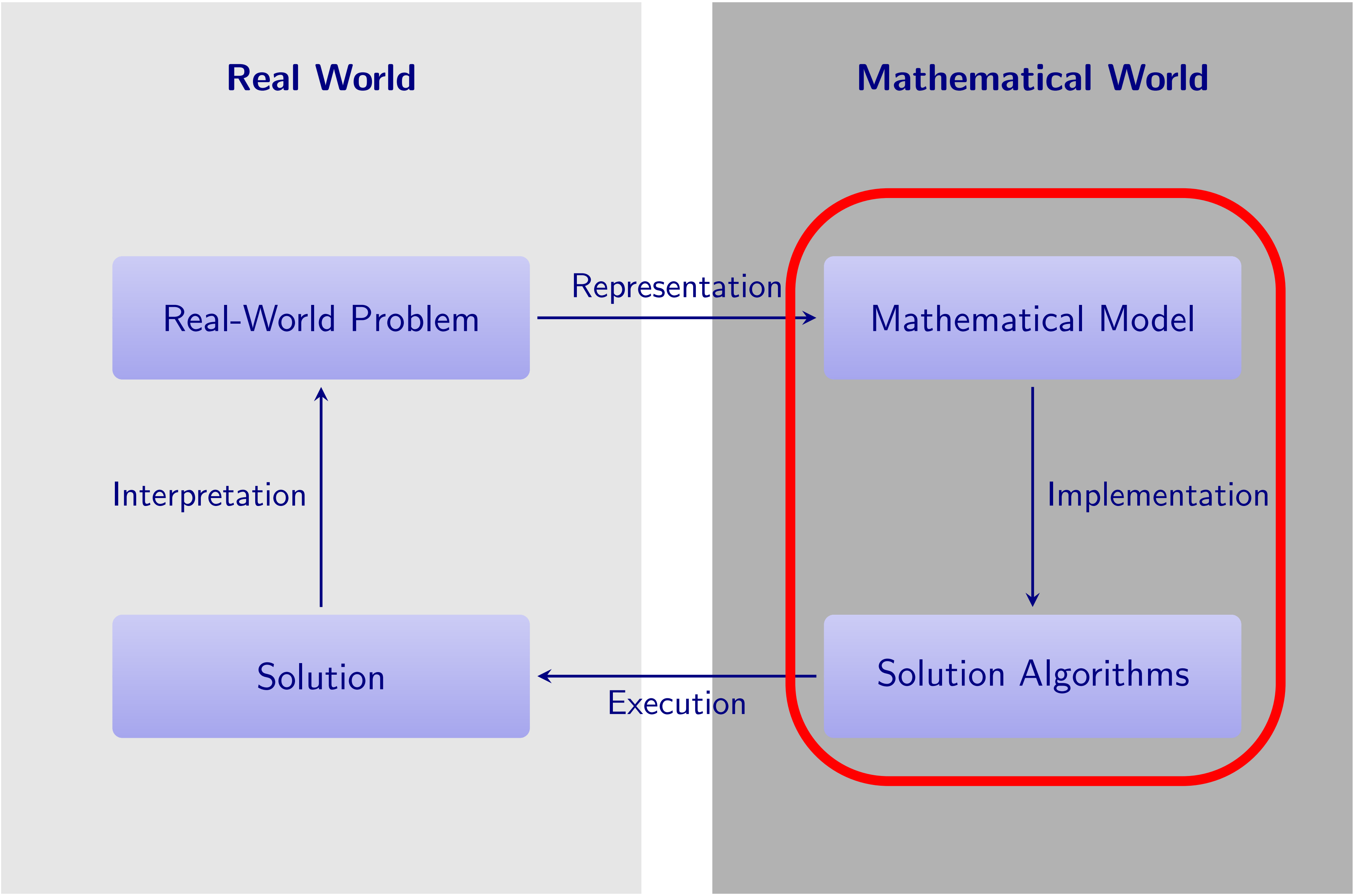
Mathematical Tools for Engineering and Management

Lecture 3

2 Nov 2011


Real World
Mathematical World
Real-World Problem
Representation
Mathematical Model
Interpretation
Implementation
Solution
Execution
Solution Algorithms

$$\text{maximize/minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \le b_i \quad \text{for all } i = 1, \ldots, m$$

$$\ell_j \le x_j \le u_j \quad \text{for all } j = 1, \ldots, n$$

General form of LPs

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General form of LPs

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maximize/minimize $\sum_{j=1}^{n} c_j x_j$

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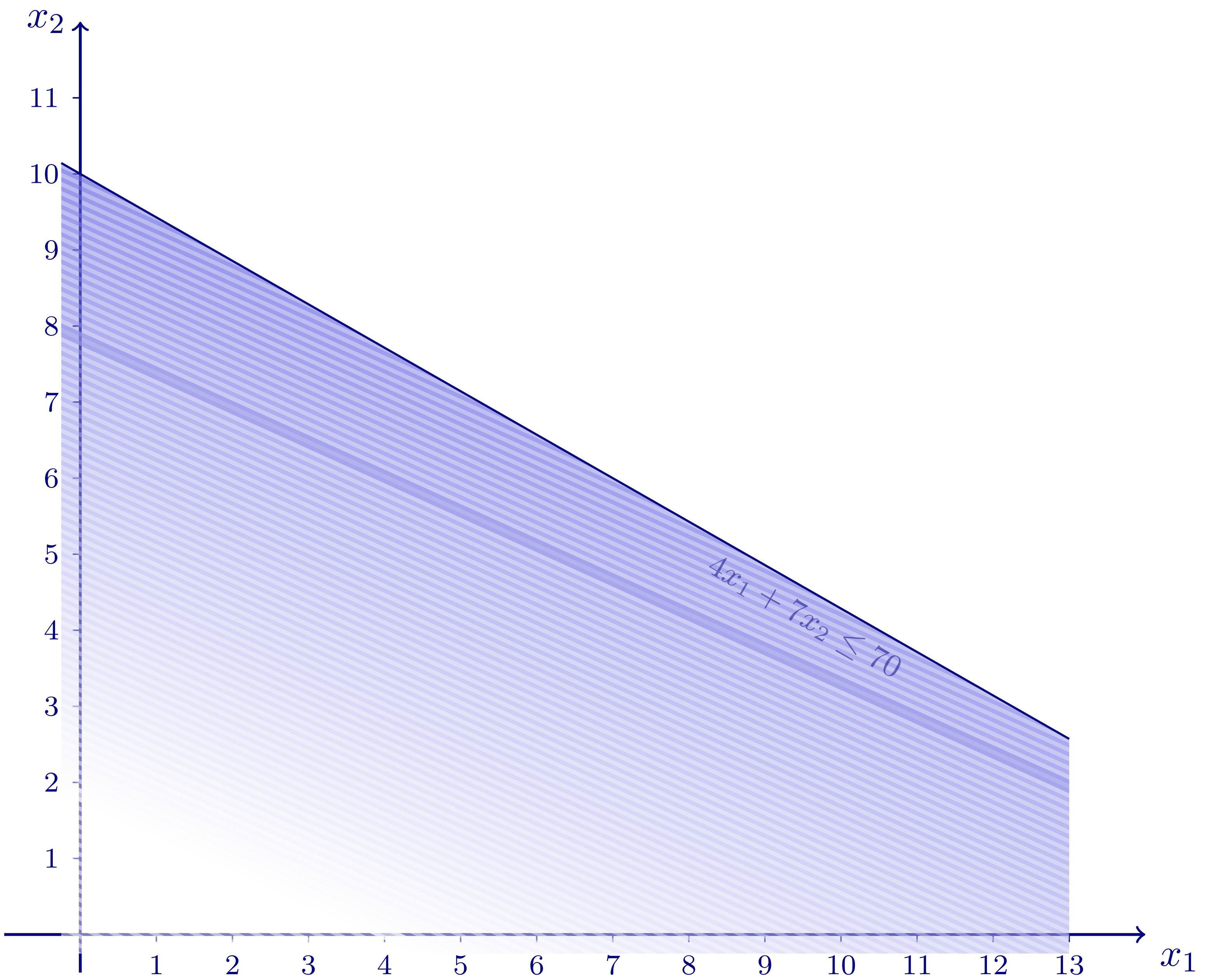
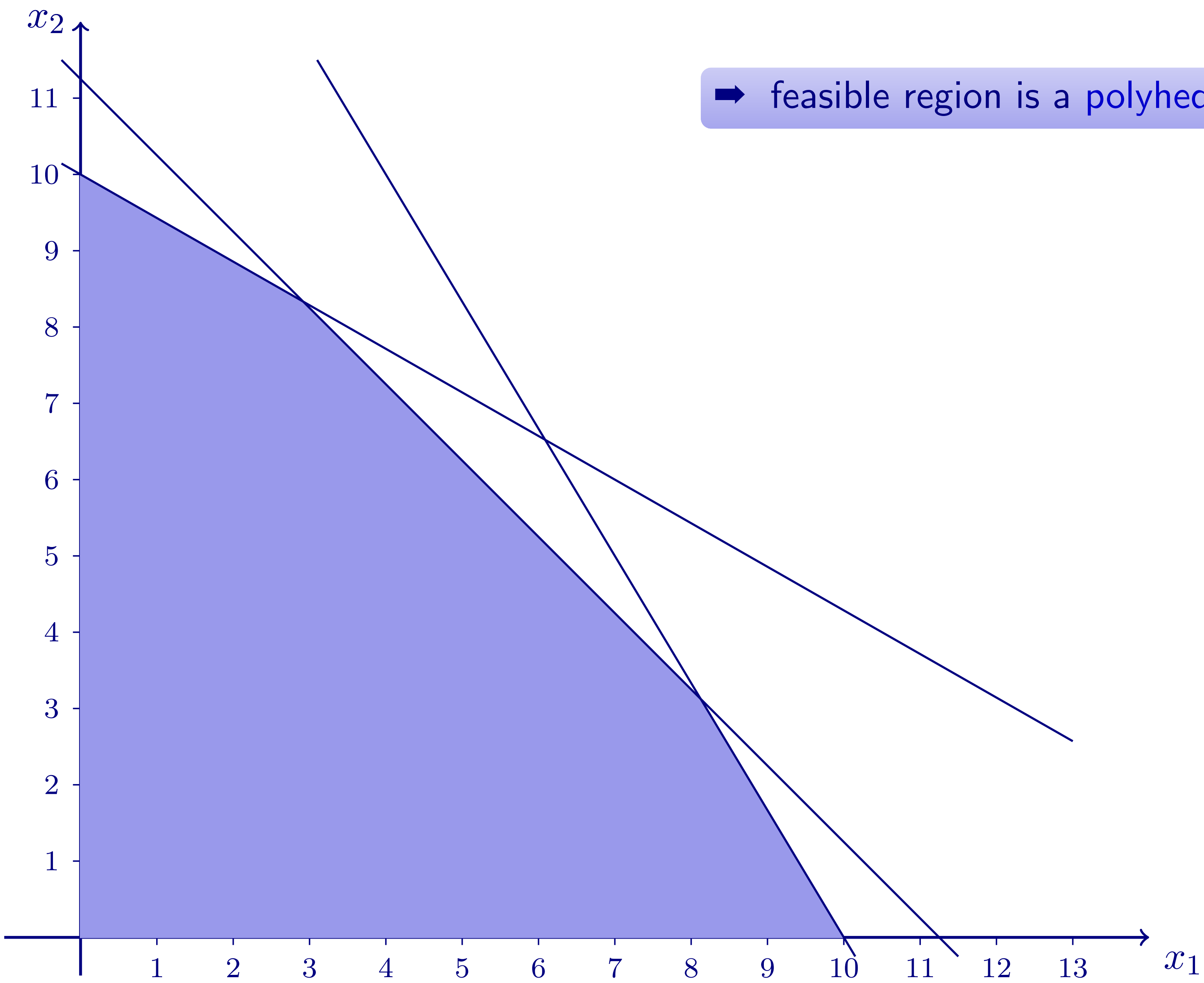
maximize $\sum_{j=1}^{n} c_j x_j$

subject to $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$ for all $i = 1, \ldots, m$

