wed, 15 Feb, 10:15 a.m. - 1:45 p.m.
in PTZ 307

GOOD LUCK!