Mathematical Tools for Engineering and Management

Lecture 10

04 Jan 2012

- ▷ Models, Data and Algorithms
- ▷ Linear Optimization
- ▷ Mathematical Background: Polyhedra, Simplex-Algorithm
- ▷ Sensitivity Analysis; (Mixed) Integer Programming
- ▷ MIP Modelling
- ▷ MIP Modelling: More Examples; Branch & Bound
- ▷ Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- ▷ TSP-Heuristics
- ▷ Network Flows
- ▷ Shortest Path Problem
- ▷ Complexity Theory
- ▷ Nonlinear Optimization, Scheduling
- ▷ Lot Sizing, Multicriteria Optimization
- ▷ Oral exam

▷ Problem: find a shortest path from PTZ to Kreuzburger

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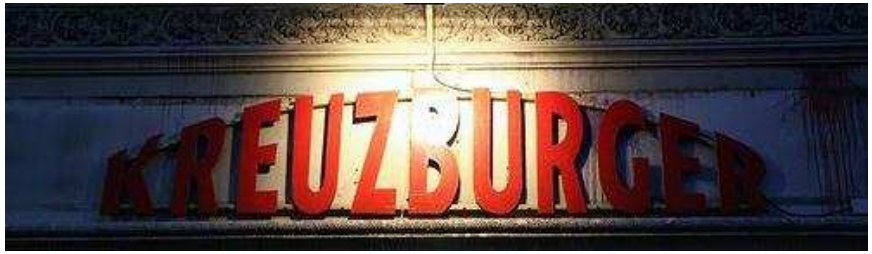

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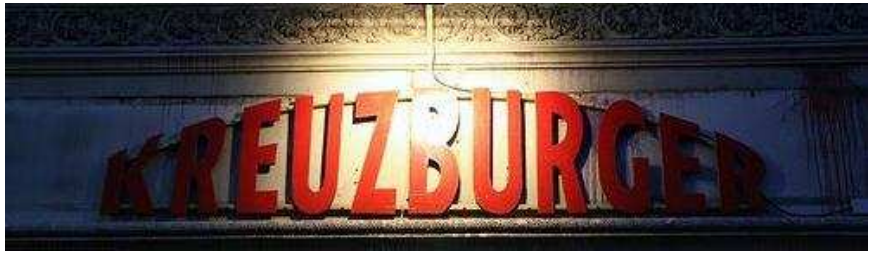

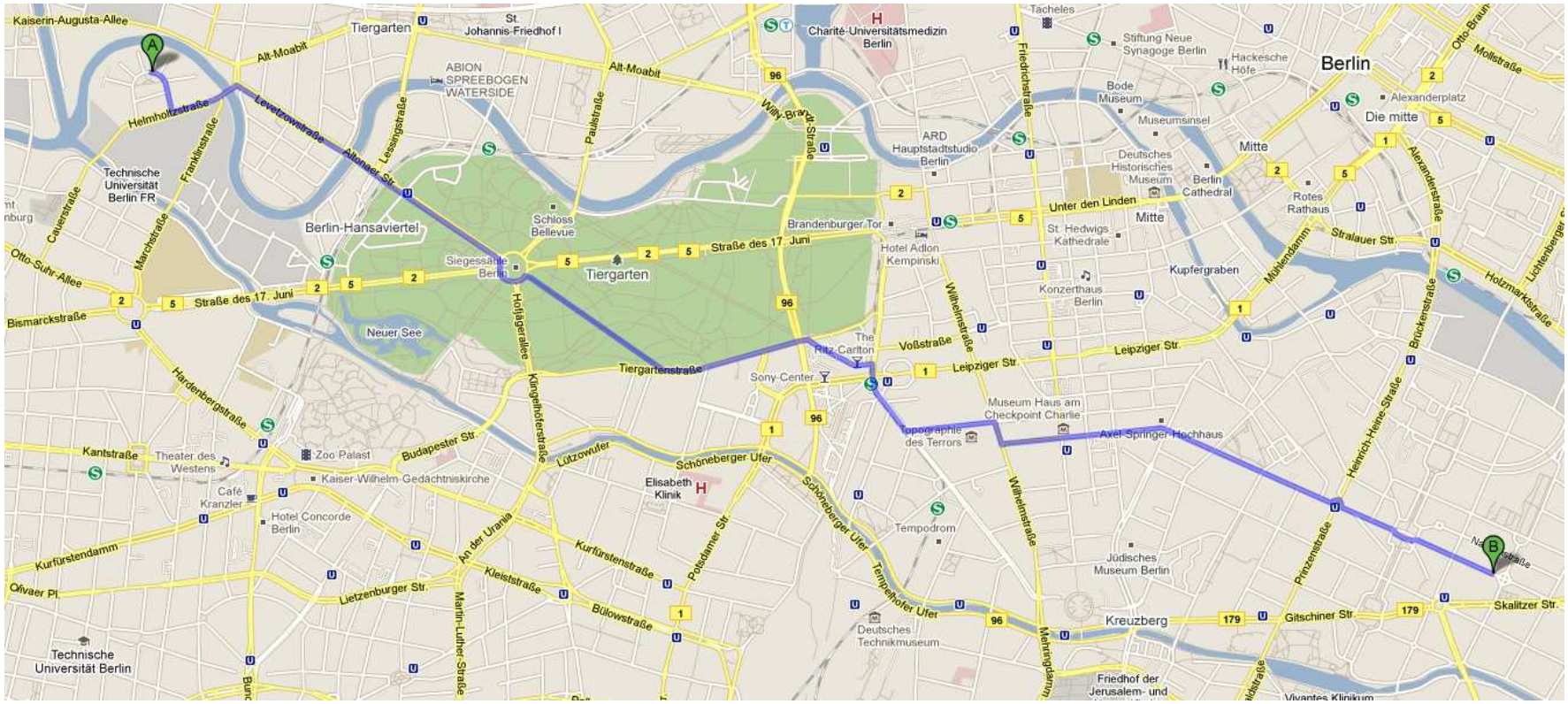

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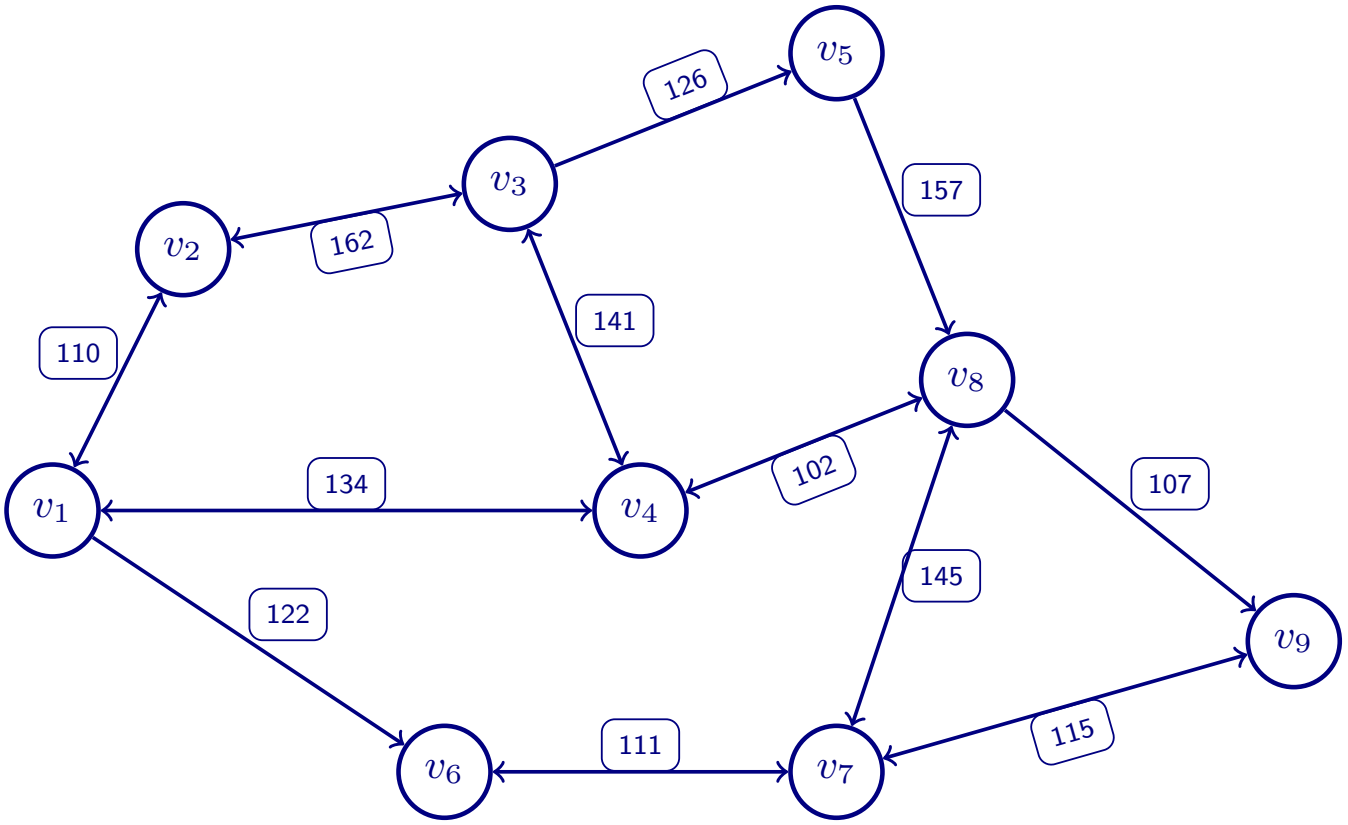

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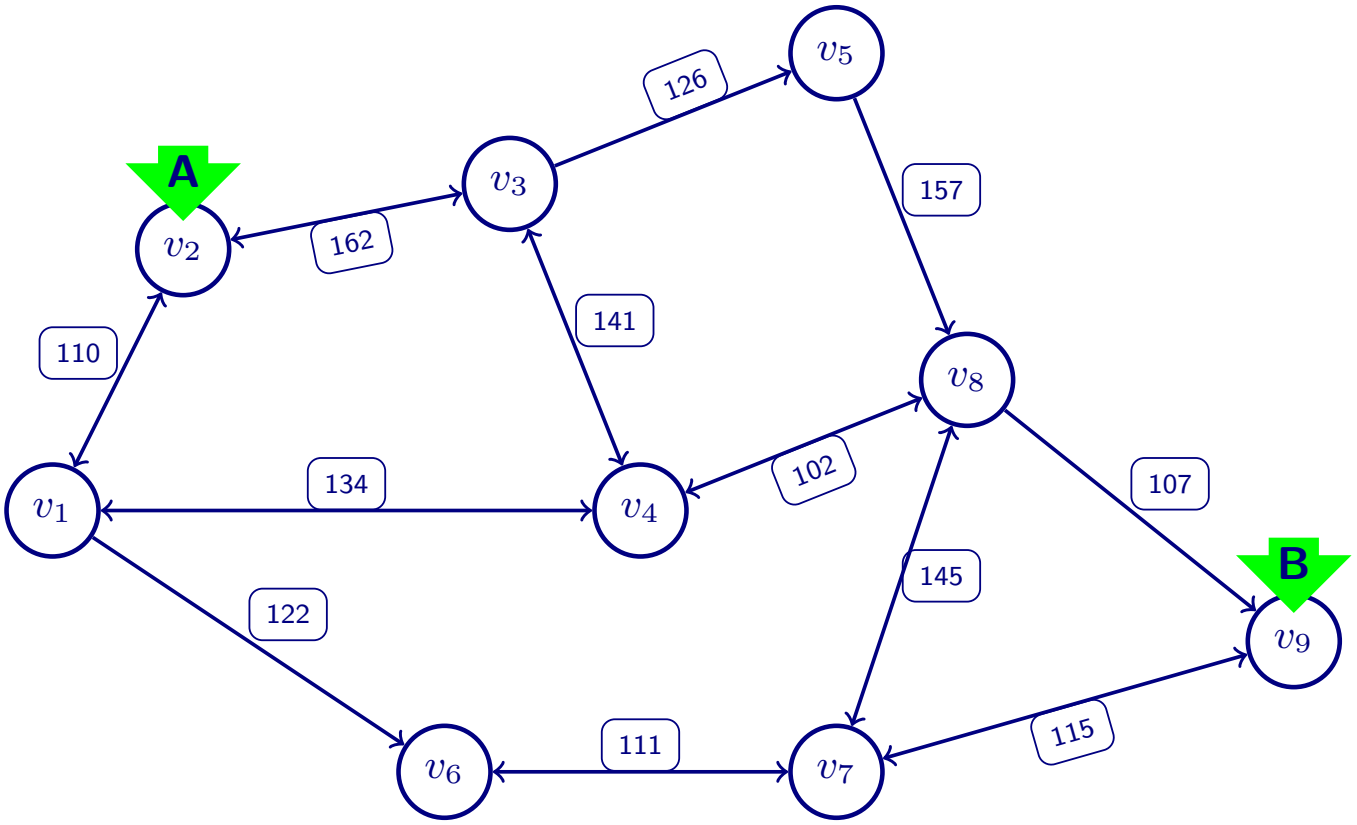

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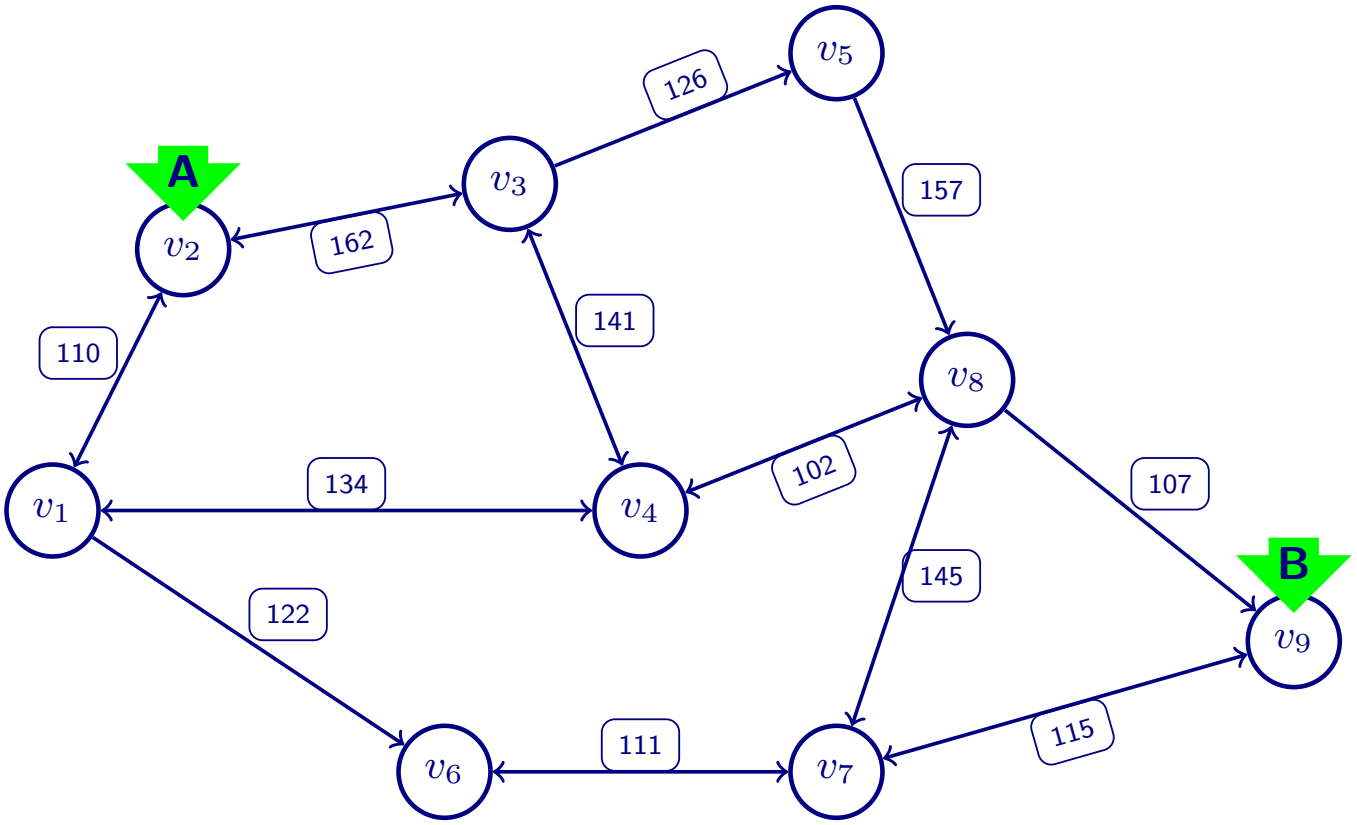