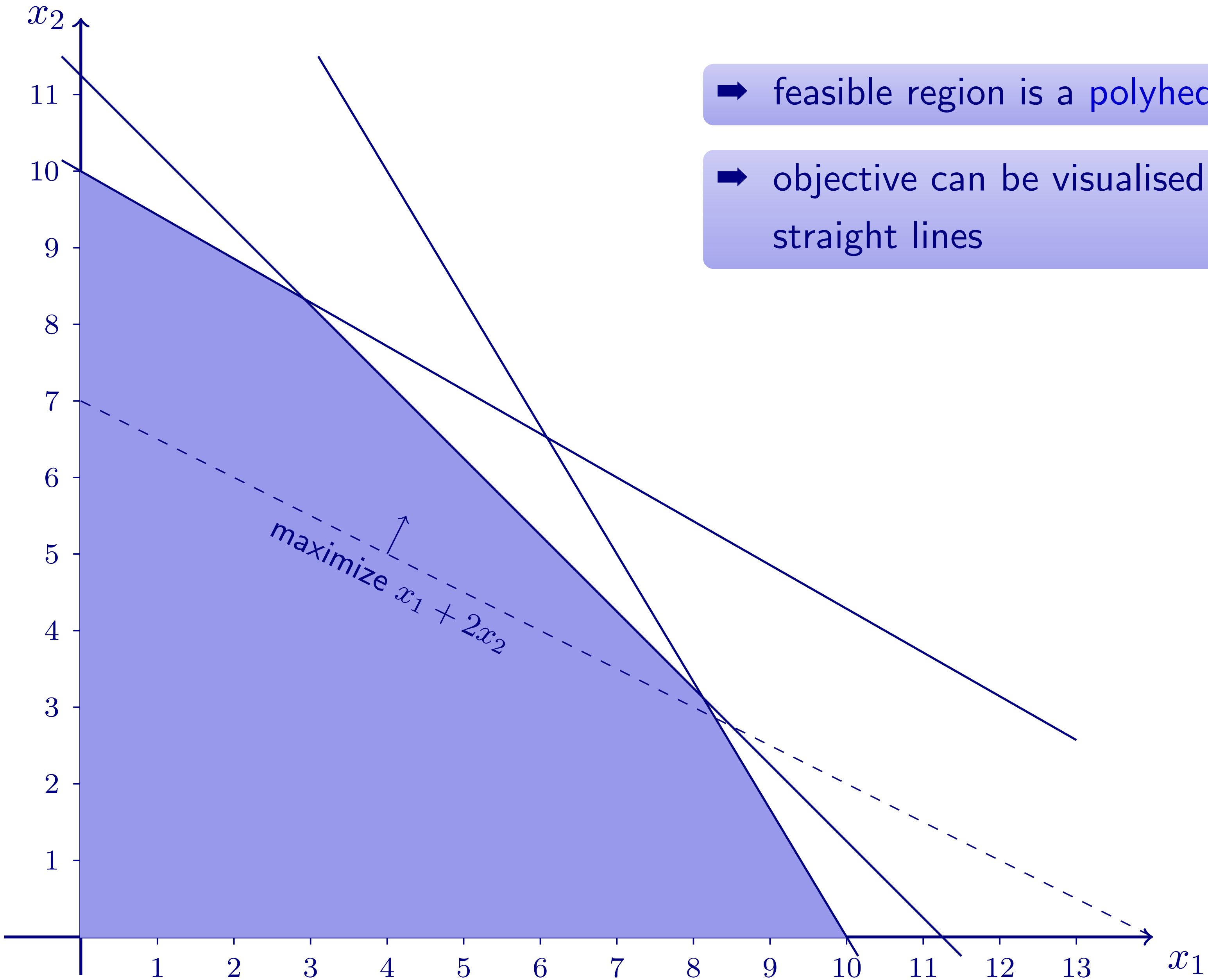
➡ NOTE:
- Every LHS and the objective are linear functions!
- Every constraint is a $\leq$-constraint!



x_2
11
10
9
8
7
6
5
4
3
2
1
$4x_1 + 7x_2 \leq 70$
1
2
3
4
5
6
7
8
9
10
11
12
13
x_1

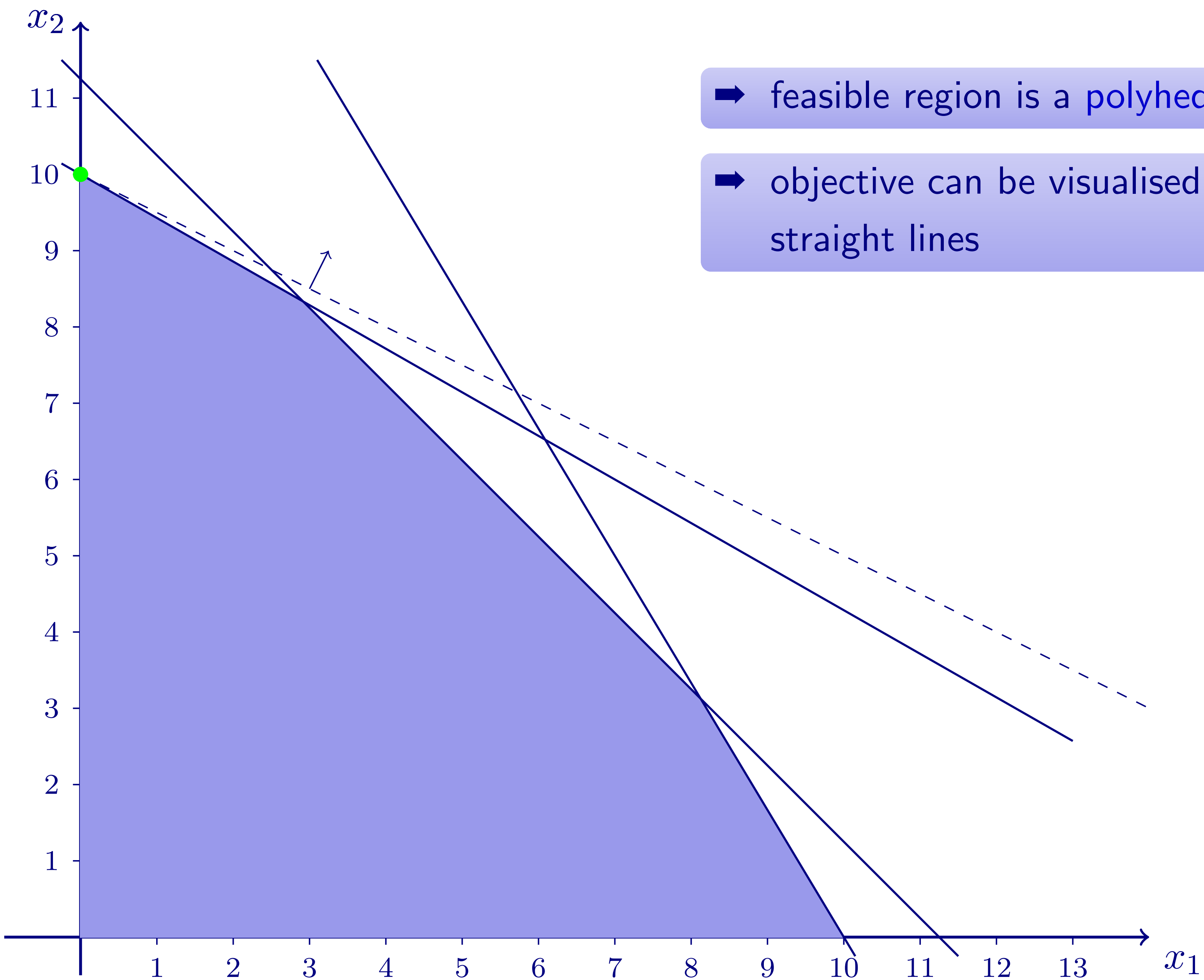


➡ feasible region is a polyhedron



➡ feasible region is a polyhedron

➡ objective can be visualised by straight lines



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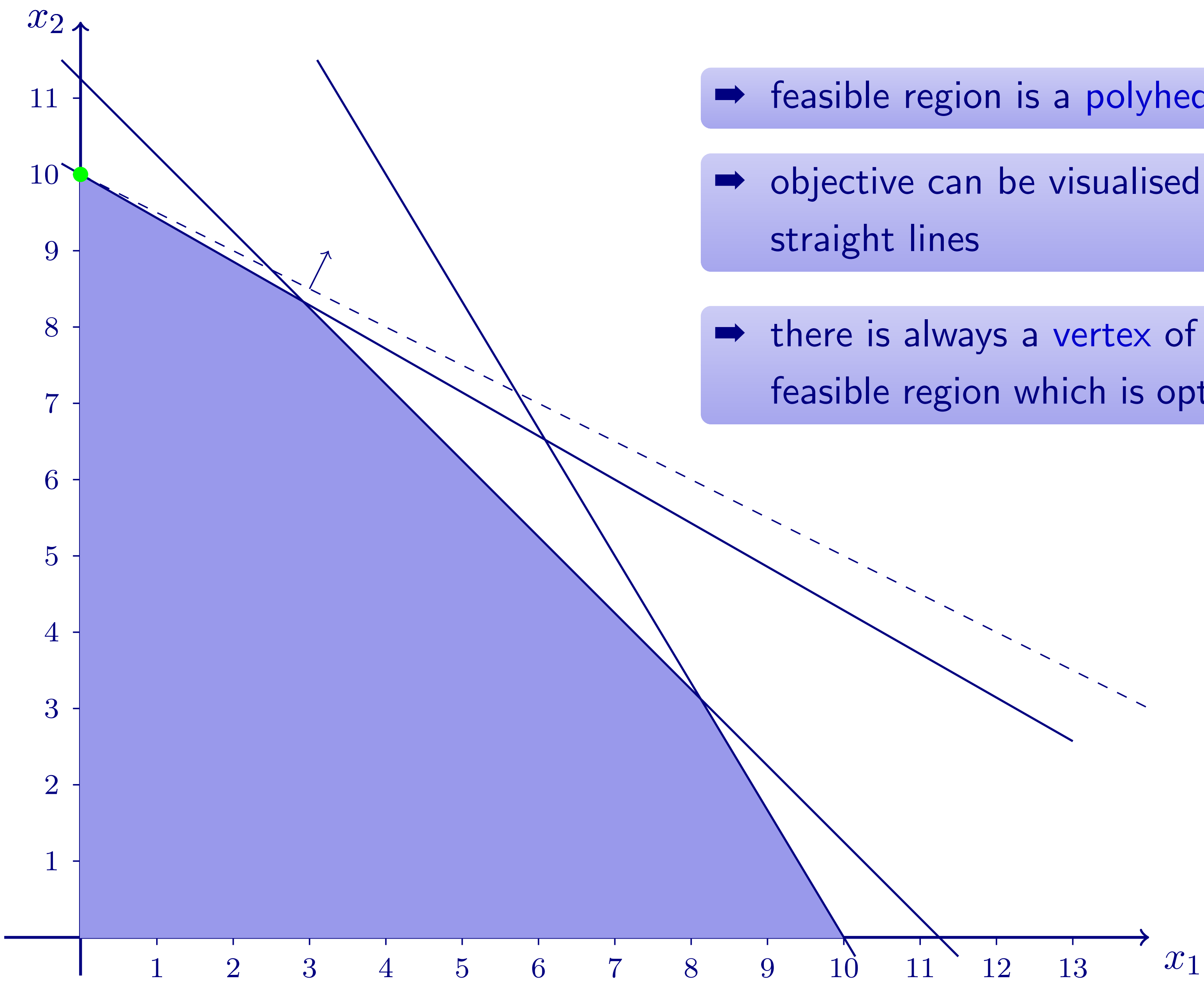
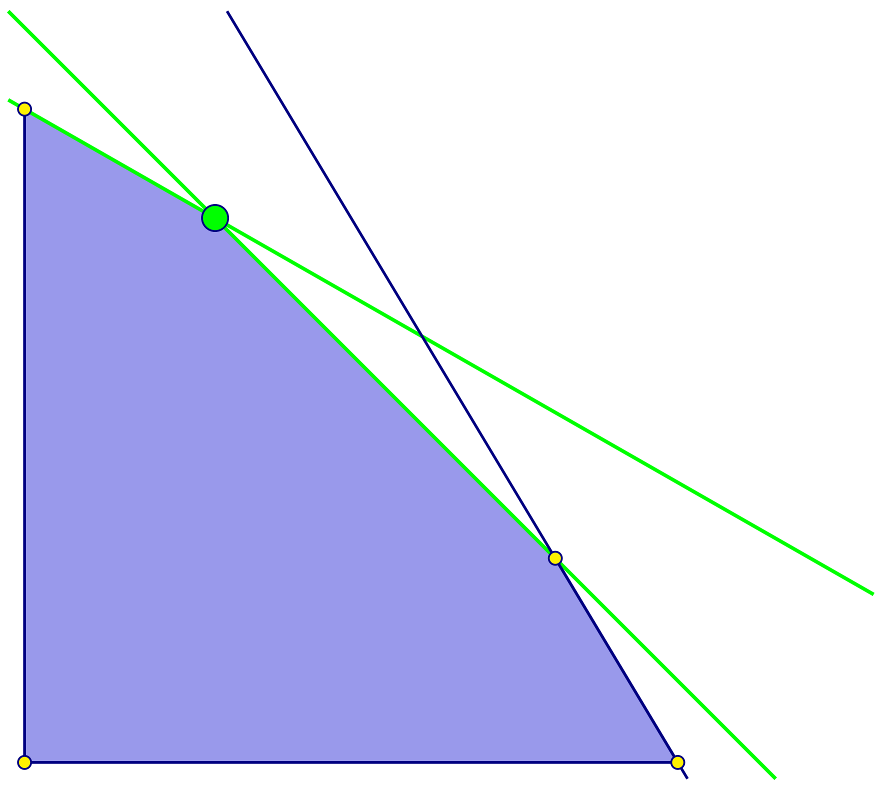
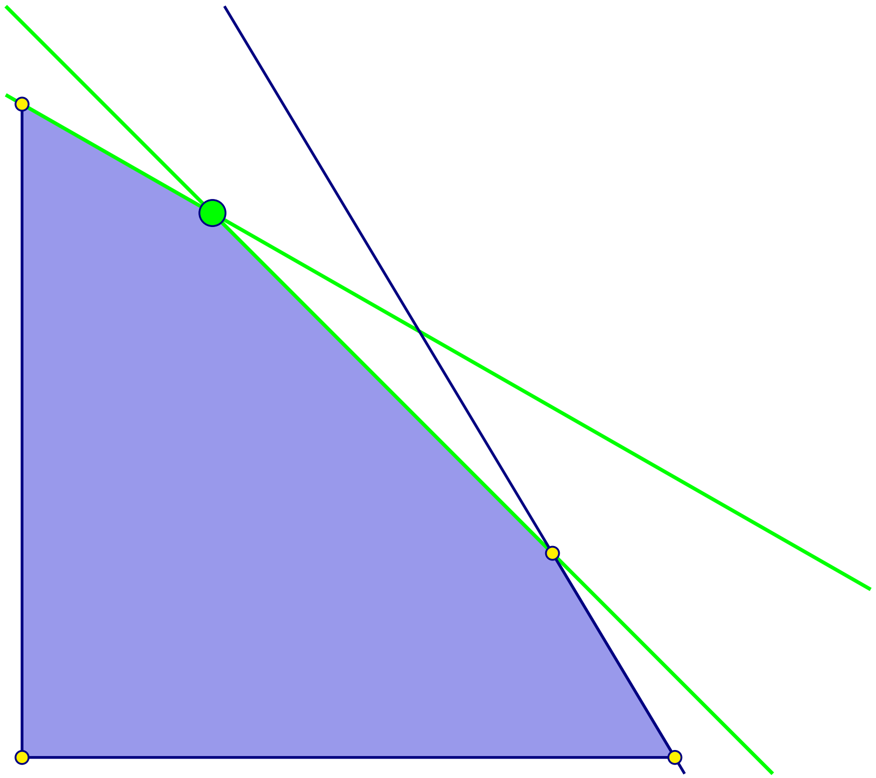


- feasible region is a polyhedron
- objective can be visualised by straight lines
- there is always a vertex of the feasible region which is optimal

- LP with 2 variables



- LP with 2 variables
  - ➡ Vertex is the intersection of 2 lines, given by 2 binding constraints



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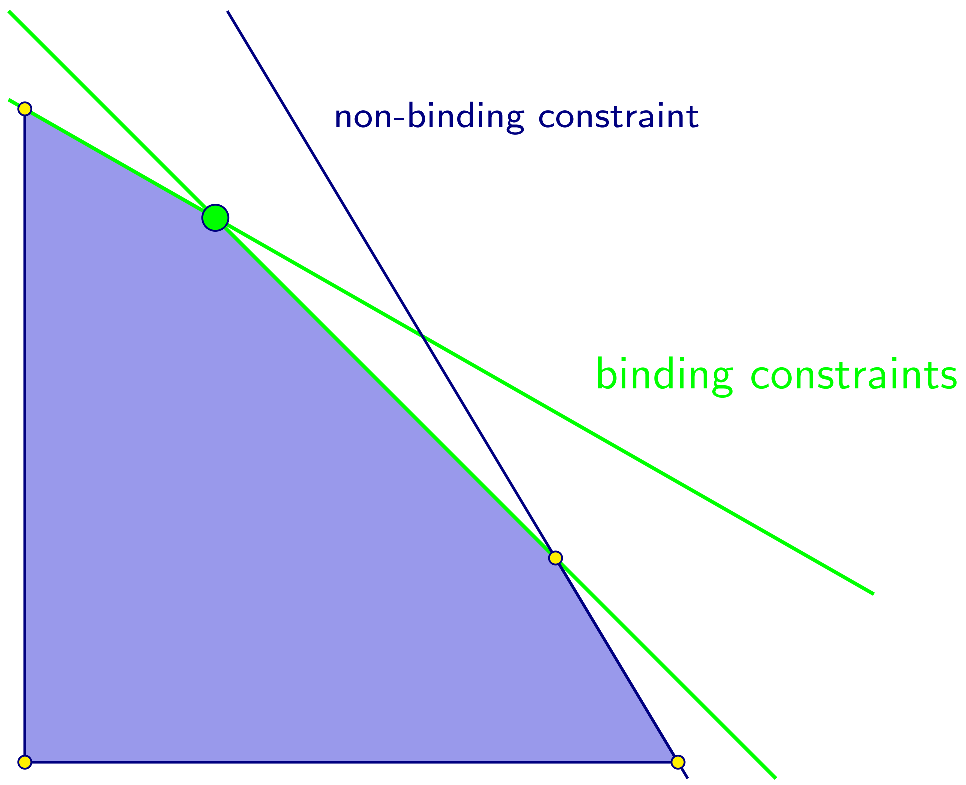
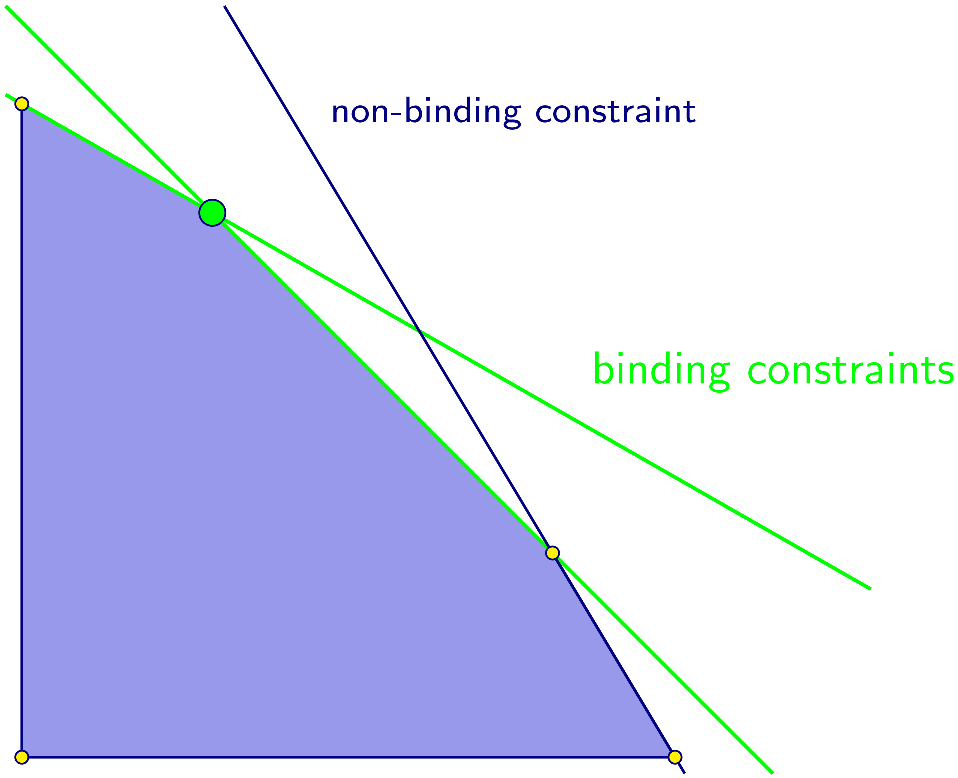


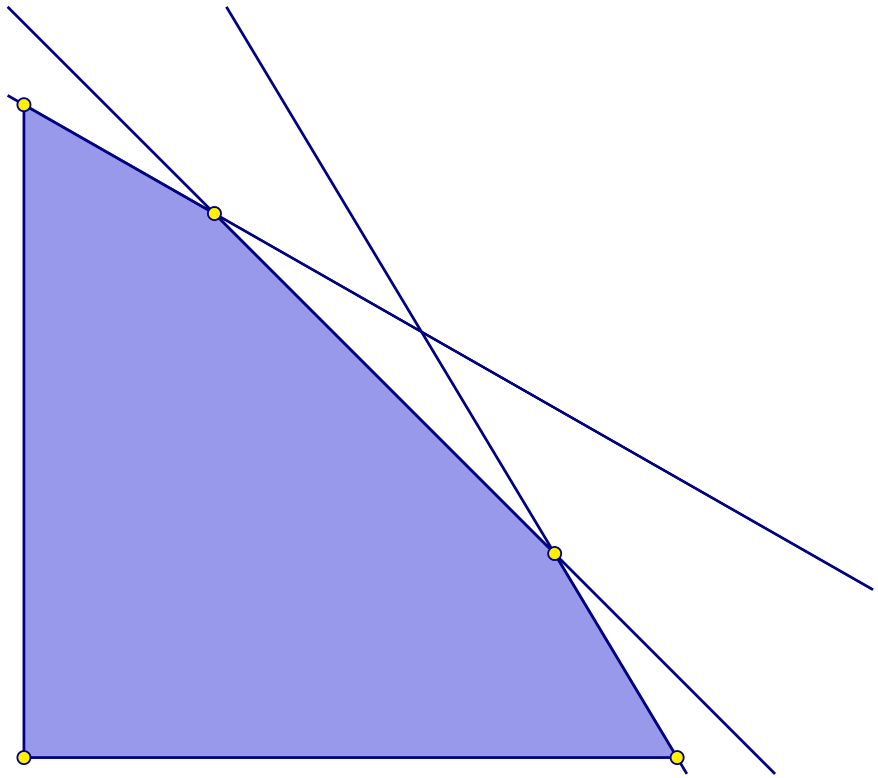
▷ LP with 2 variables

➡ Vertex is the intersection of 2 lines, given by 2 binding constraints

➡ Compute its coordinates by solving a system of 2 linear equations in 2 variables



▷ Solutions of a system of 2 linear equations given by constraints are called basic solutions