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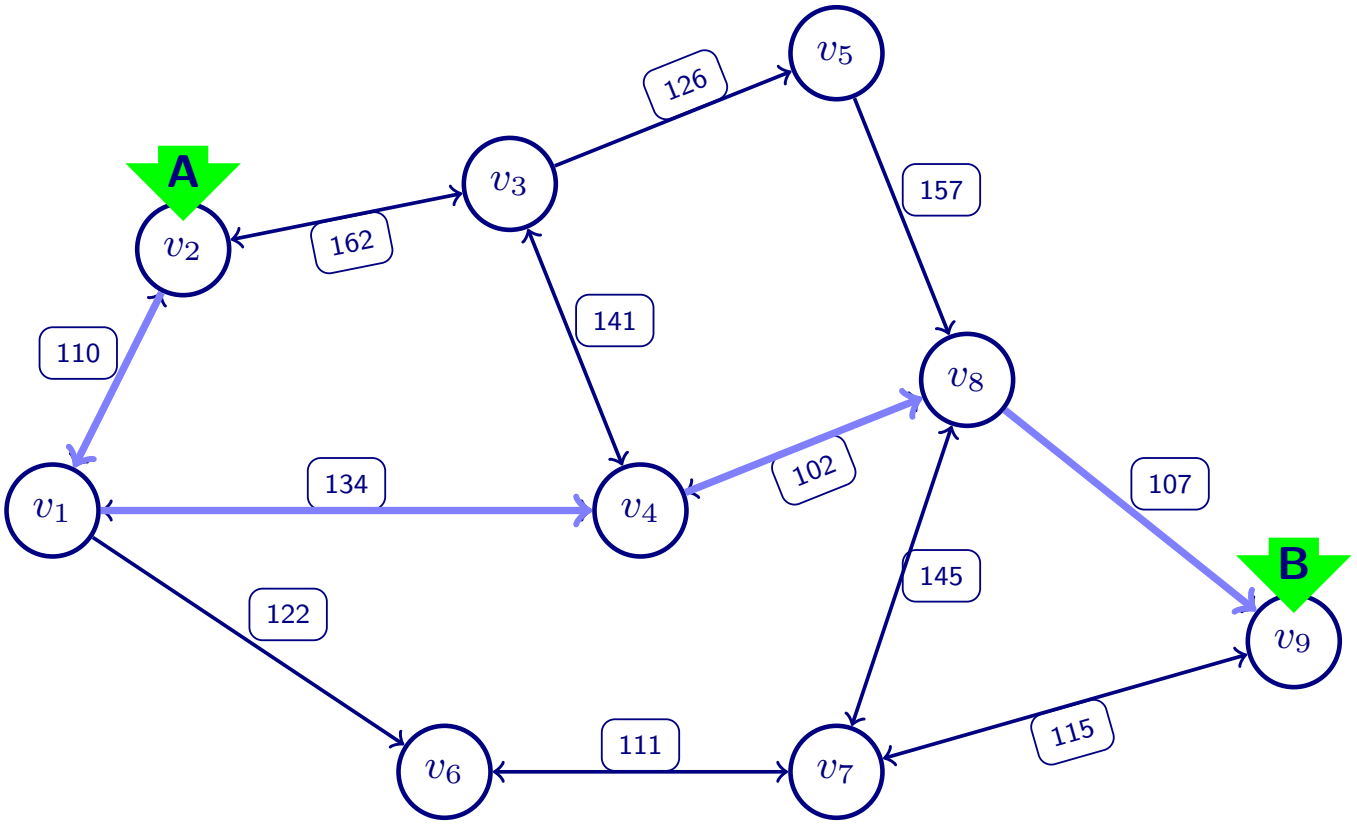

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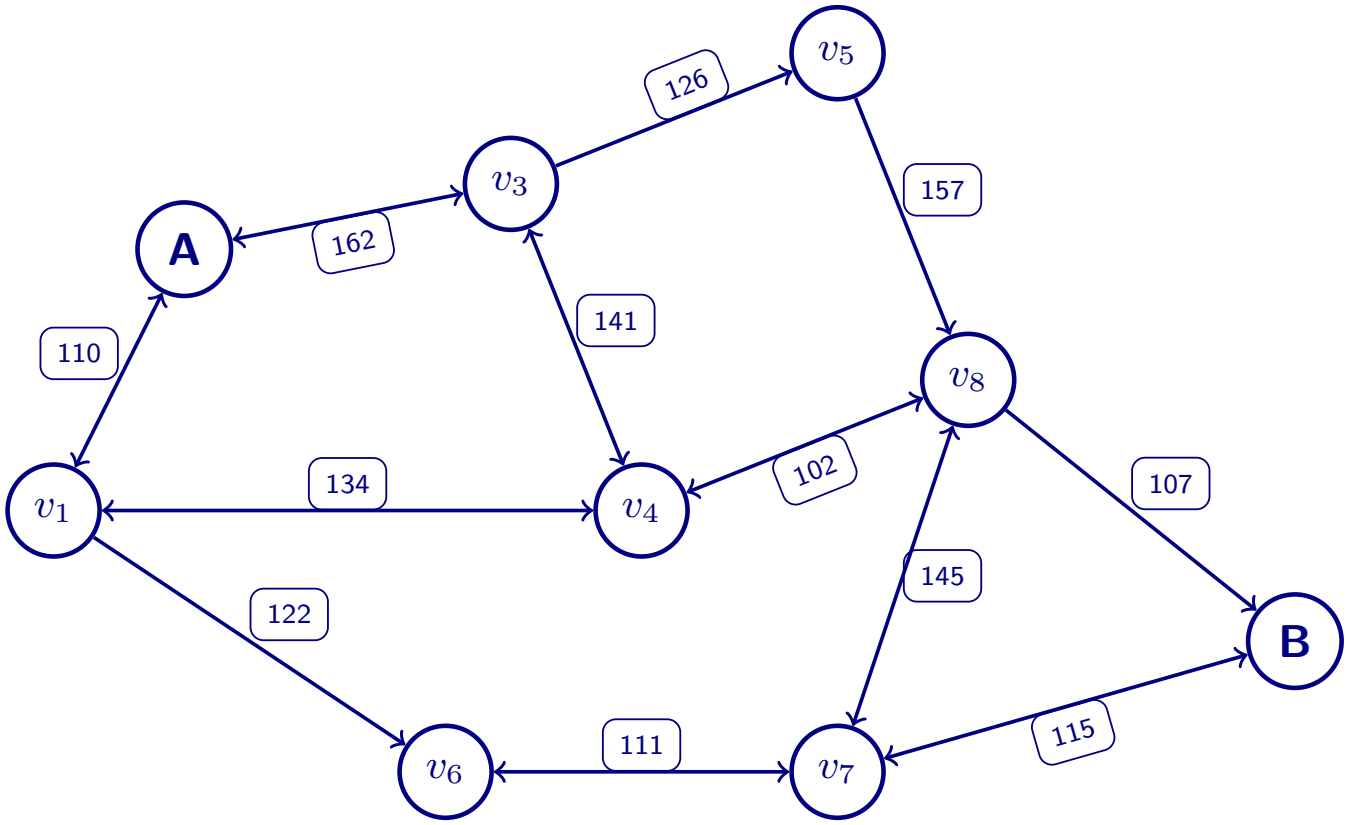

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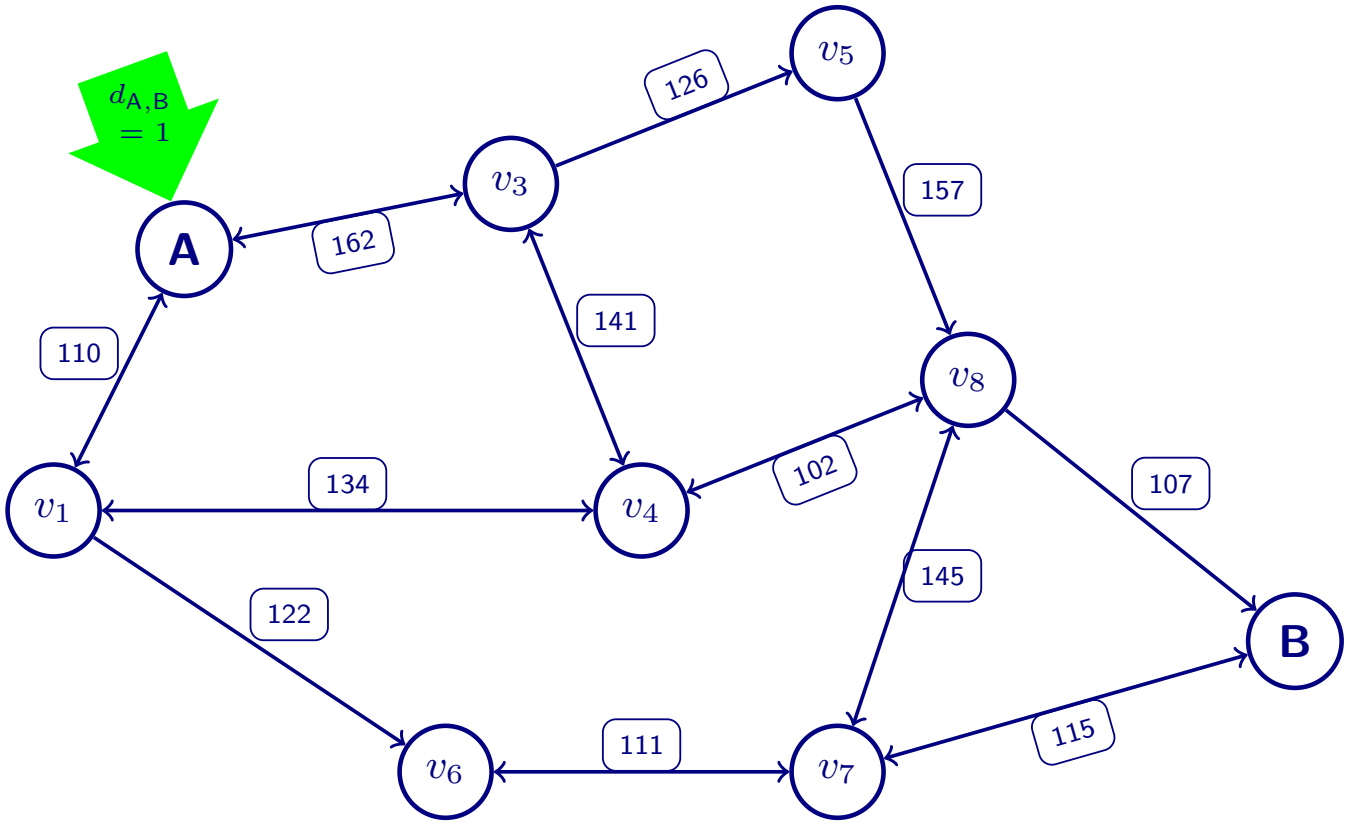

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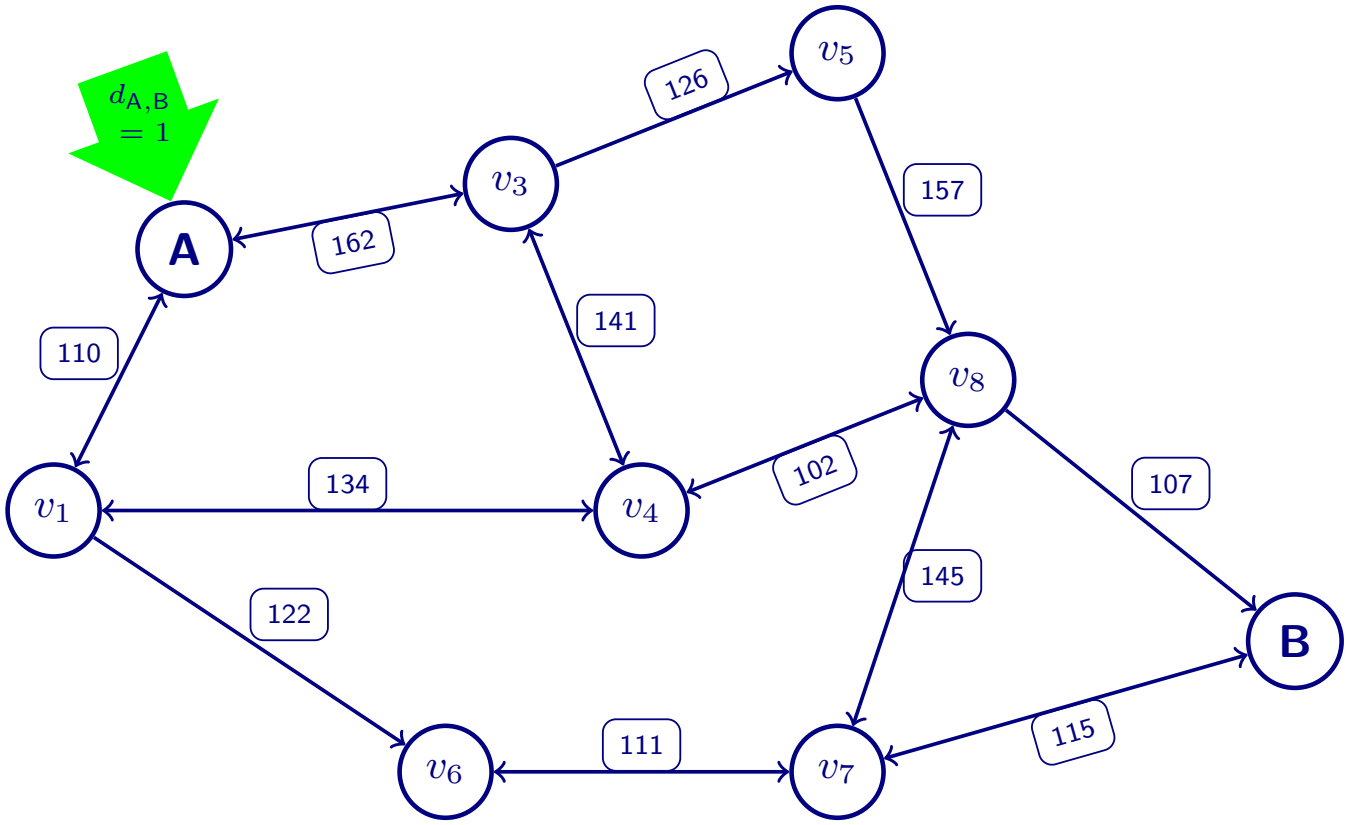