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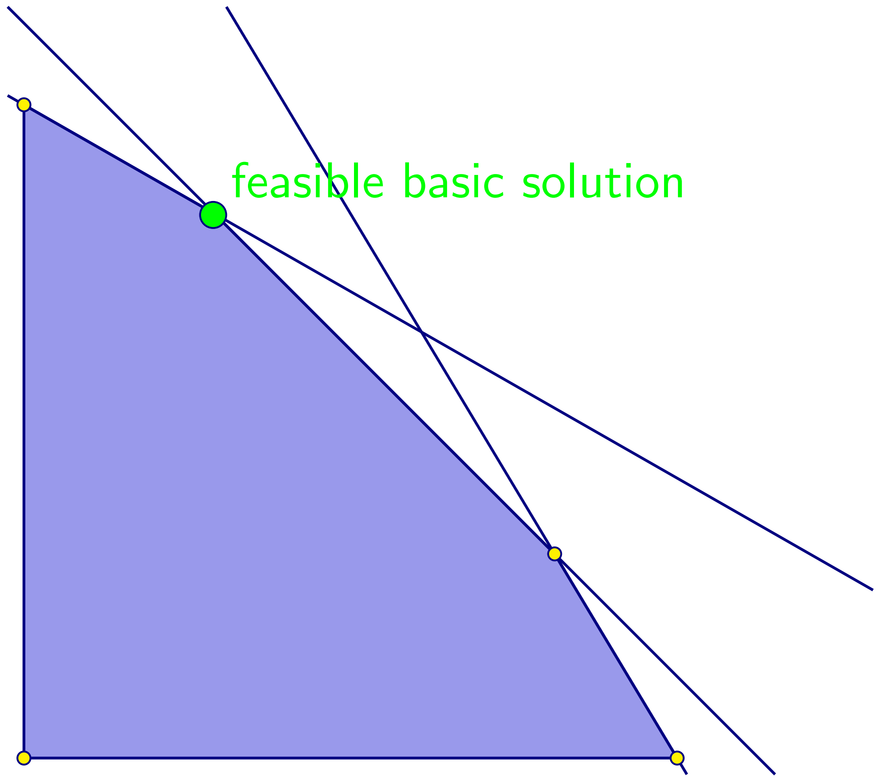

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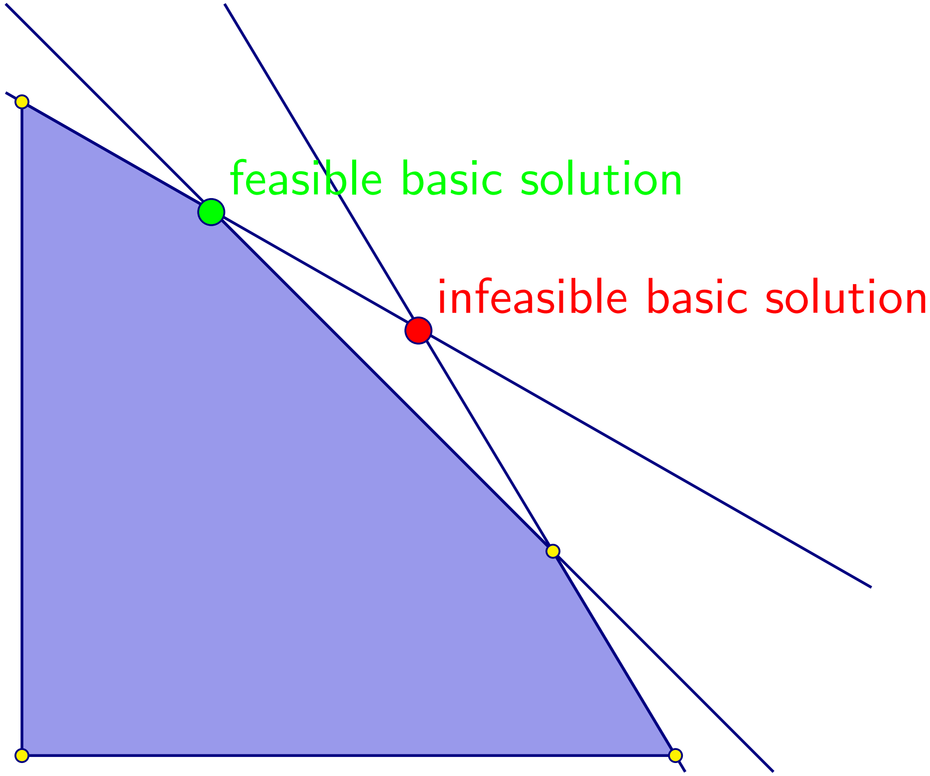
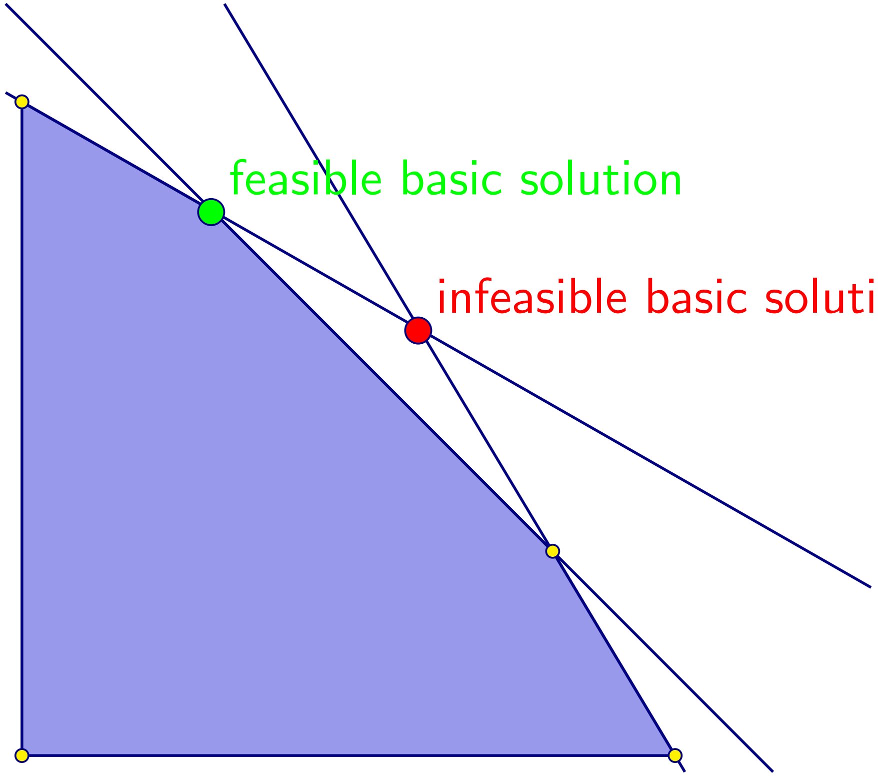


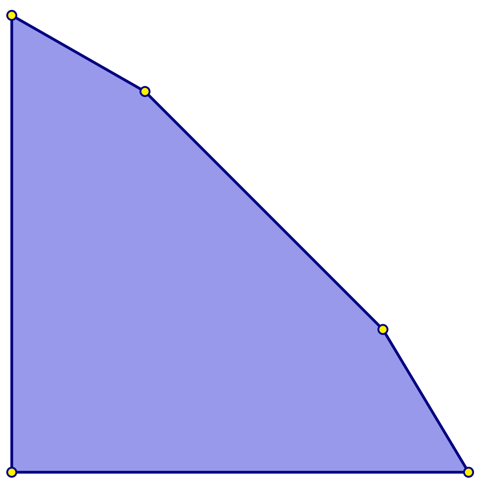
- Solutions of a system of 2 linear equations given by constraints are called **basic solutions**

- **Feasible basic solutions** are exactly the vertices of the feasible region

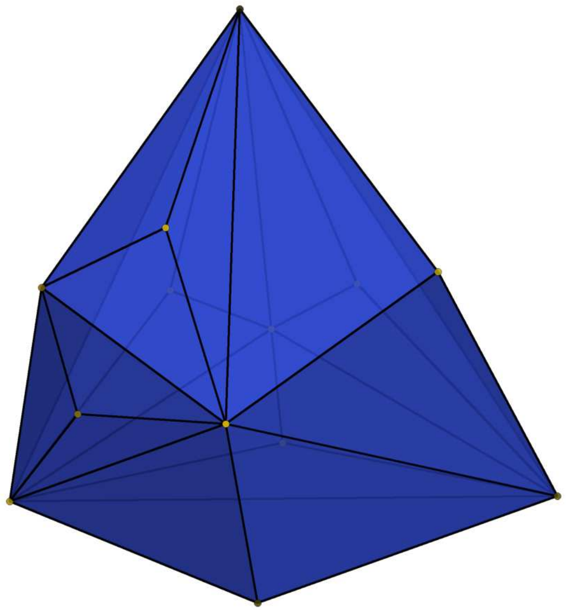


▷ LP with n variables

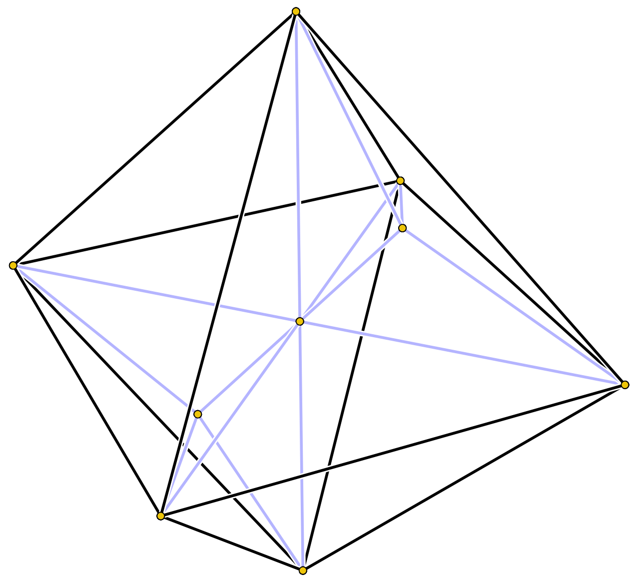
2-dim



3-dim



4-dim



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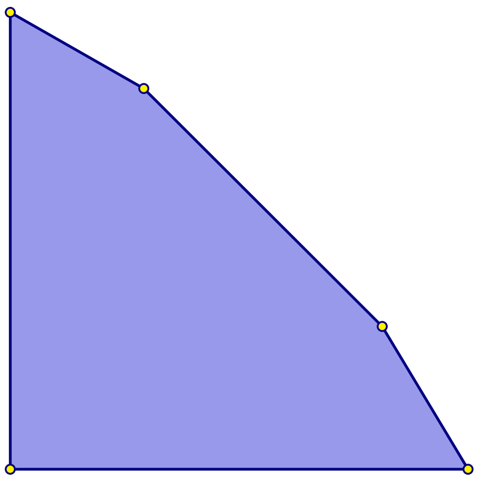

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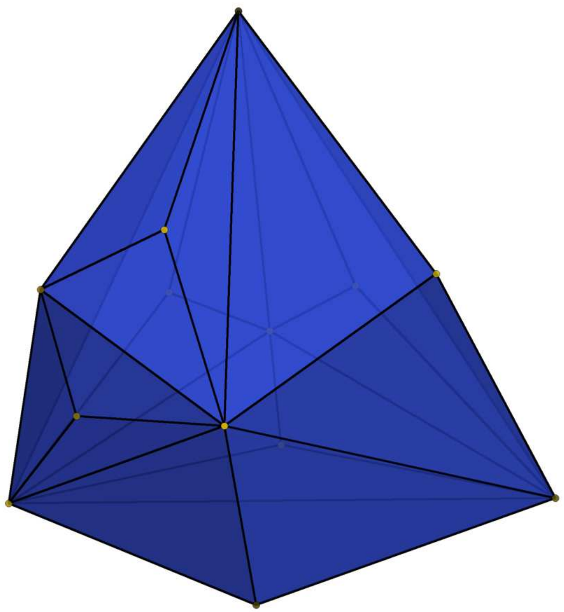

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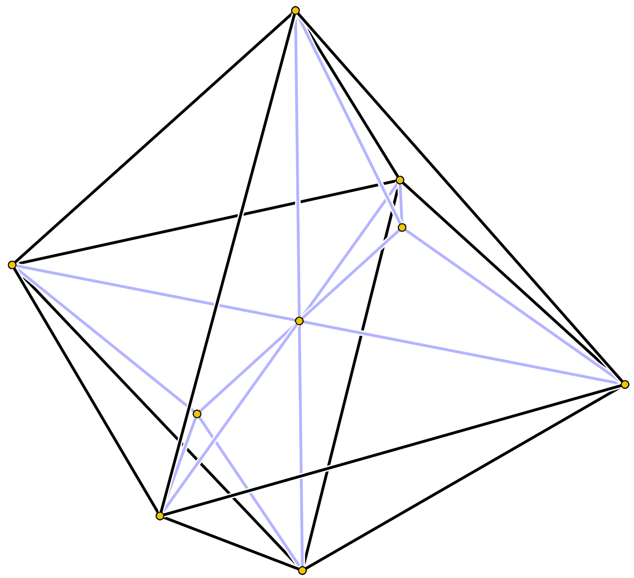

- LP with n variables
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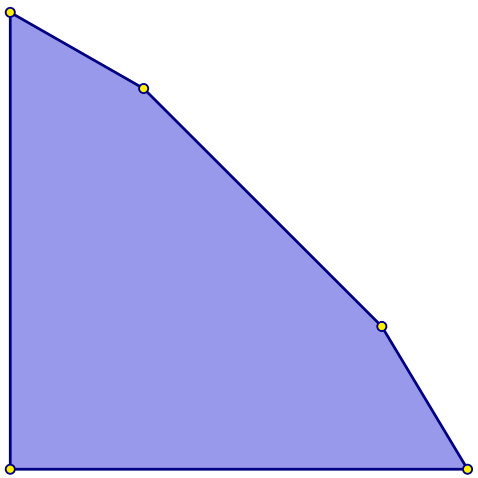

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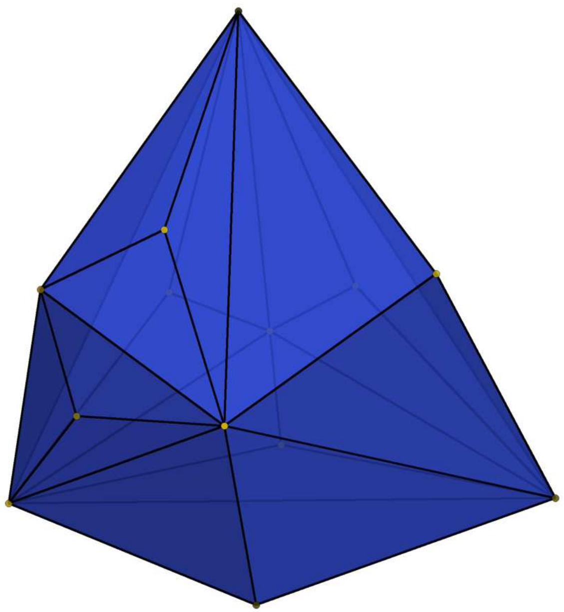

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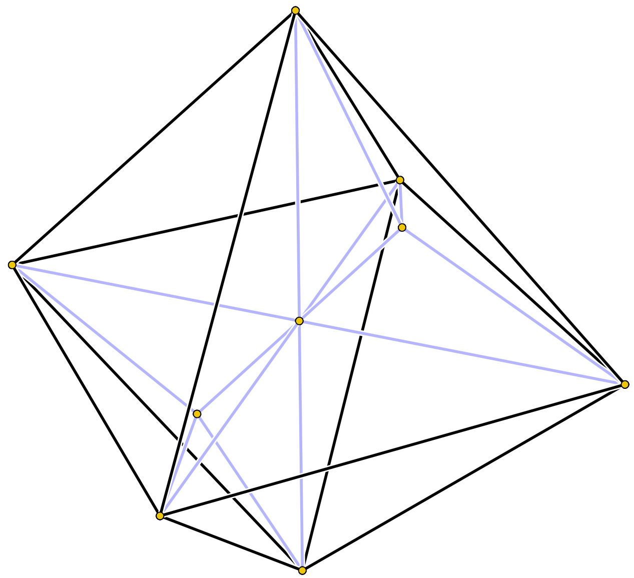

* constraints have to be linearly independent!

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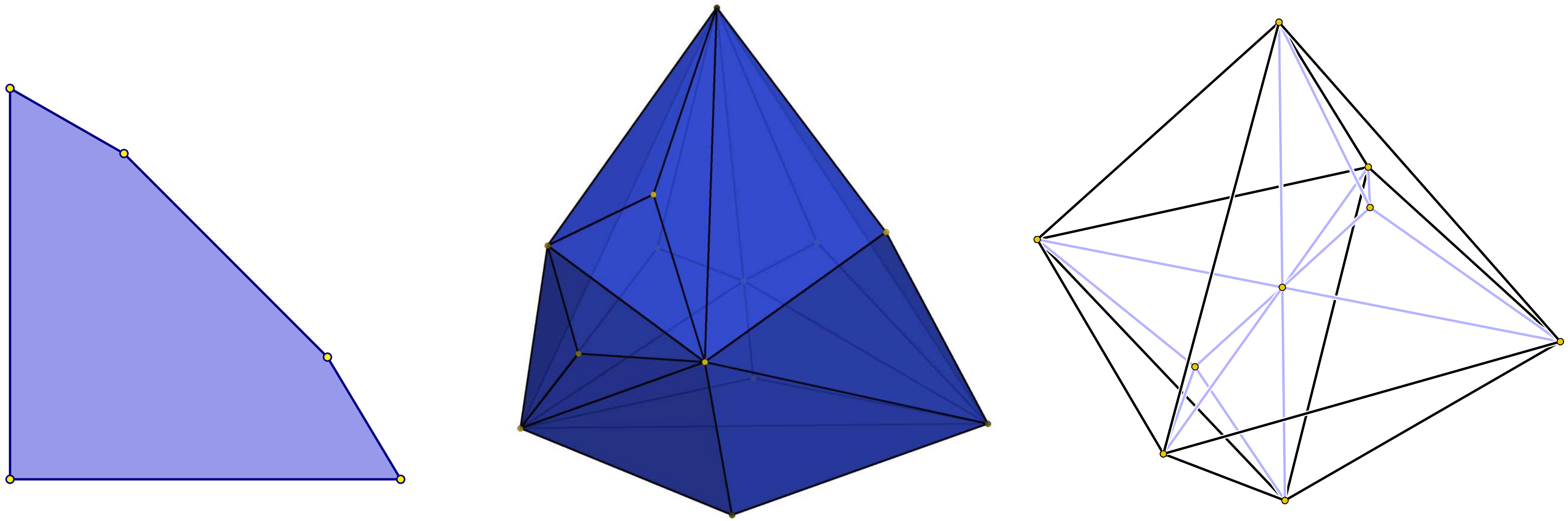
2-dim 3-dim 4-dim



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How to get to a neighbouring vertex?

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How to realise that the optimum is reached?

What if there is no optimum?

Or no feasible solution at all?!

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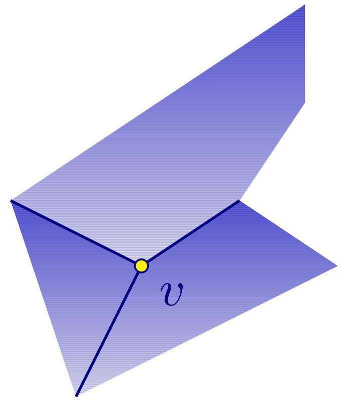
- Two vertices v, w are neighbours of each other if they are connected by an edge



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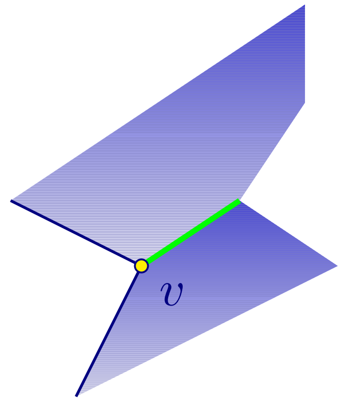

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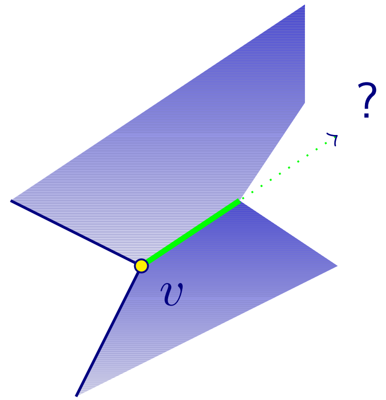

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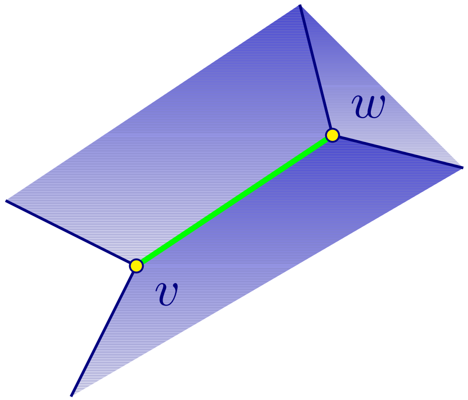
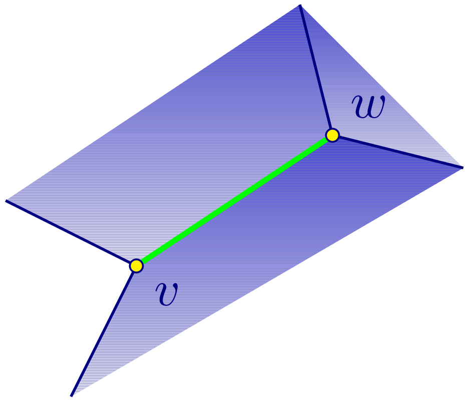


- ▷ Two vertices v, w are neighbours of each other if they are connected by an edge
  - ➡ To compute w from v only one binding constraint has to be exchanged!



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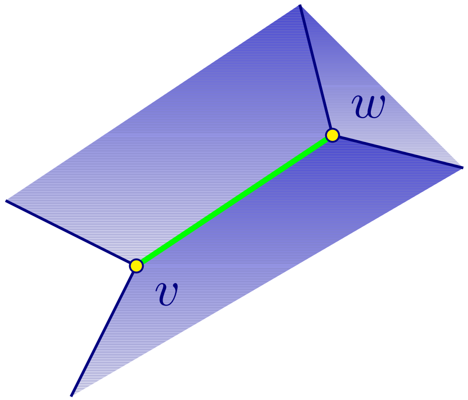
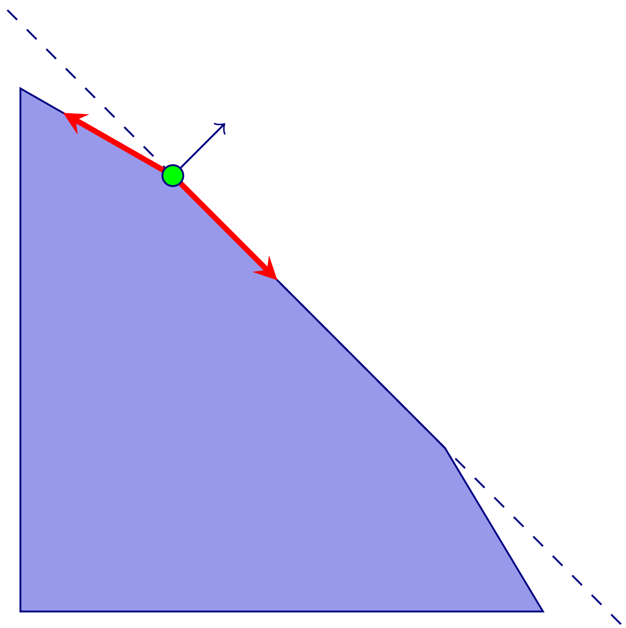
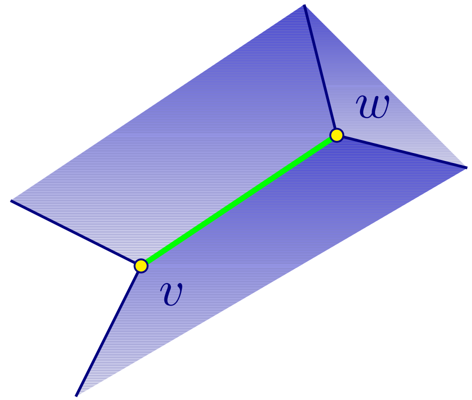


- If a vertex has no neighbours with a better objective function value then an optimum is reached!