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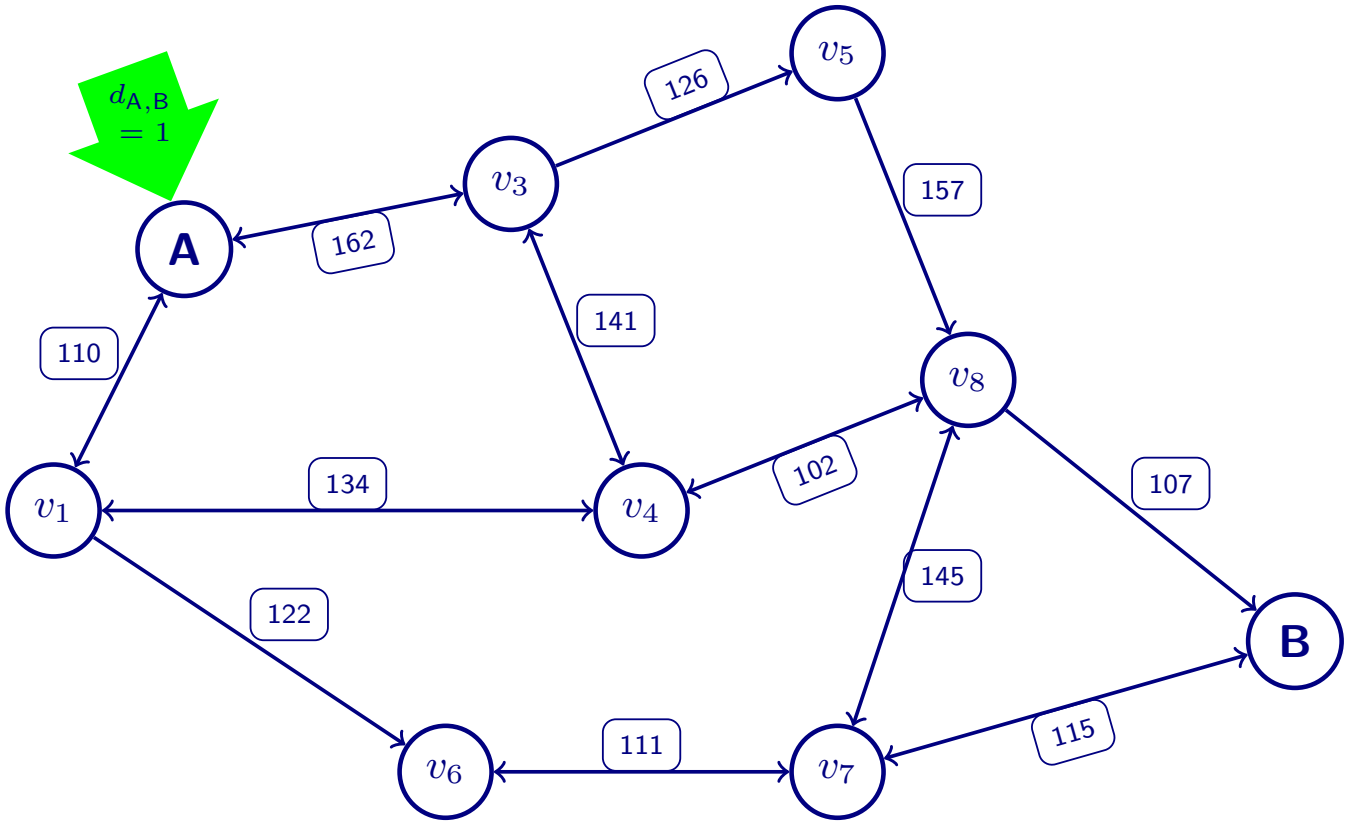

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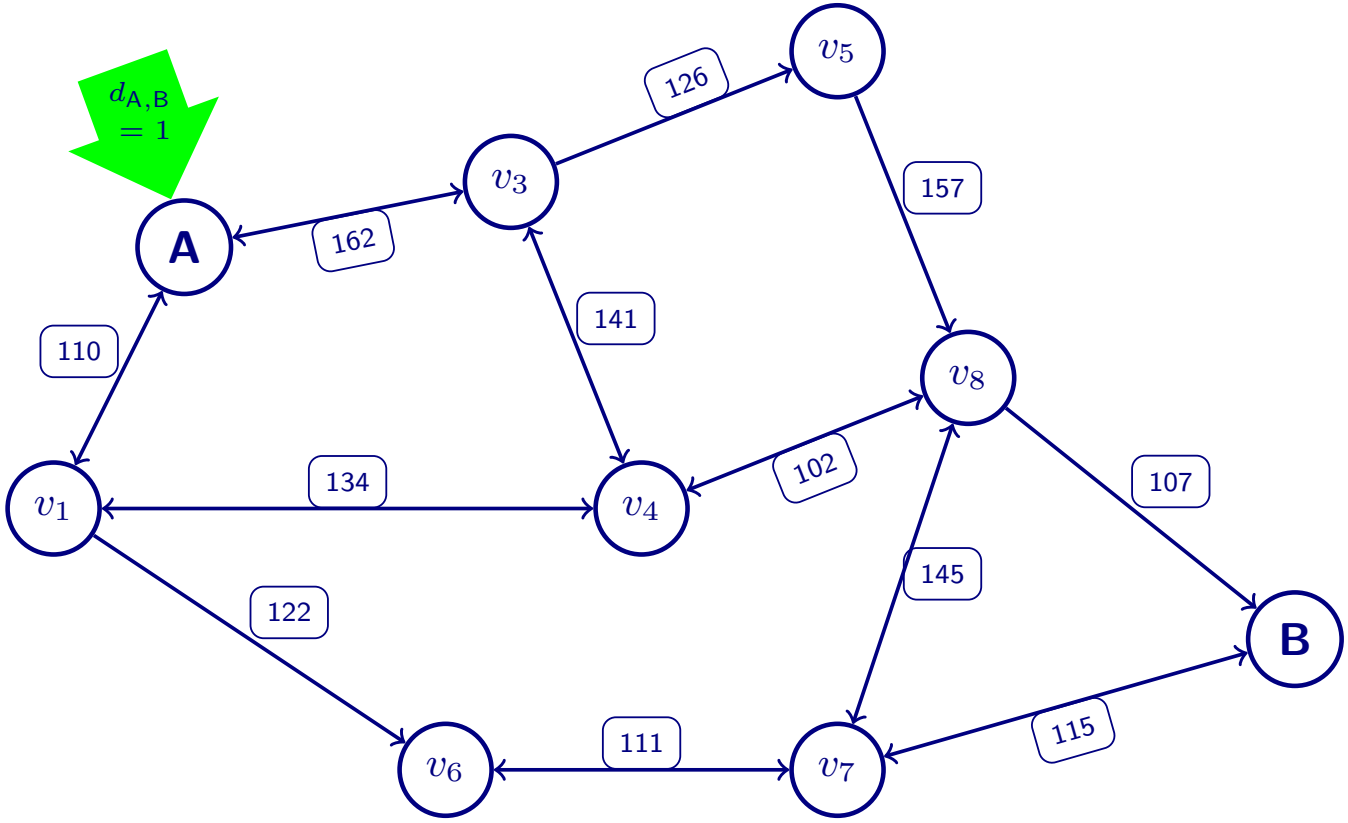

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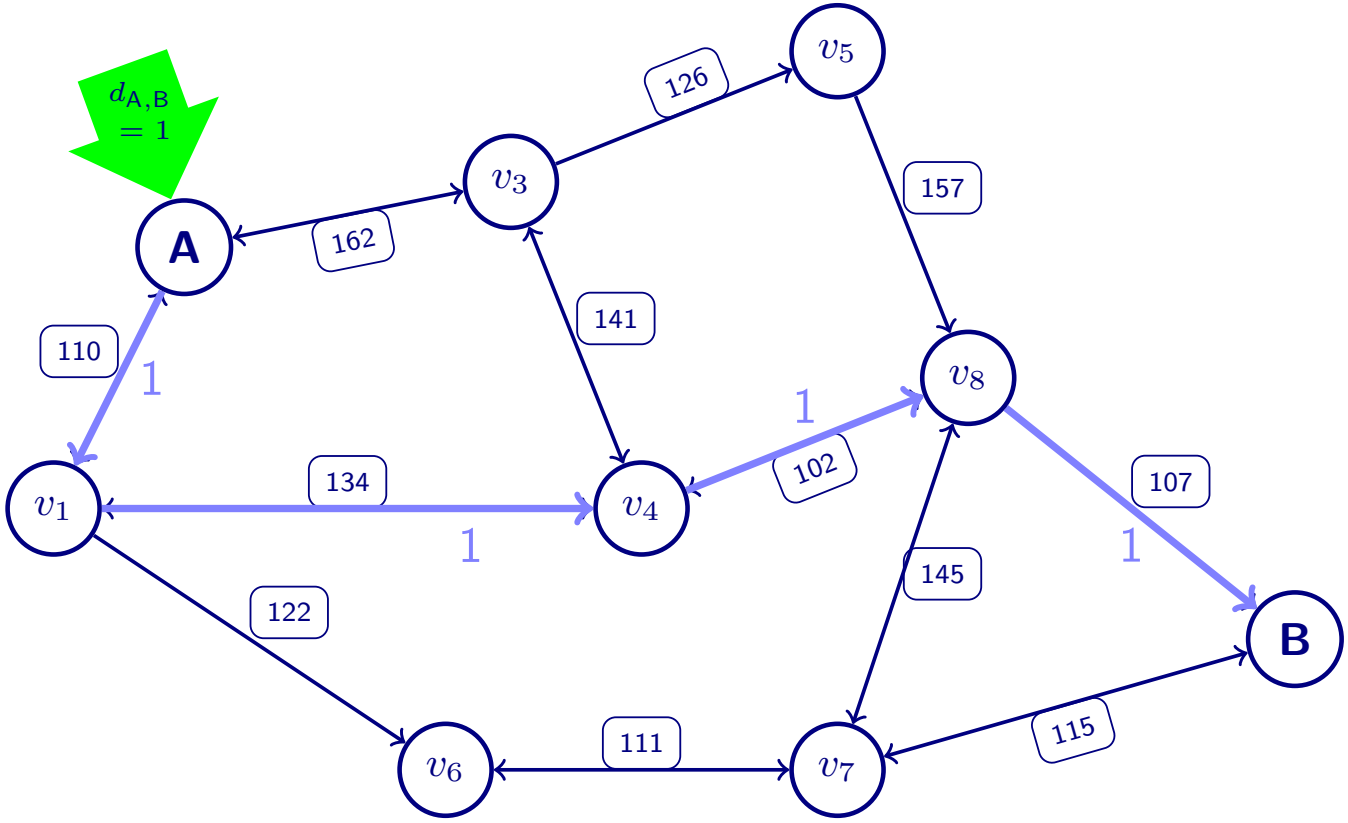

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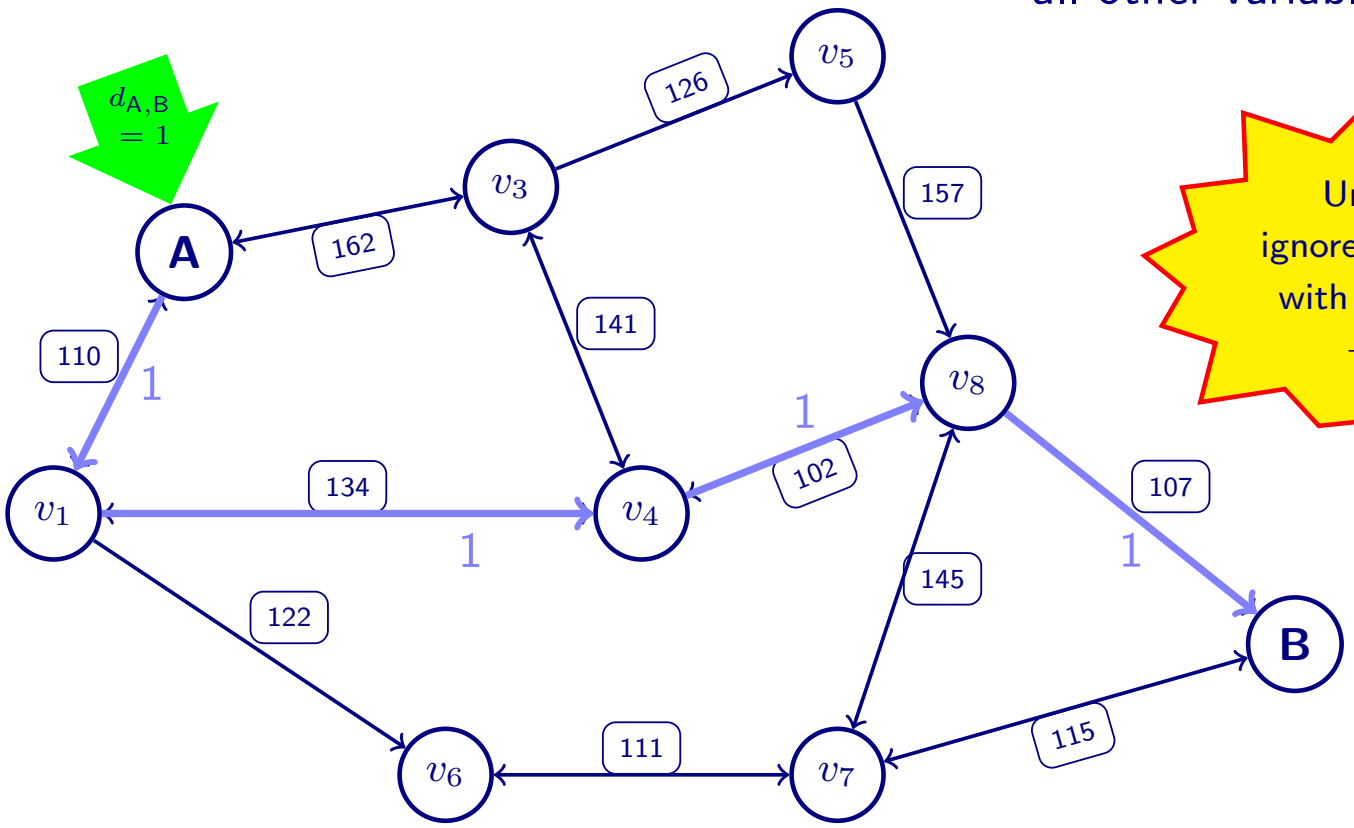

Unsplittability can be ignored if the model is solved with the simplex algorithm → linear program!

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Edsger Wybe Dijkstra
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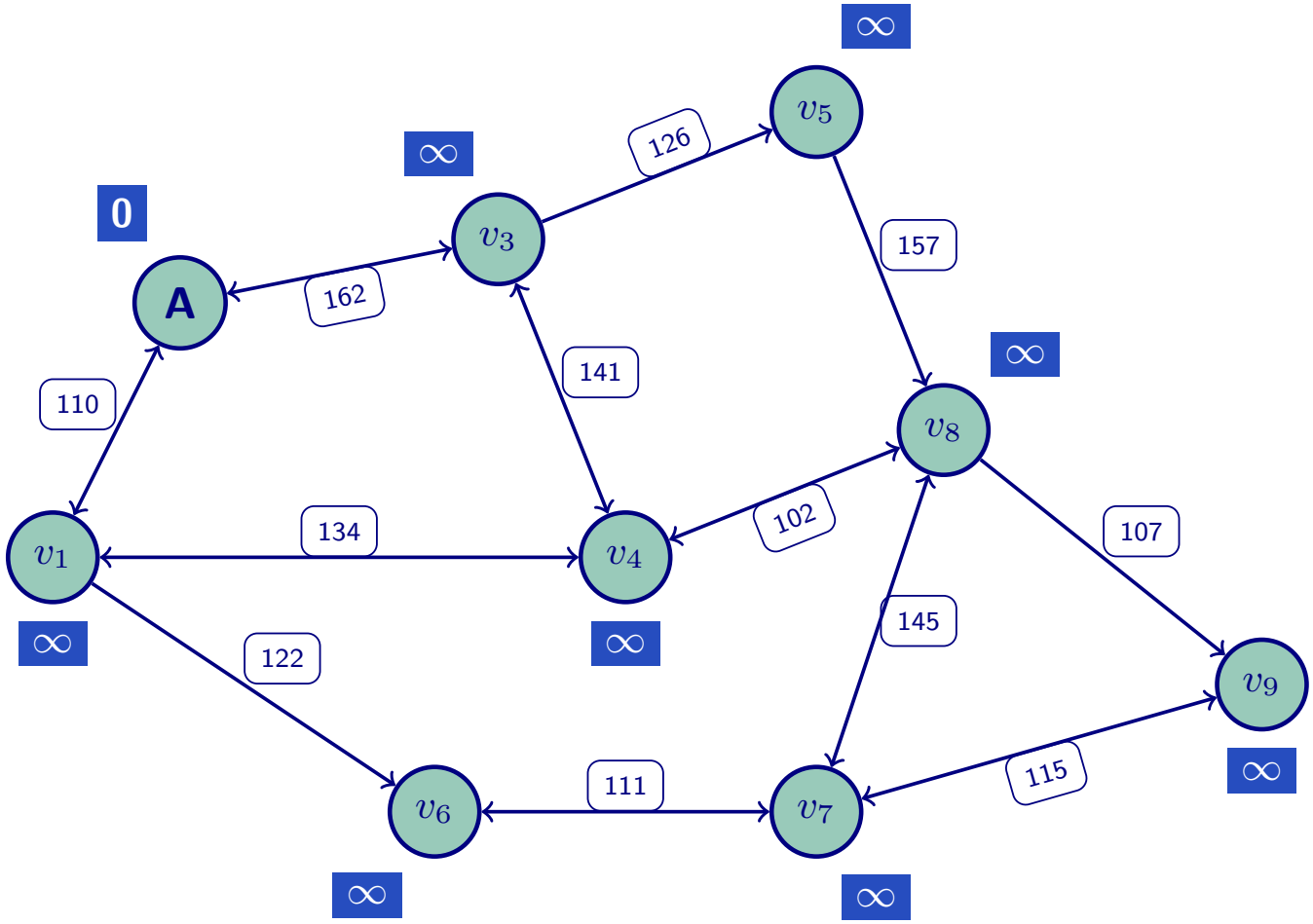