- Two vertices v, w are neighbours of each other if they are connected by an edge
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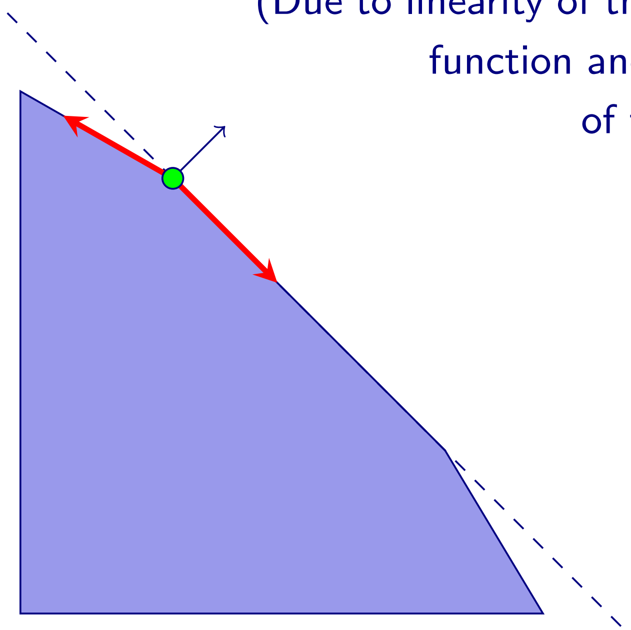

(Due to linearity of the objective function and convexity of the feasible region!)

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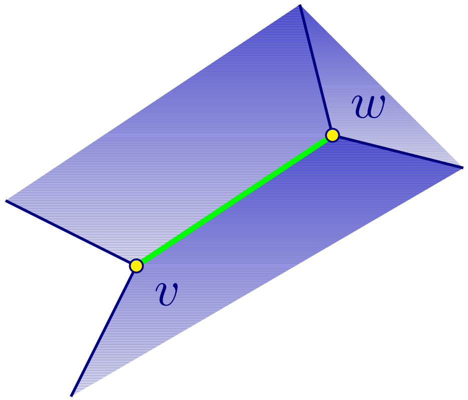

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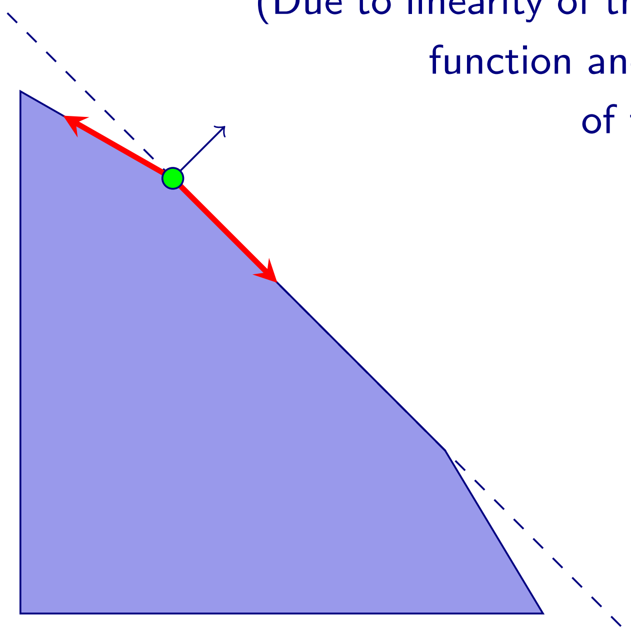

(Due to linearity of the objective function and convexity of the feasible region!)

▷ If no opposite vertex can be computed (i.e. no “opposite” constraint found), then the problem is unbounded!

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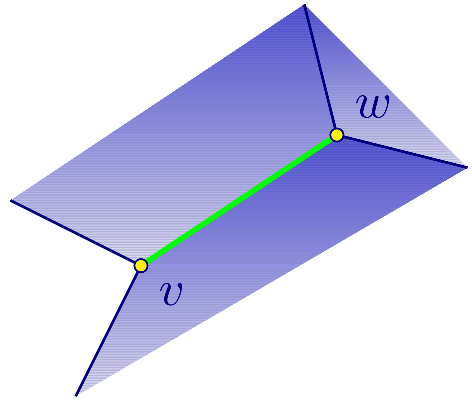

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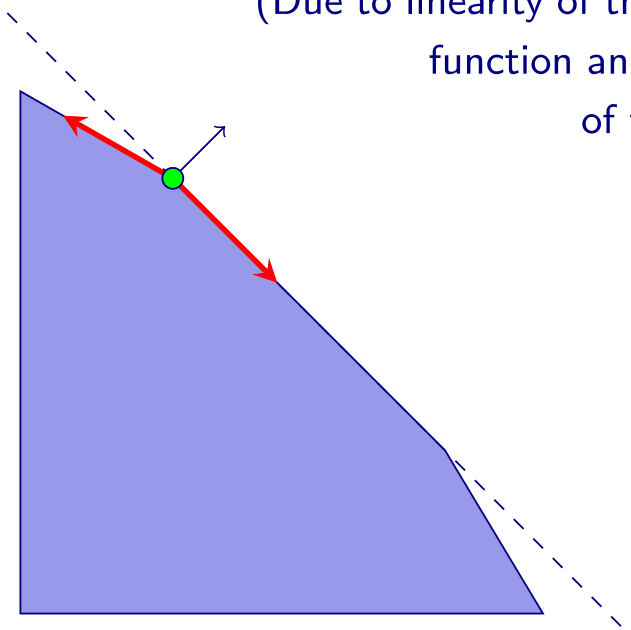

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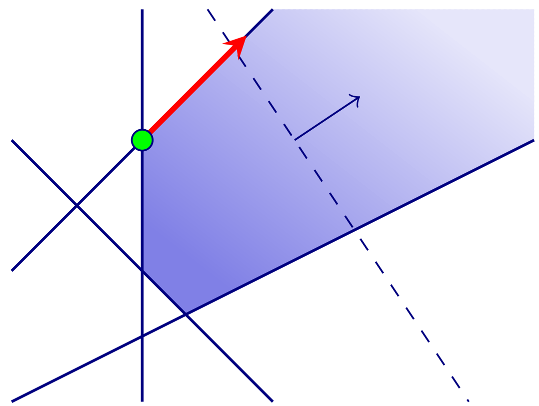
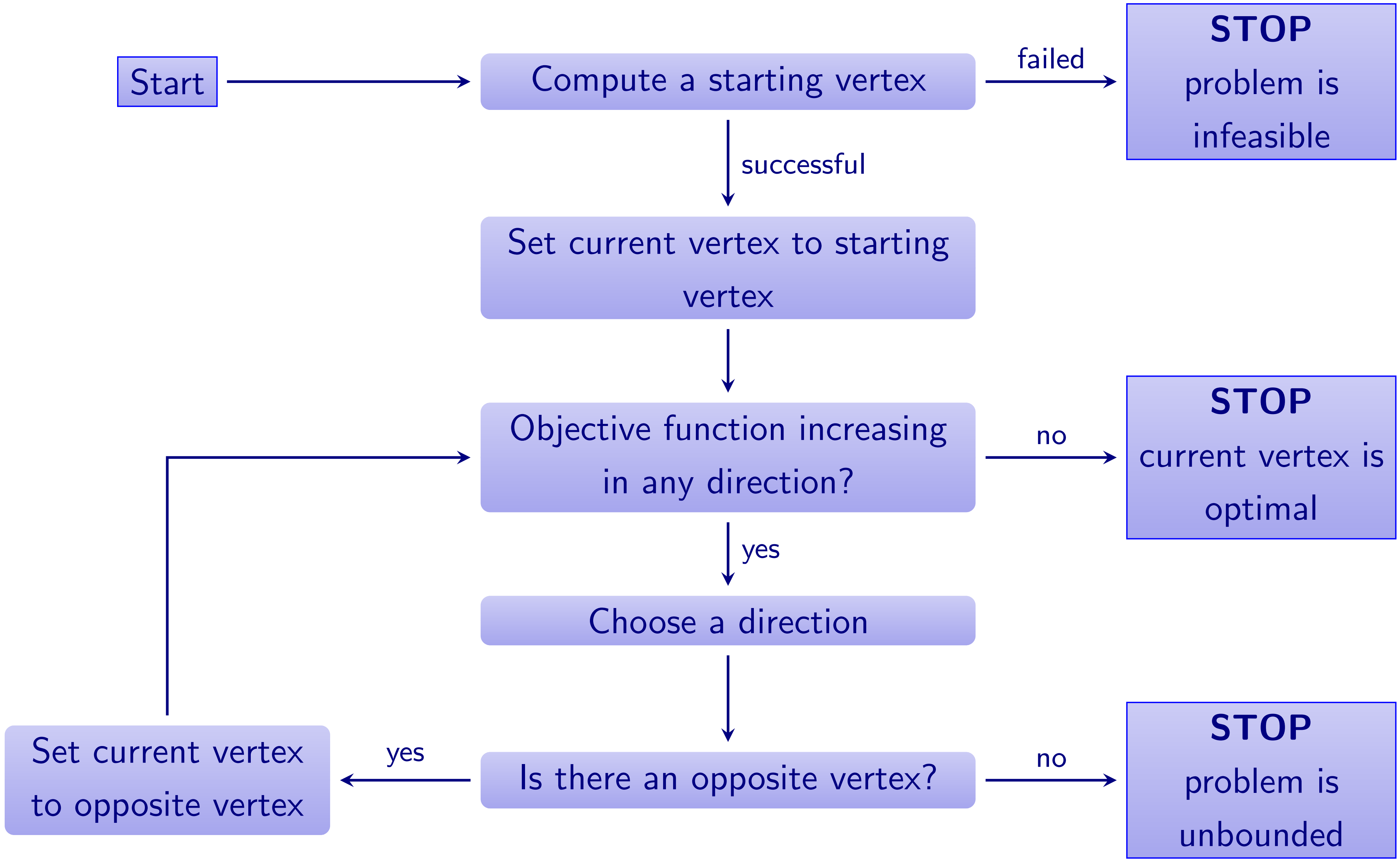
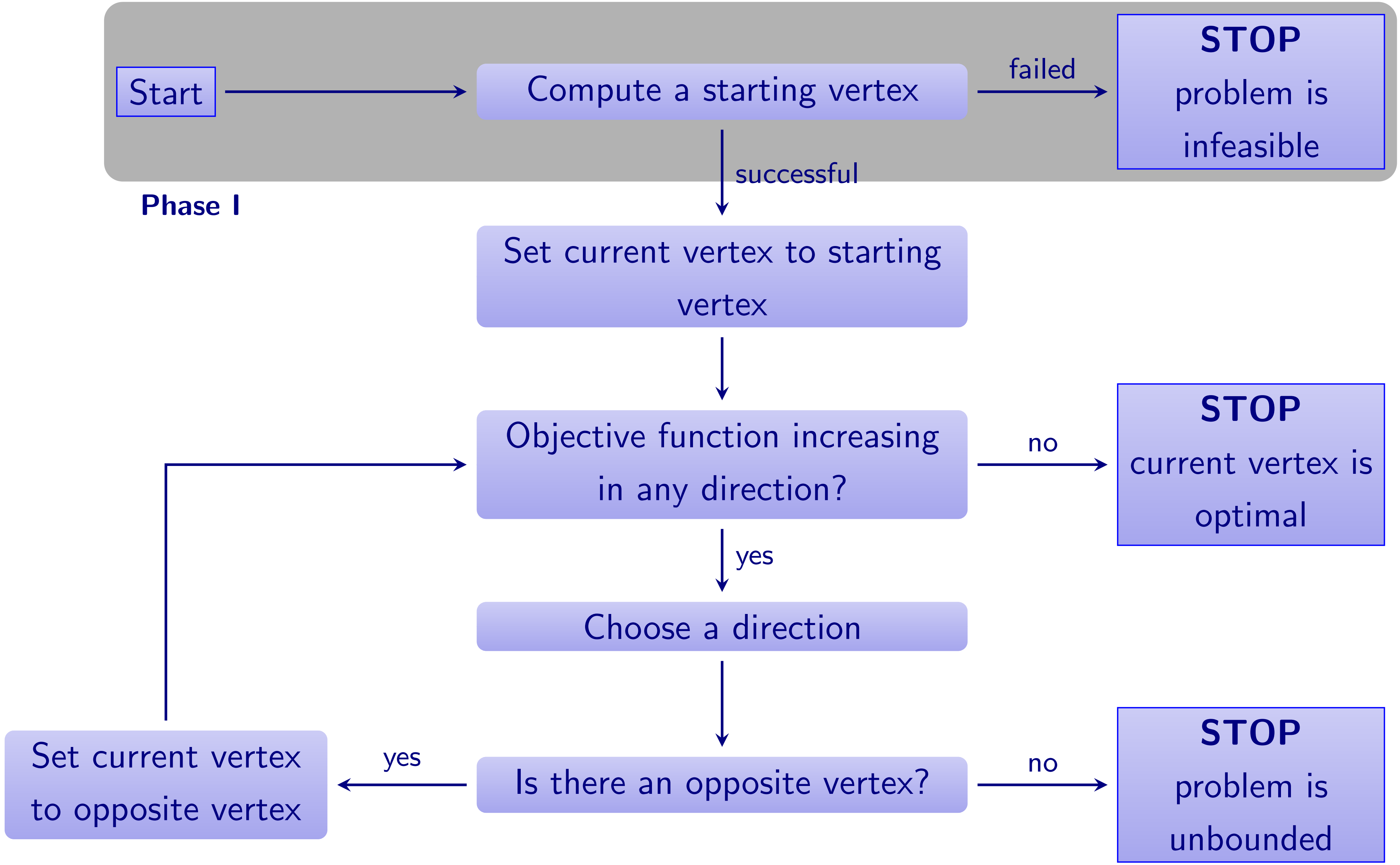



Start
Compute a starting vertex
failed
STOP
problem is infeasible
successful
Set current vertex to starting vertex
Objective function increasing in any direction?
no
STOP
current vertex is optimal
yes
Choose a direction
Is there an opposite vertex?
yes
Set current vertex to opposite vertex
no
STOP
problem is unbounded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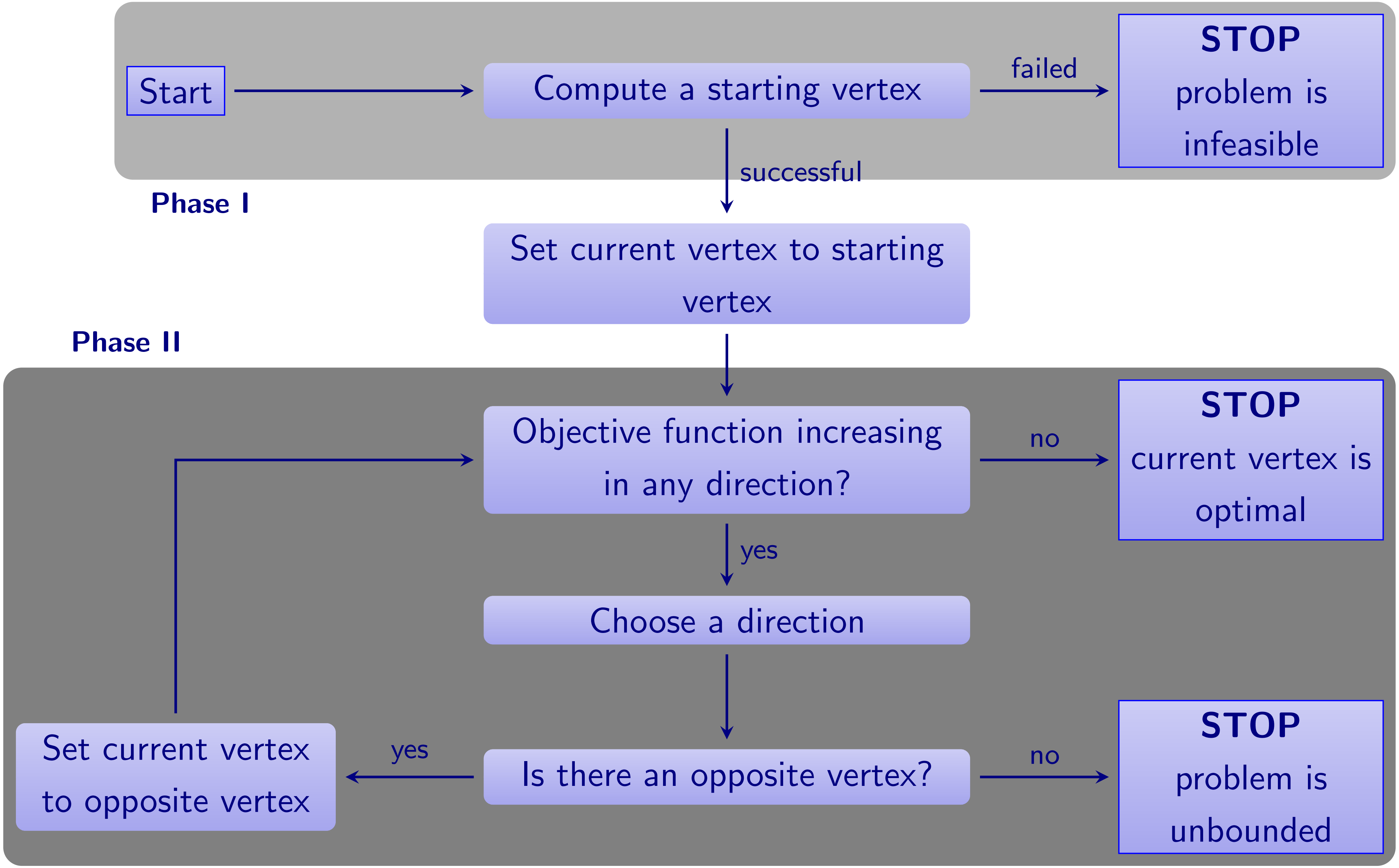
Start
Compute a starting vertex
failed
STOP
problem is infeasible
successful
Phase I
Set current vertex to starting vertex
Phase II
Objective function increasing in any direction?
no
STOP
current vertex is optimal
yes
Choose a direction
Set current vertex to opposite vertex
yes
Is there an opposite vertex?
no
STOP
problem is unbounded

▷ Problem: Compute a starting vertex

- ▷ Problem: Compute a starting vertex

  ➡ Phase I: Formulate an auxiliary LP from the original with

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and solve it by using (Phase II) of the simplex algorithm itself!

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▷ Problem: **Choose a direction**

▷ Problem: **Compute a starting vertex**

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▷ Bad Pivot rules may lead to inefficient behaviour

▷ Problem: **Compute a starting vertex**

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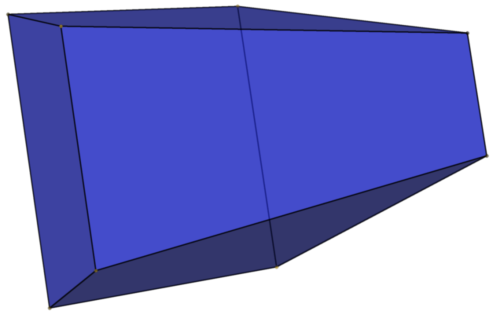

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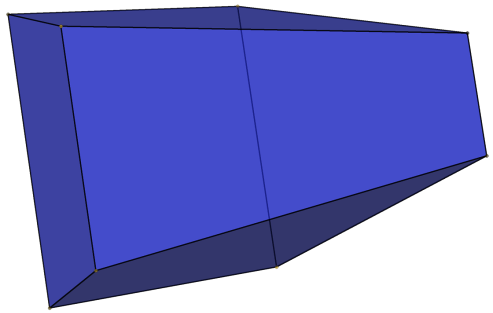

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➡ Phase I: Formulate an auxiliary LP from the original with

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and solve it by using (Phase II) of the simplex algorithm itself!

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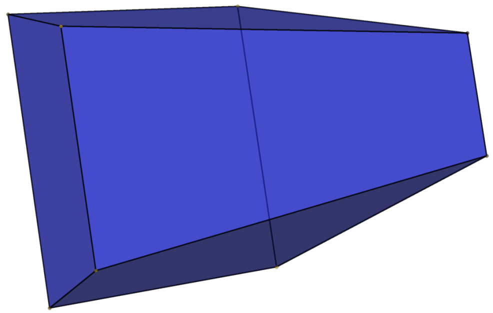

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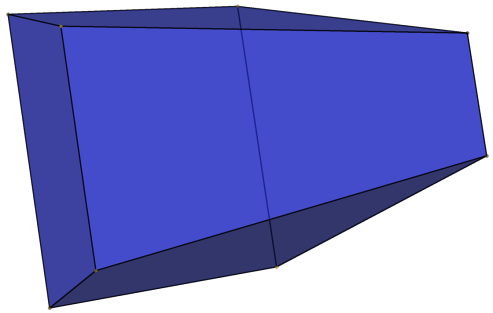

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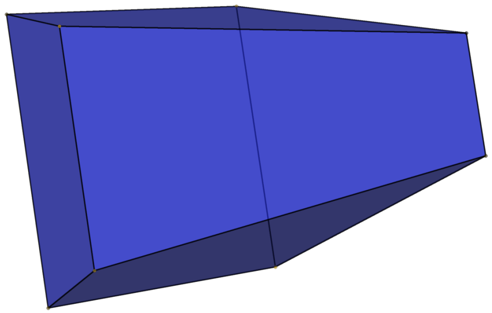

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➡ May lead to unprecise solution values, or even completely wrong solutions, or also cycling! ➡ Use numerically stable computations!

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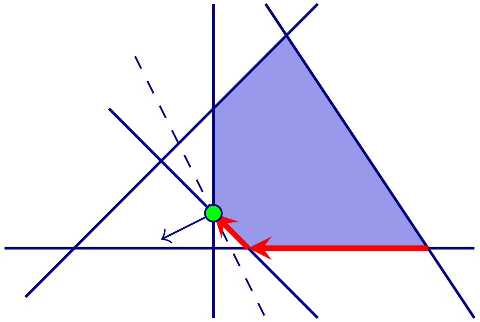
(Primal) Simplex Algorithm: Step from primal feasible solution to neighbouring primal feasible solution in direction of the objective until an optimal solution is reached



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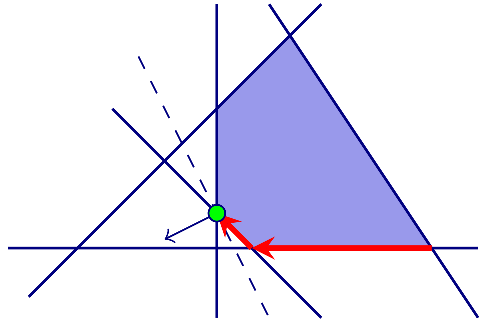
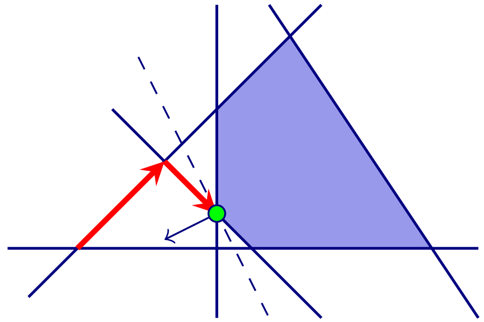


Dual Simplex Algorithm: Step from dual feasible solution to neighbouring dual feasible solution until a primal feasible solution is reached (which is then also optimal!)



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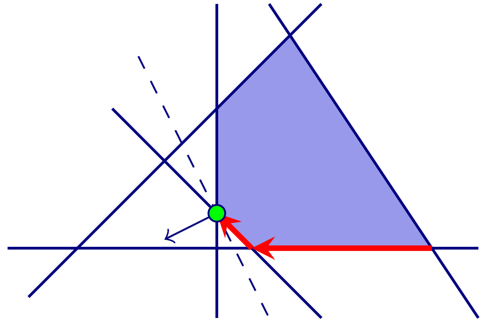

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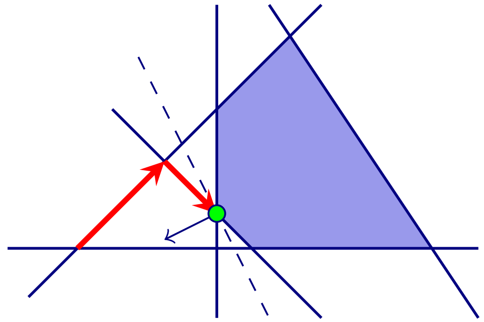

- Dual simplex seems to be much more efficient (on average on real-world problems)!