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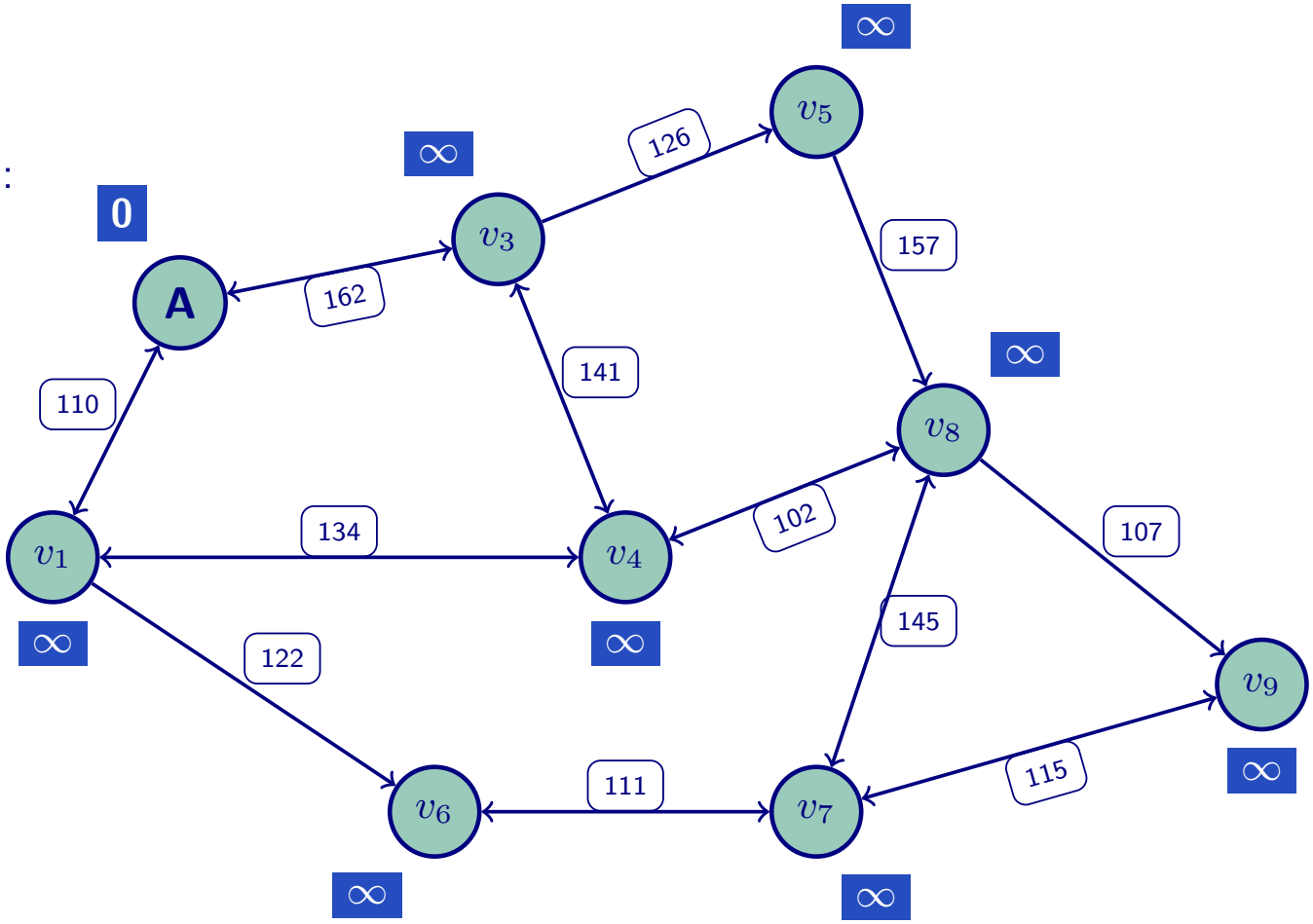

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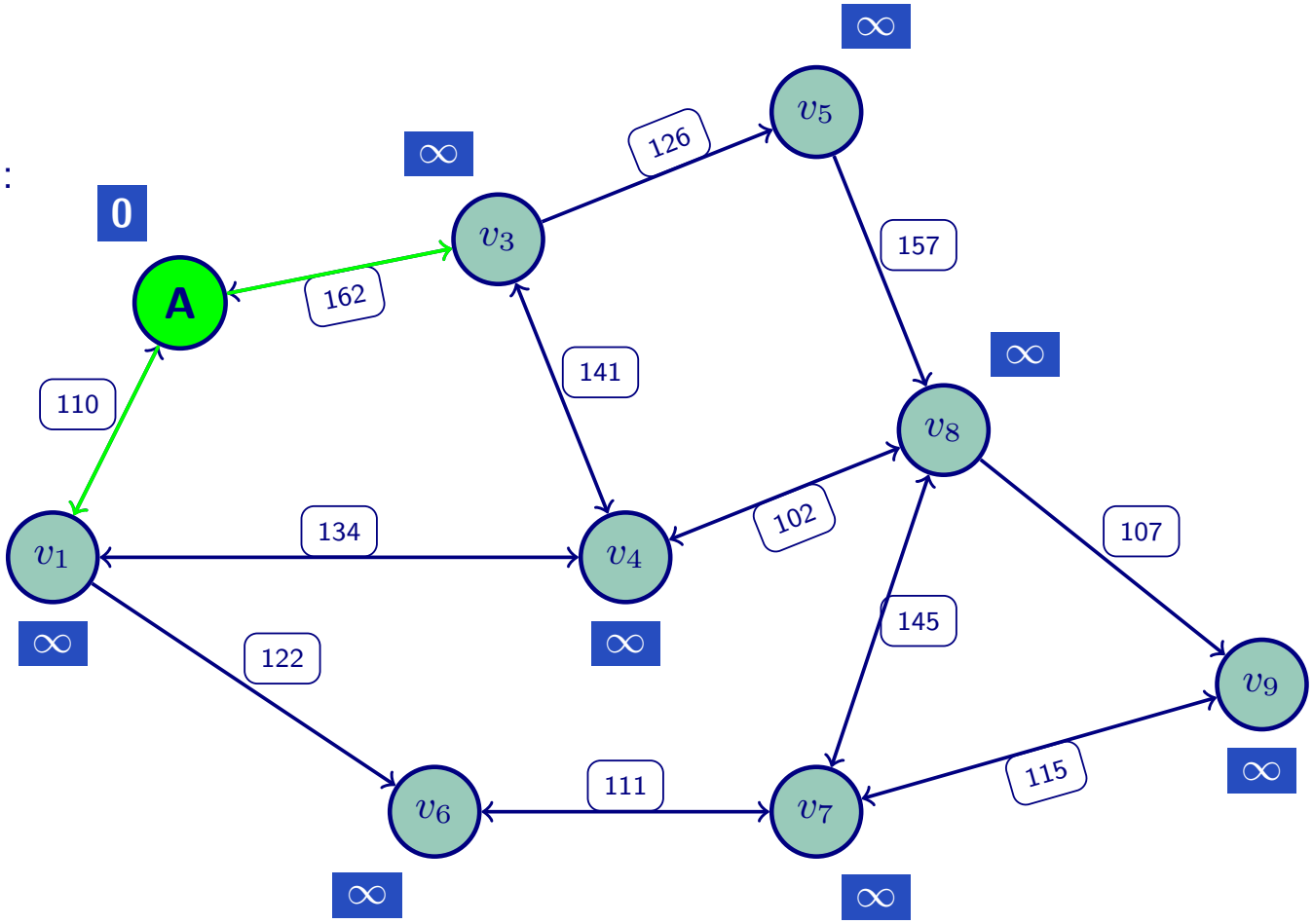

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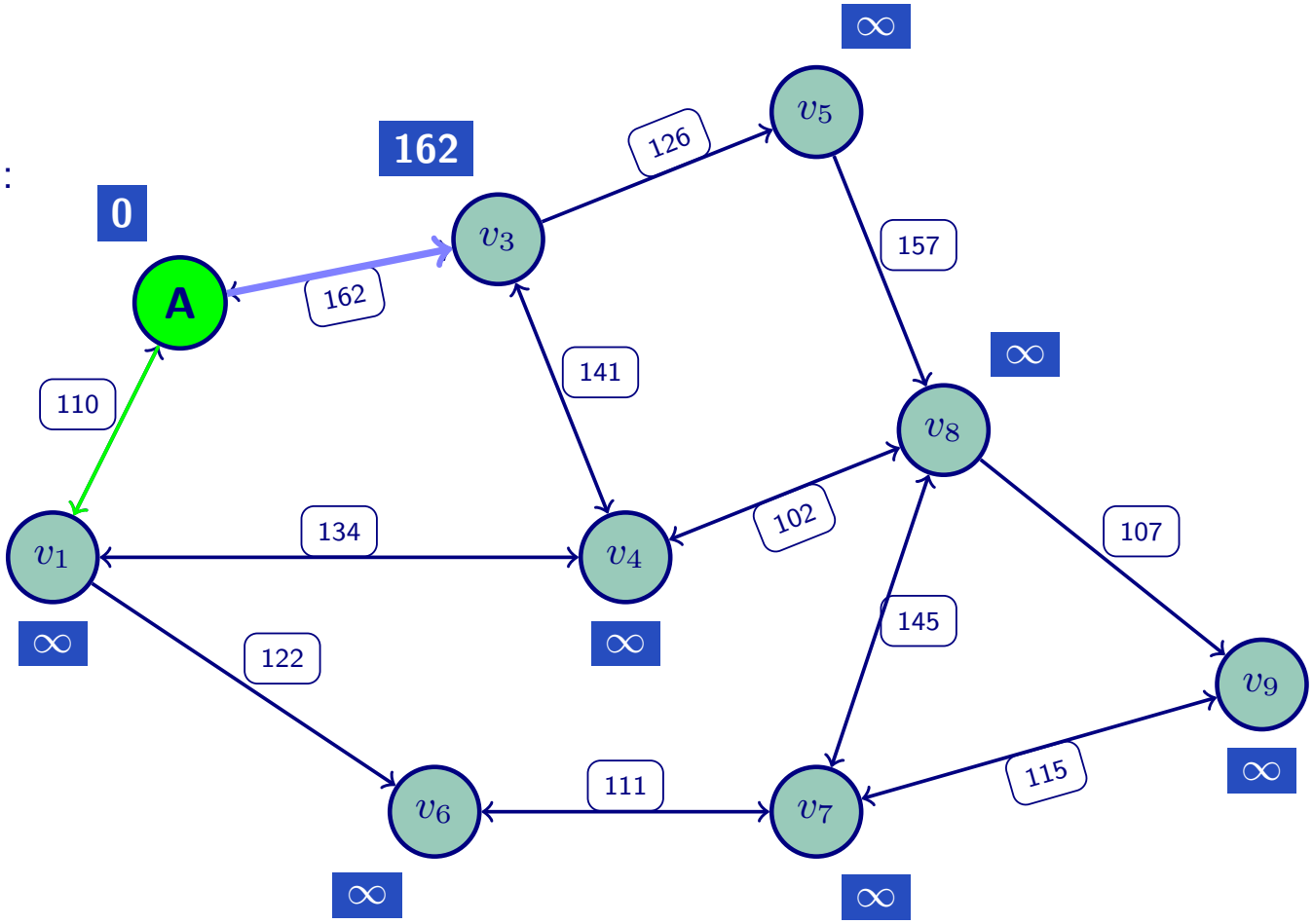

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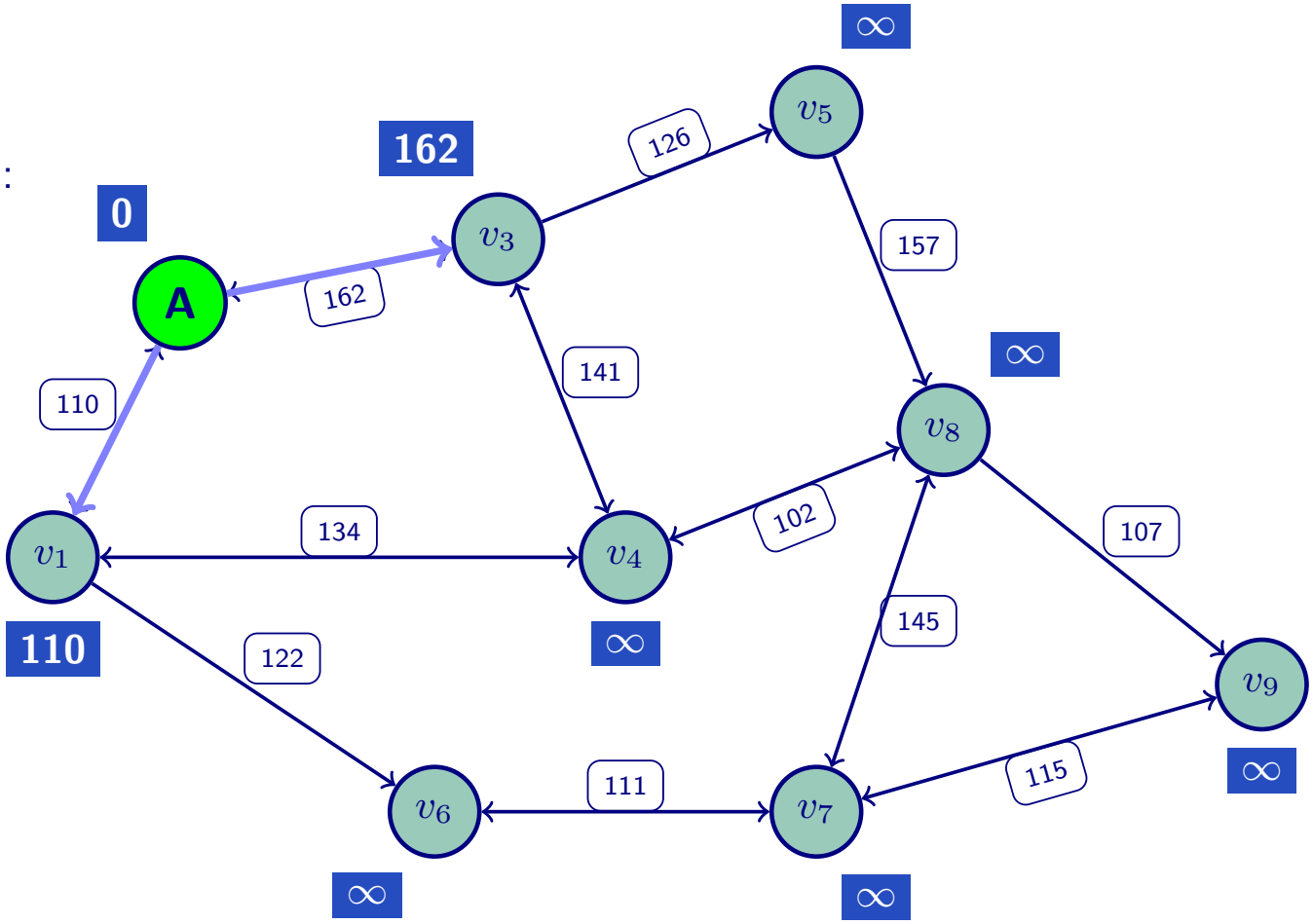