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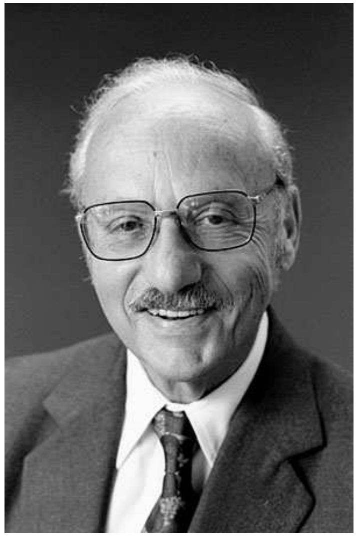

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(1914–2005)

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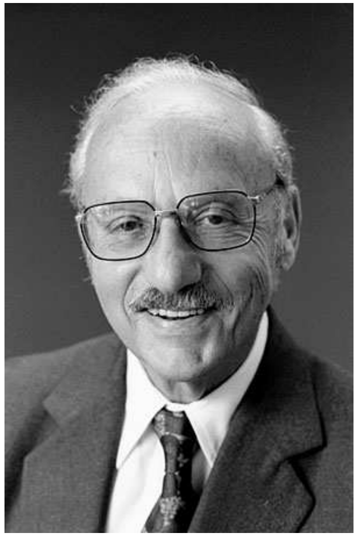

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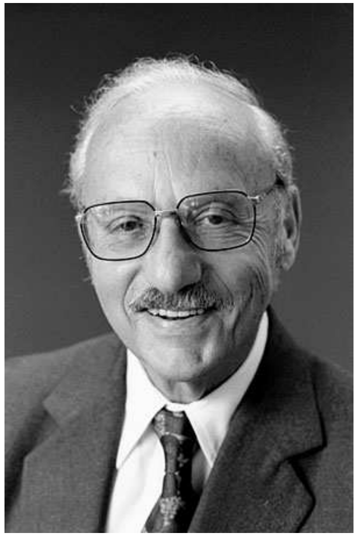

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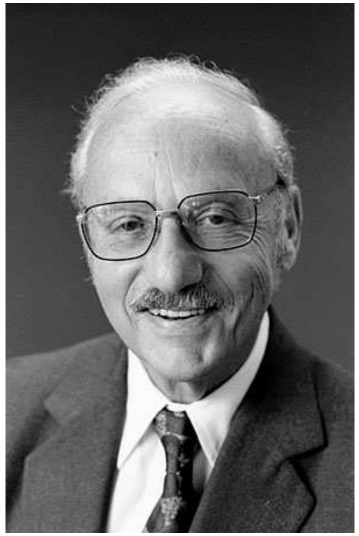

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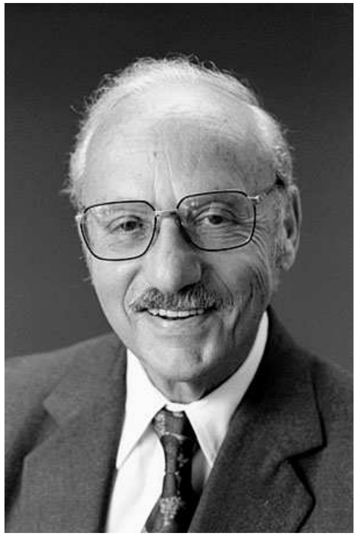

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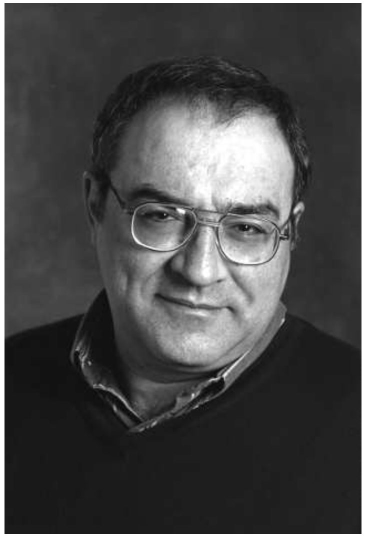

Leonid Genrikhovich Khachiyan
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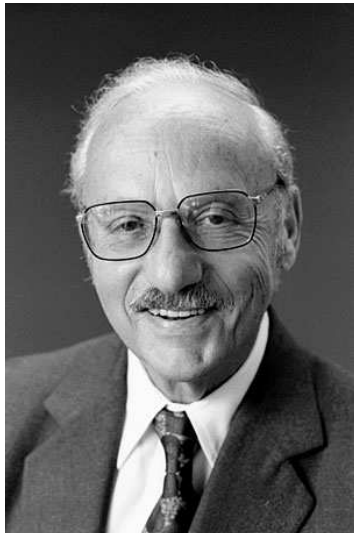
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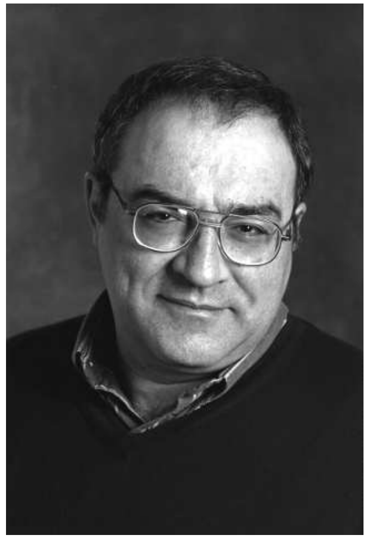
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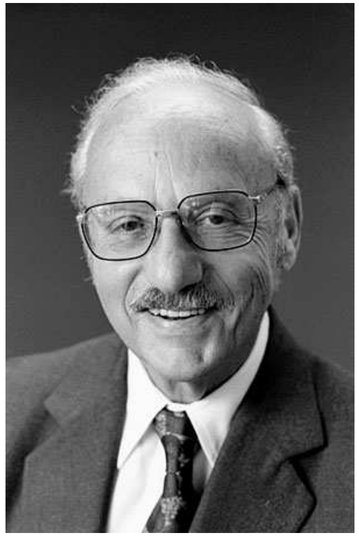

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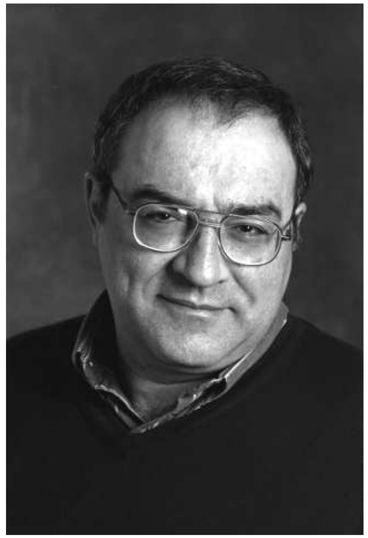

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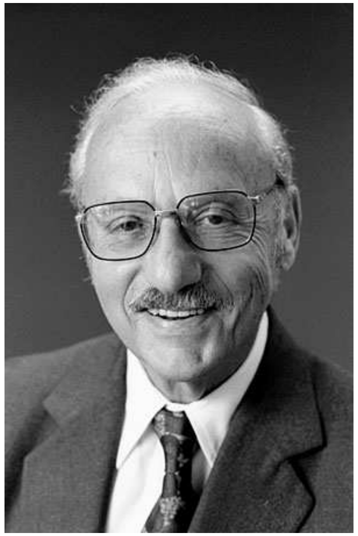

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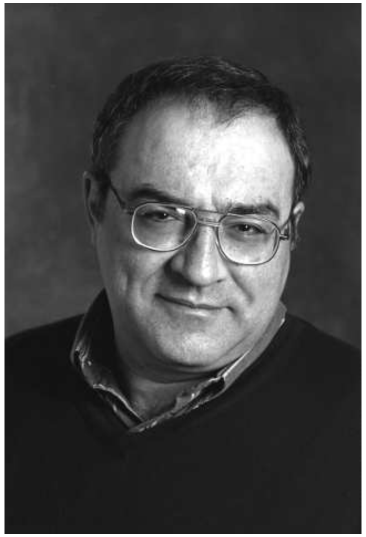

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LP algorithm zoo

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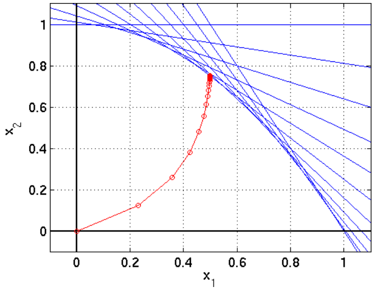

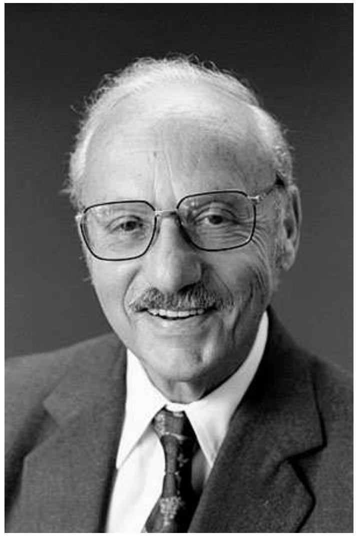

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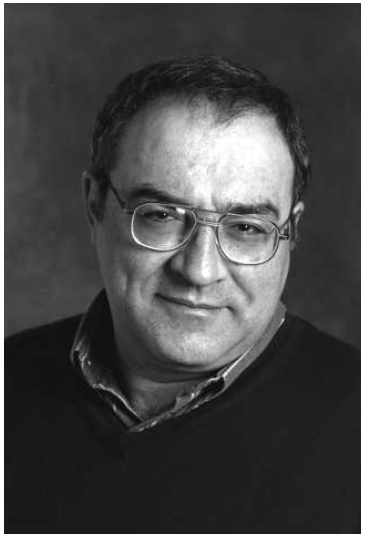

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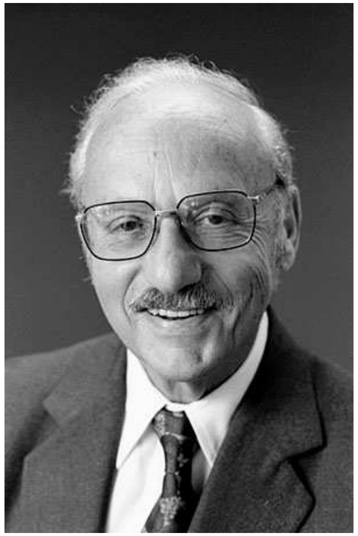

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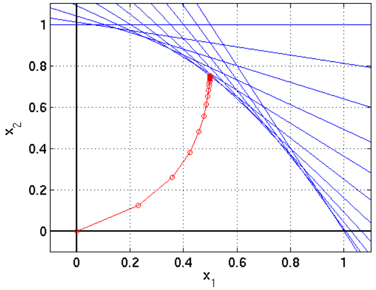

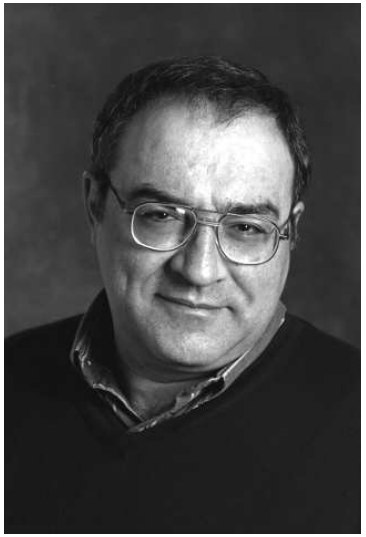

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*from: Bixby, *Solving real-world linear programs: a decade and more of progress*, OR 50 (2002), 3–15

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Sun 3/50 vs Pentium 4, 1.7 GHz ➡ speedup factor 800

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- Conclusion (as of 2002)*:

  A model that might have taken a year to solve 10 years ago can now solve in less than 30 seconds.

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