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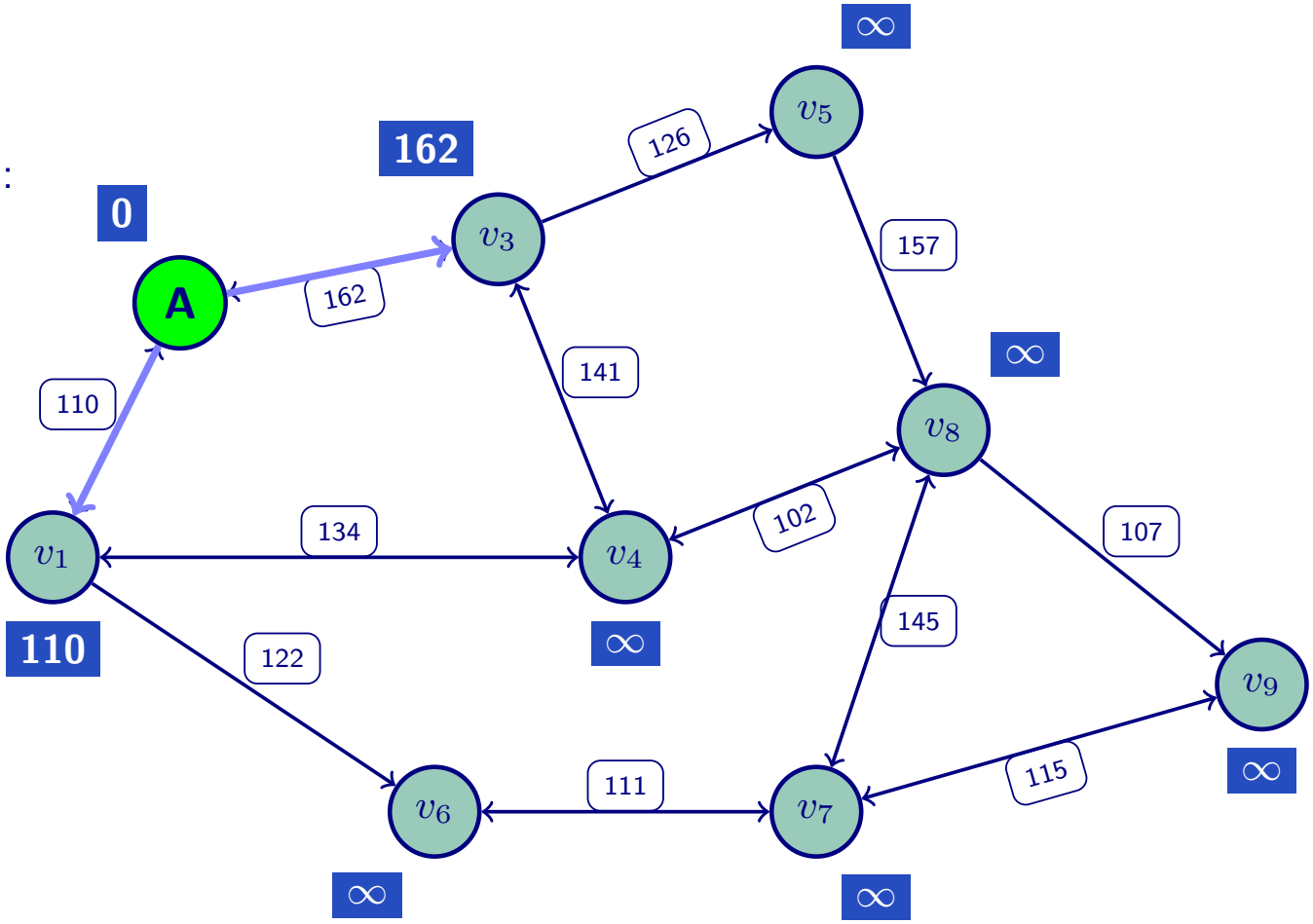

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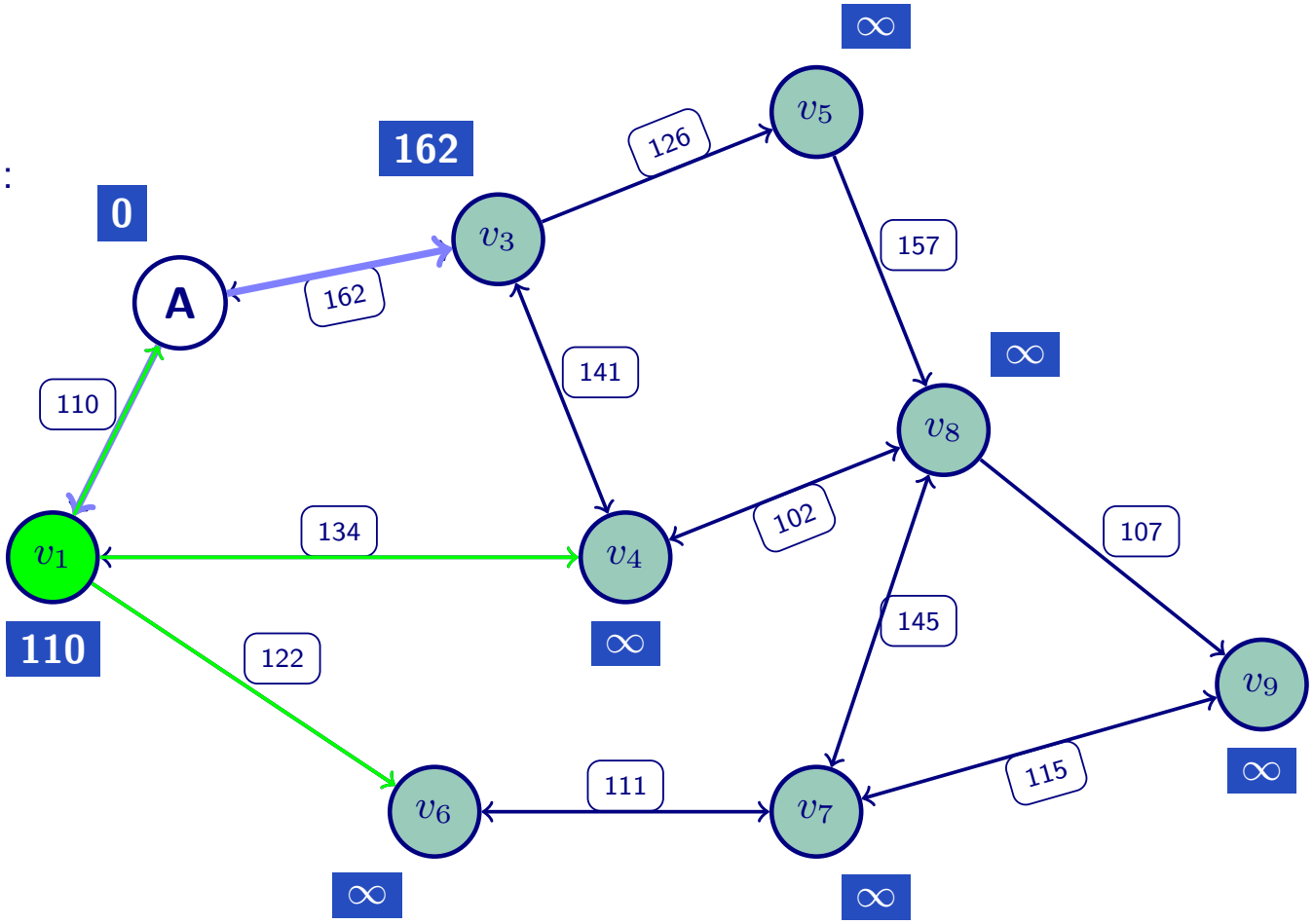

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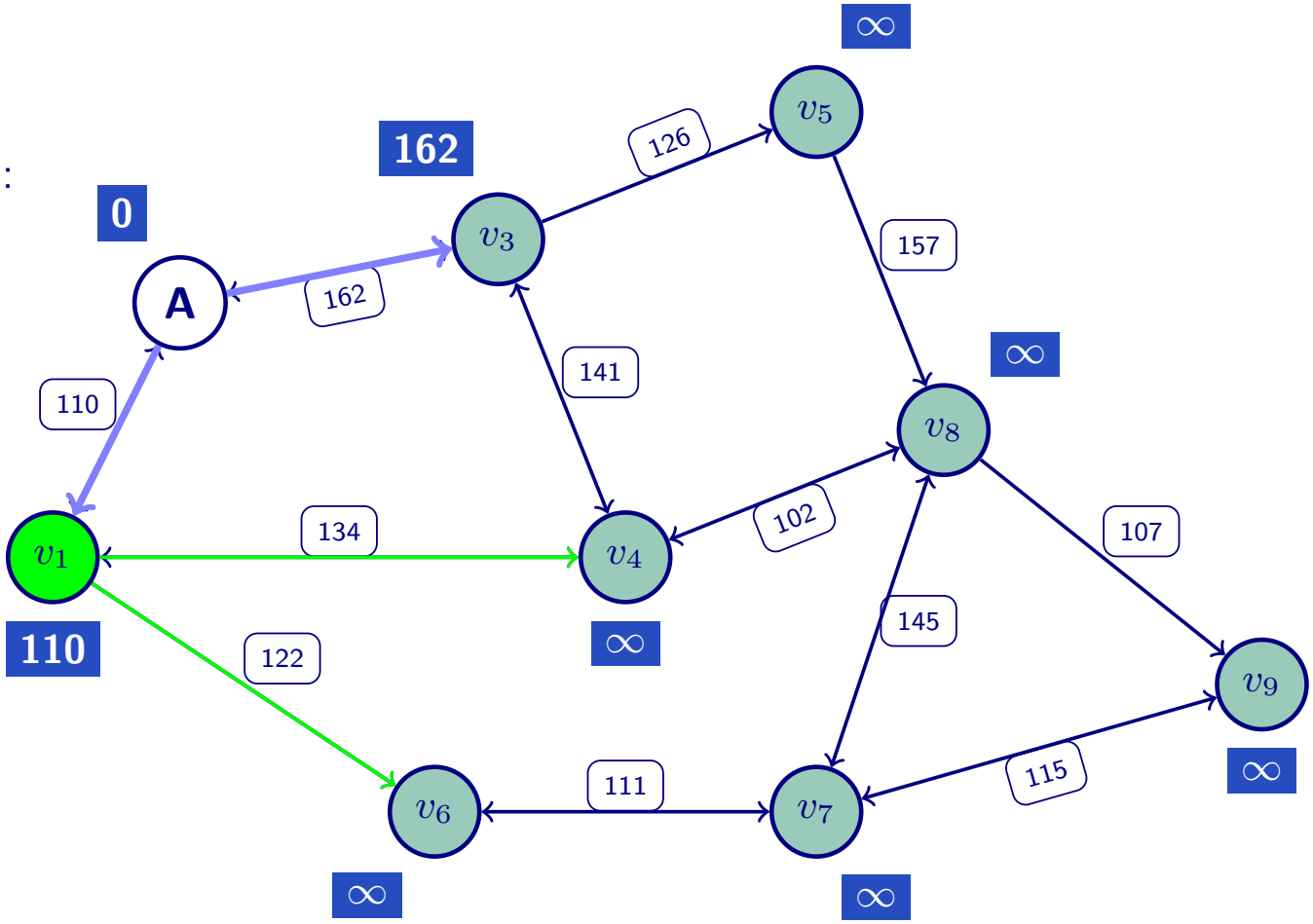

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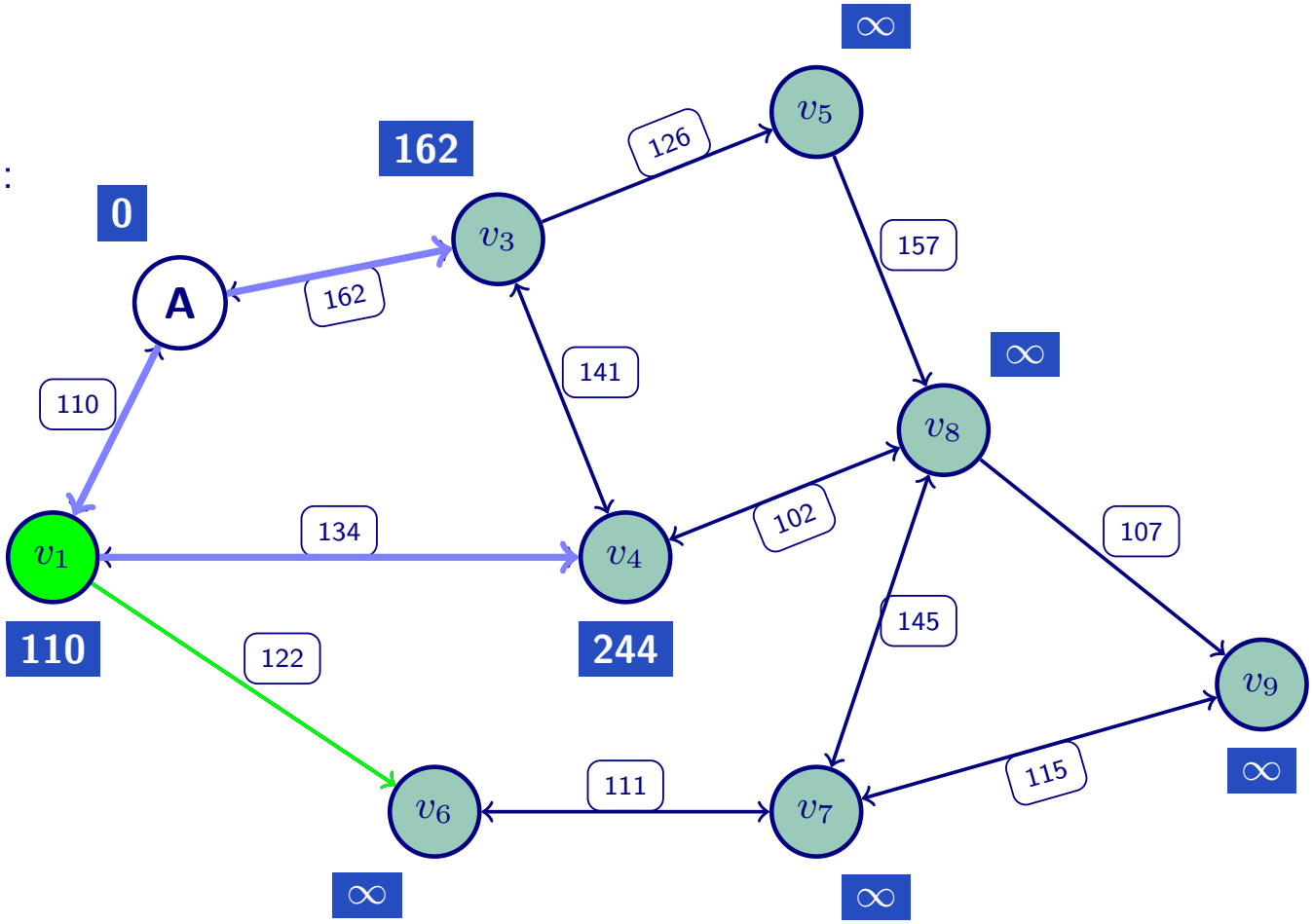

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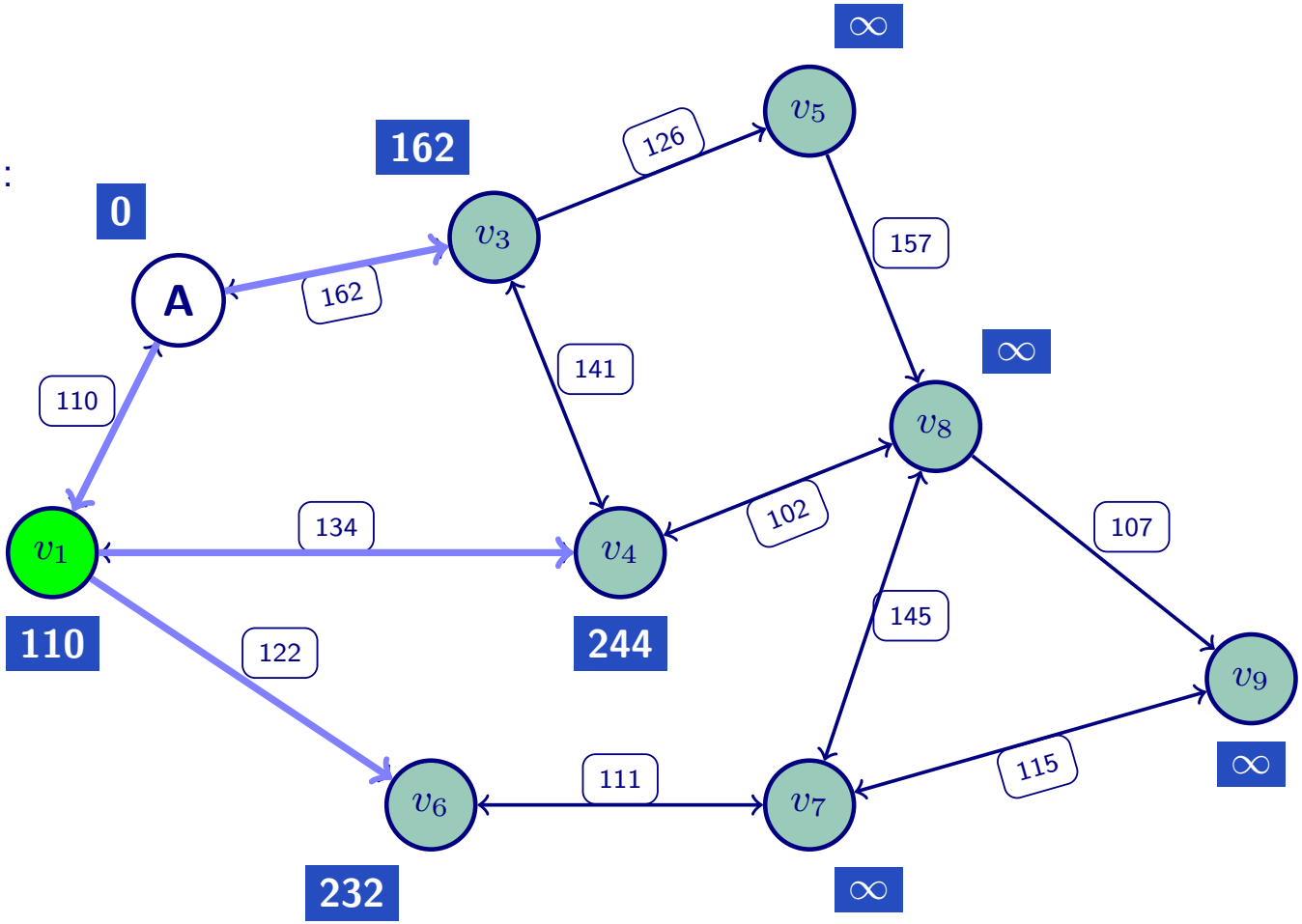

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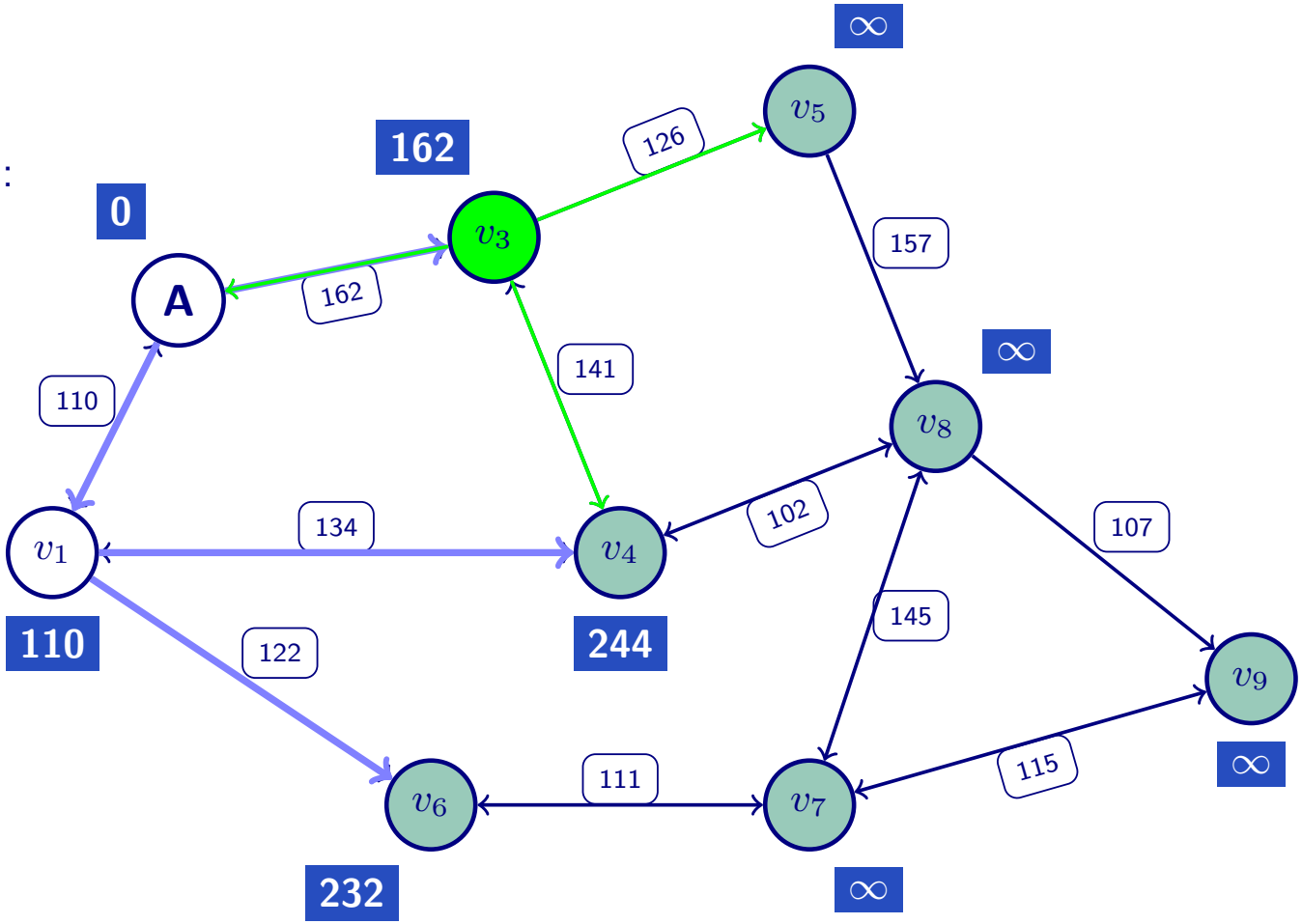

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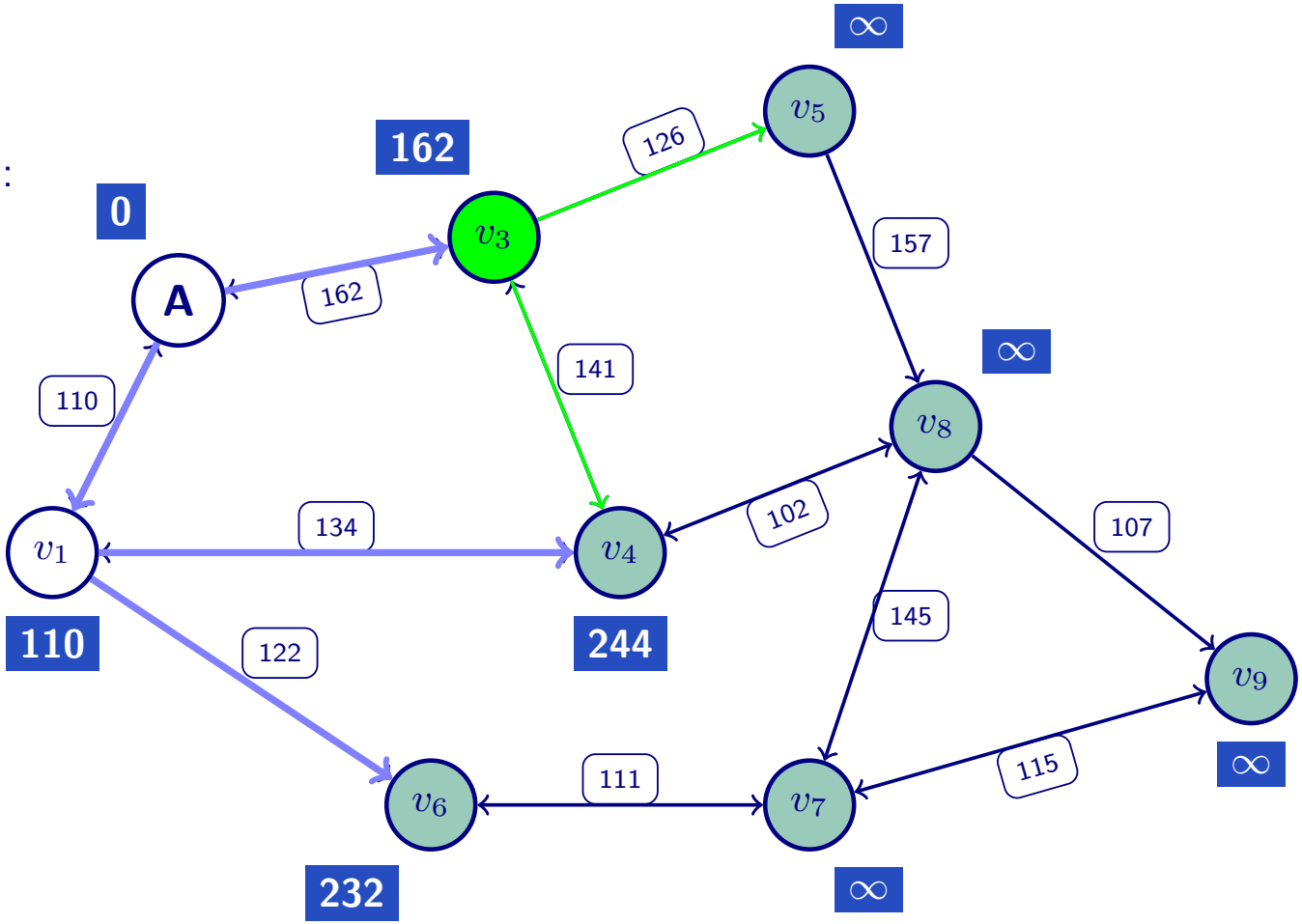

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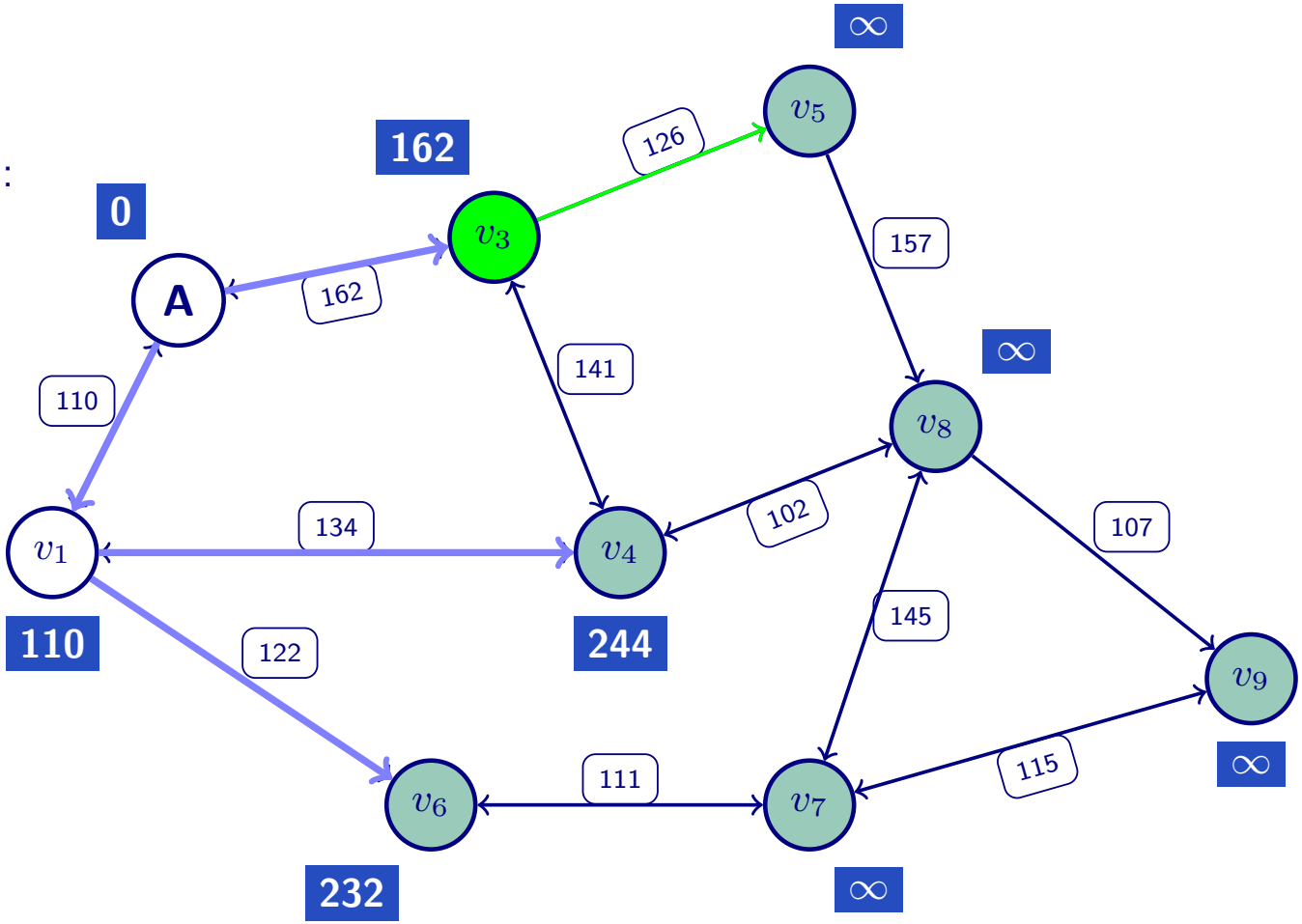

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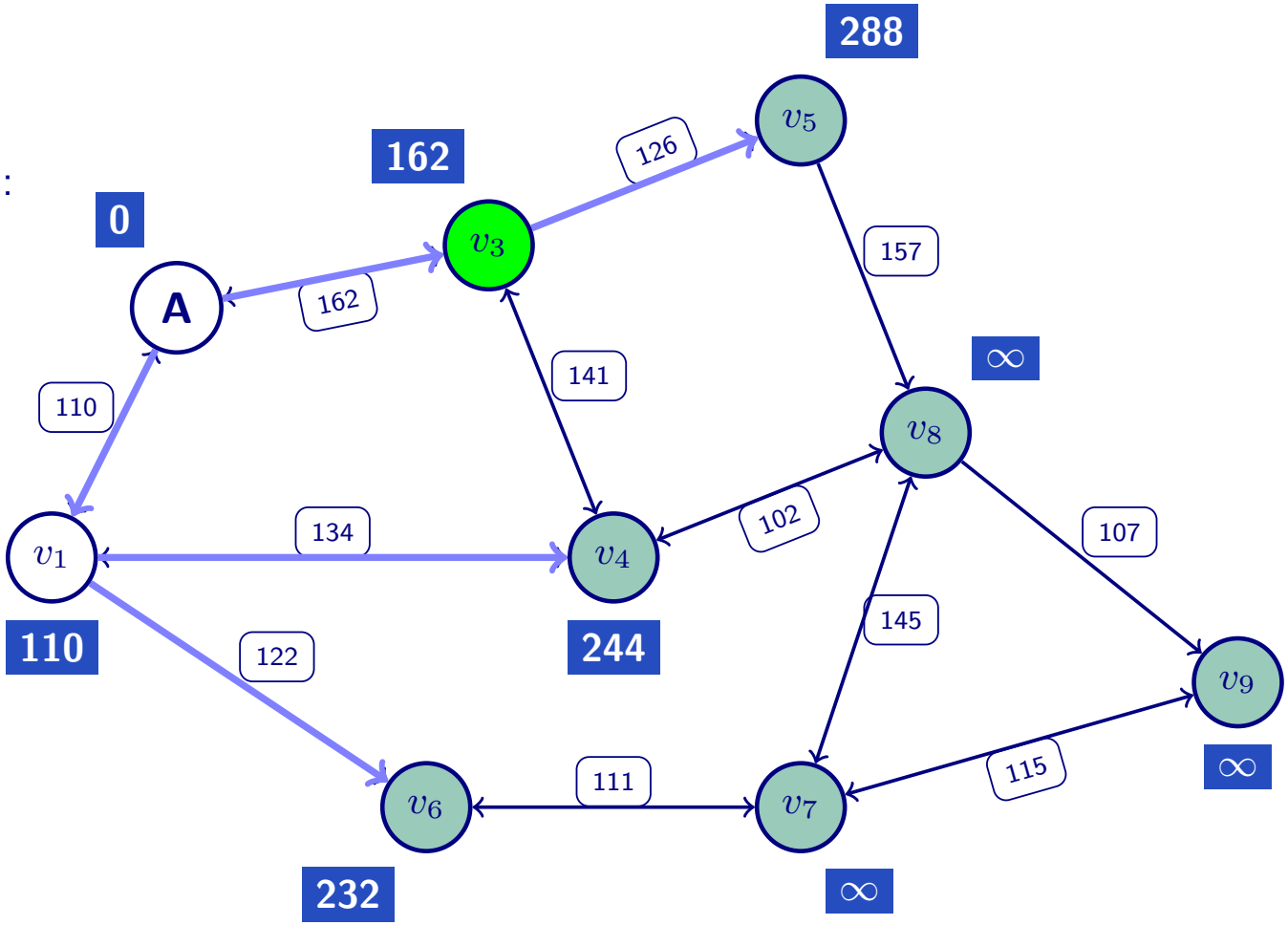

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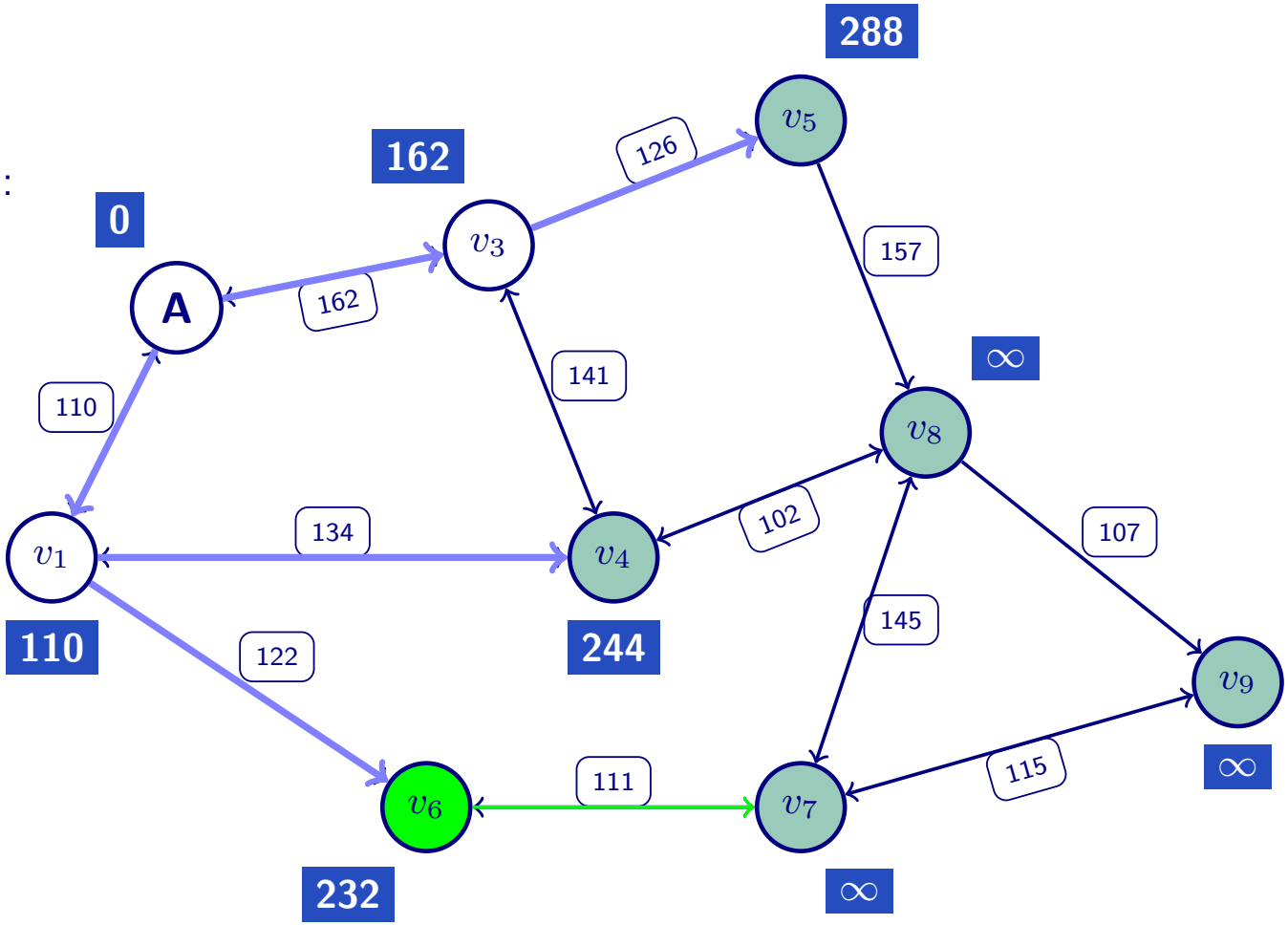

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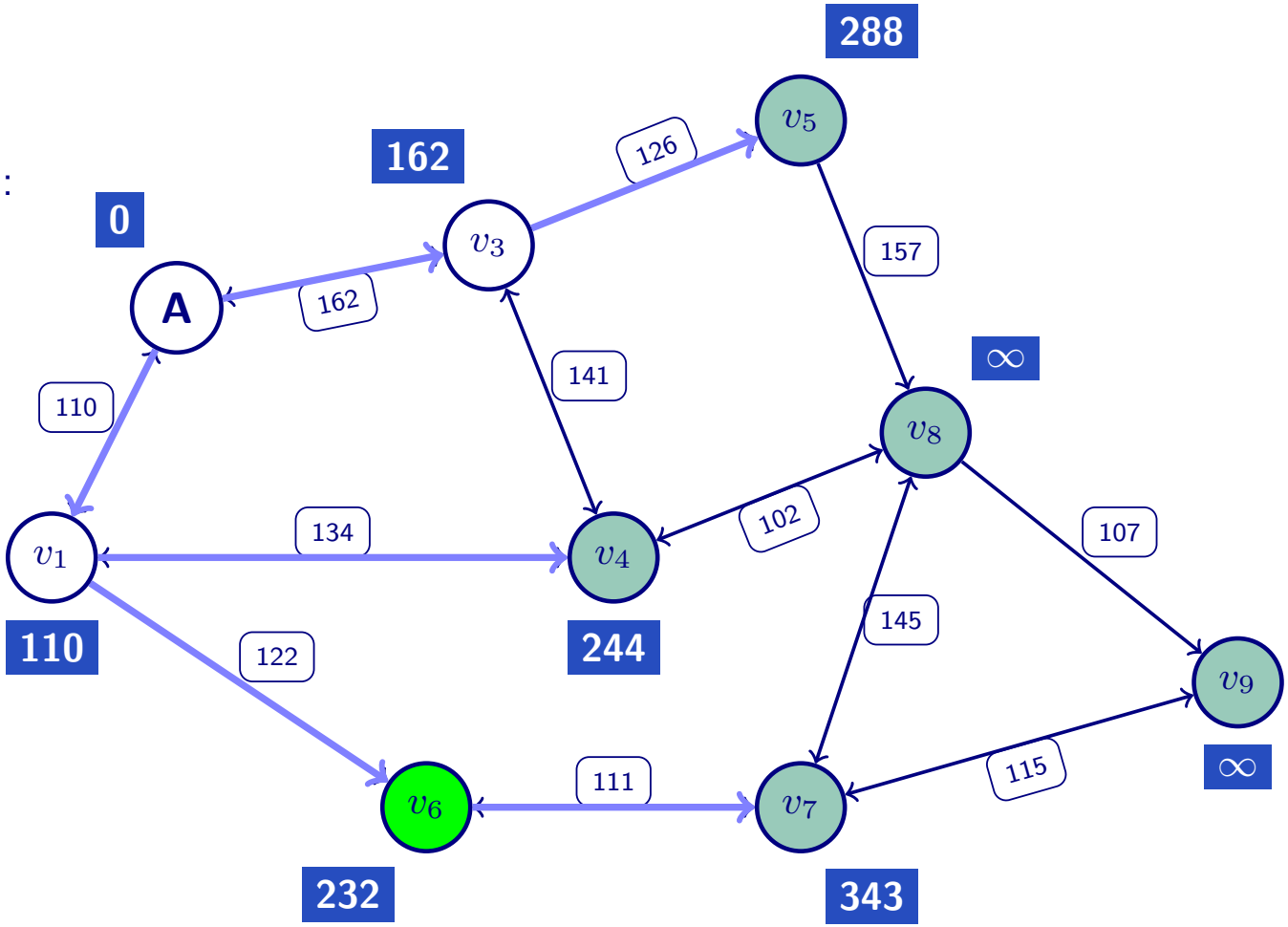

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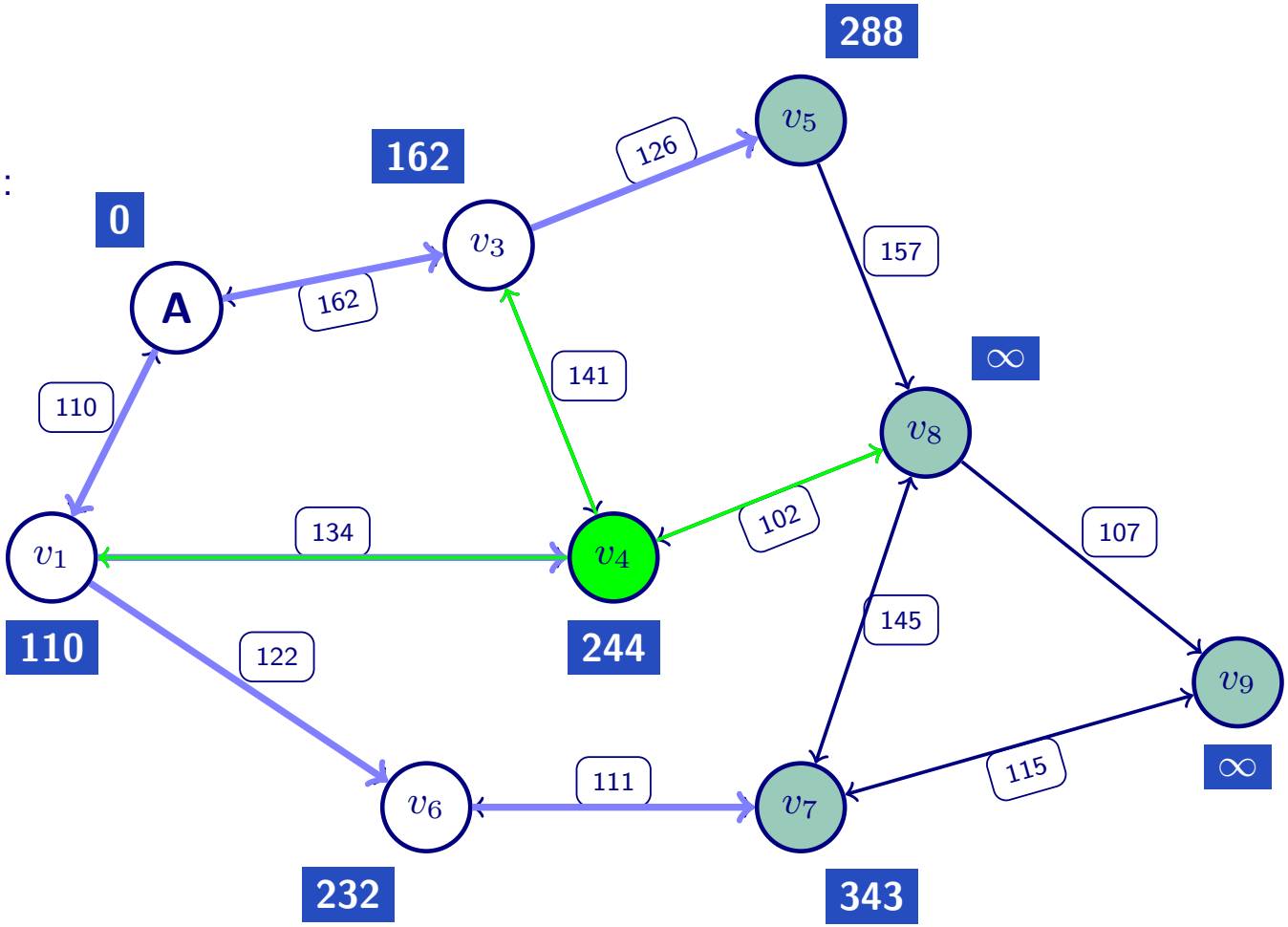
R remaining nodes

$d(v)$ min distance to A

predecessor tree



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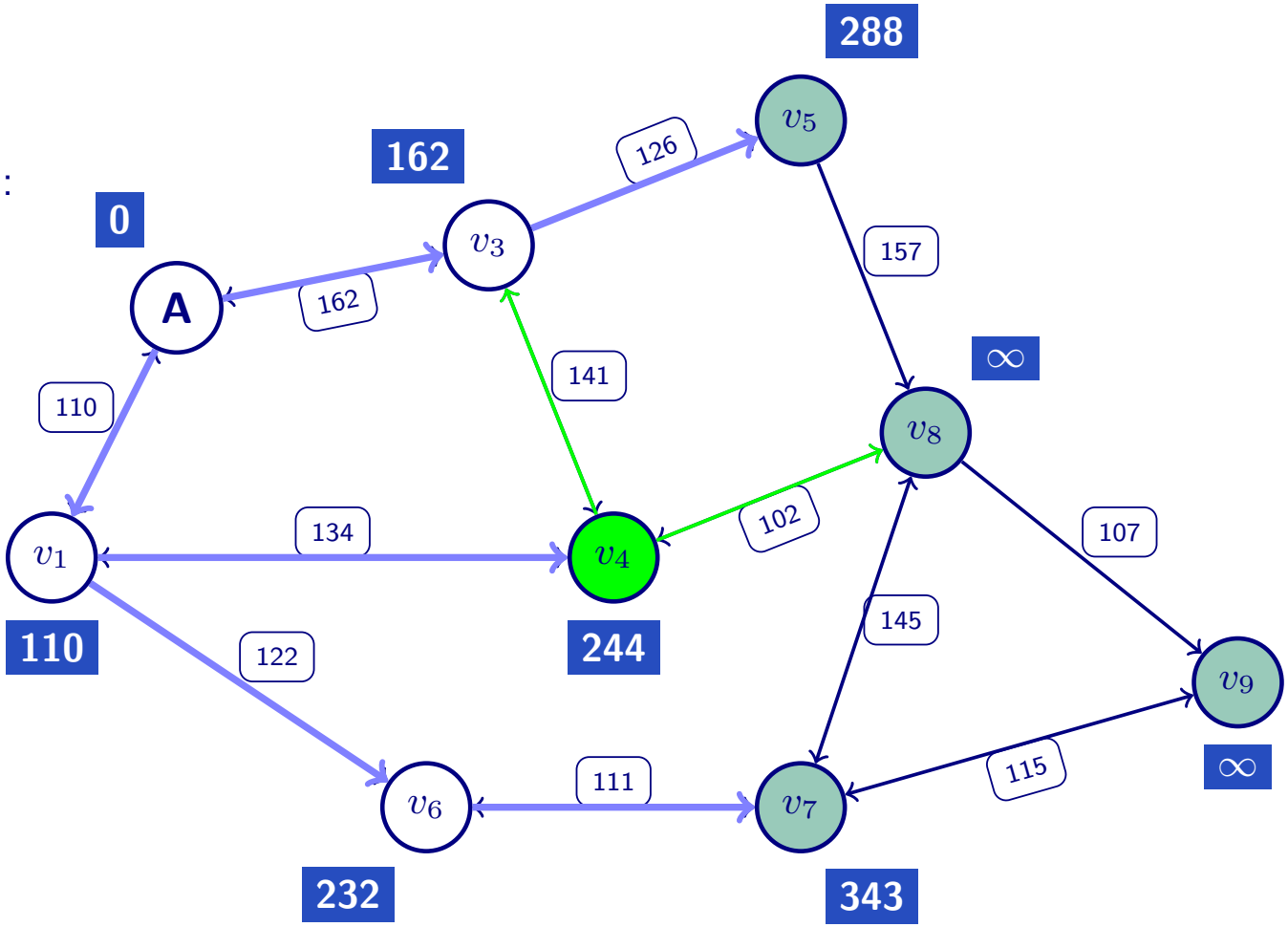

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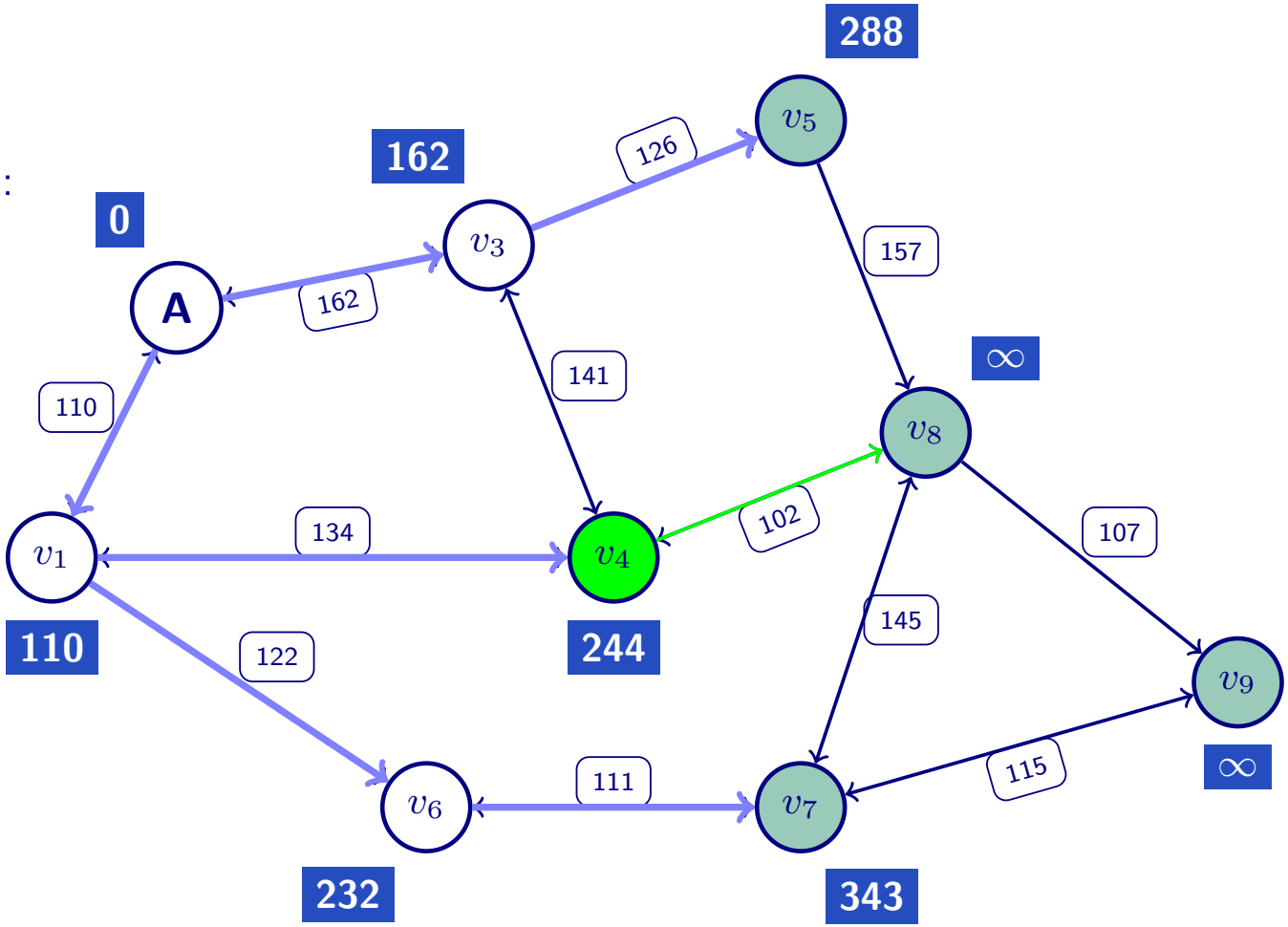

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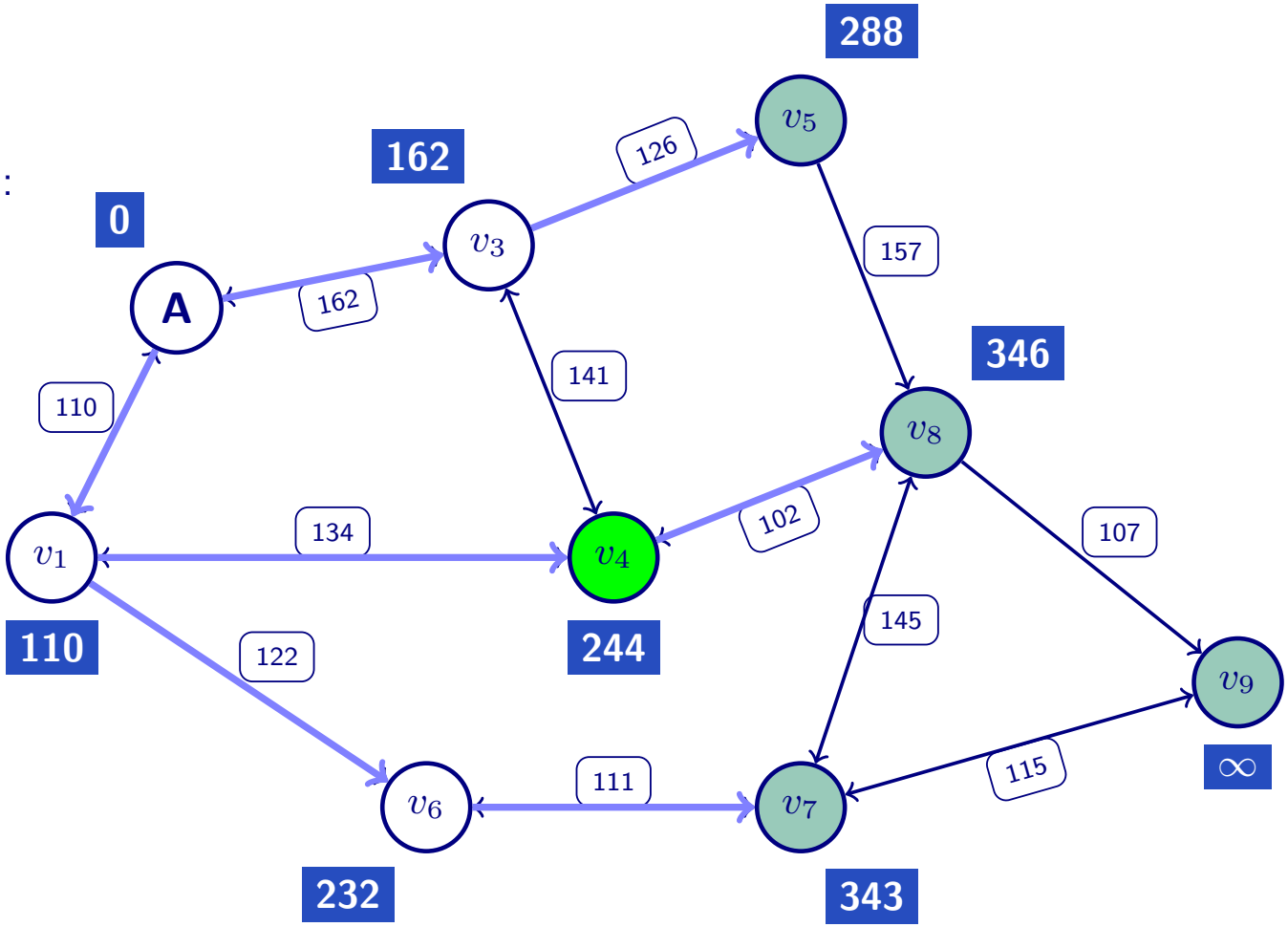

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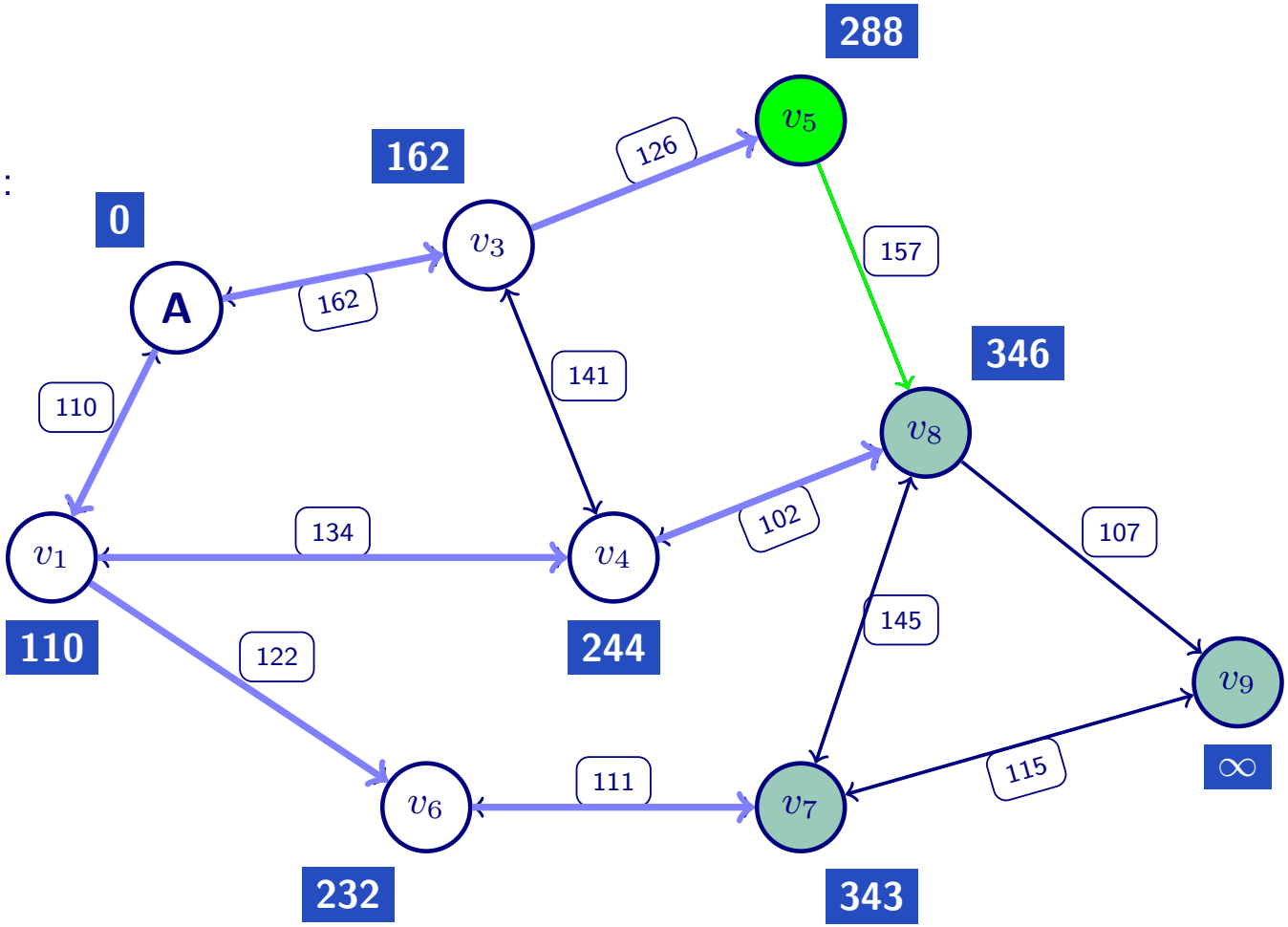

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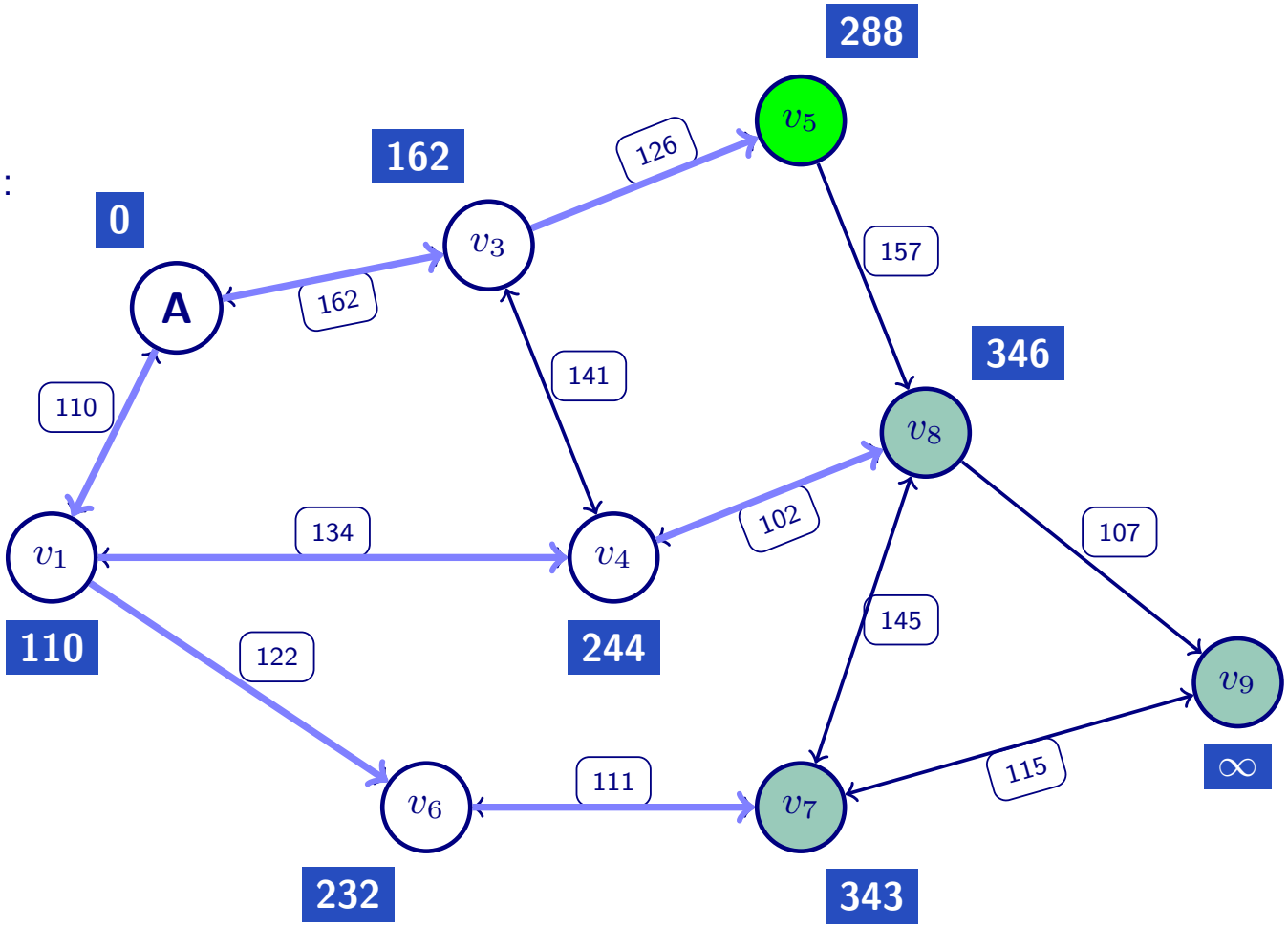

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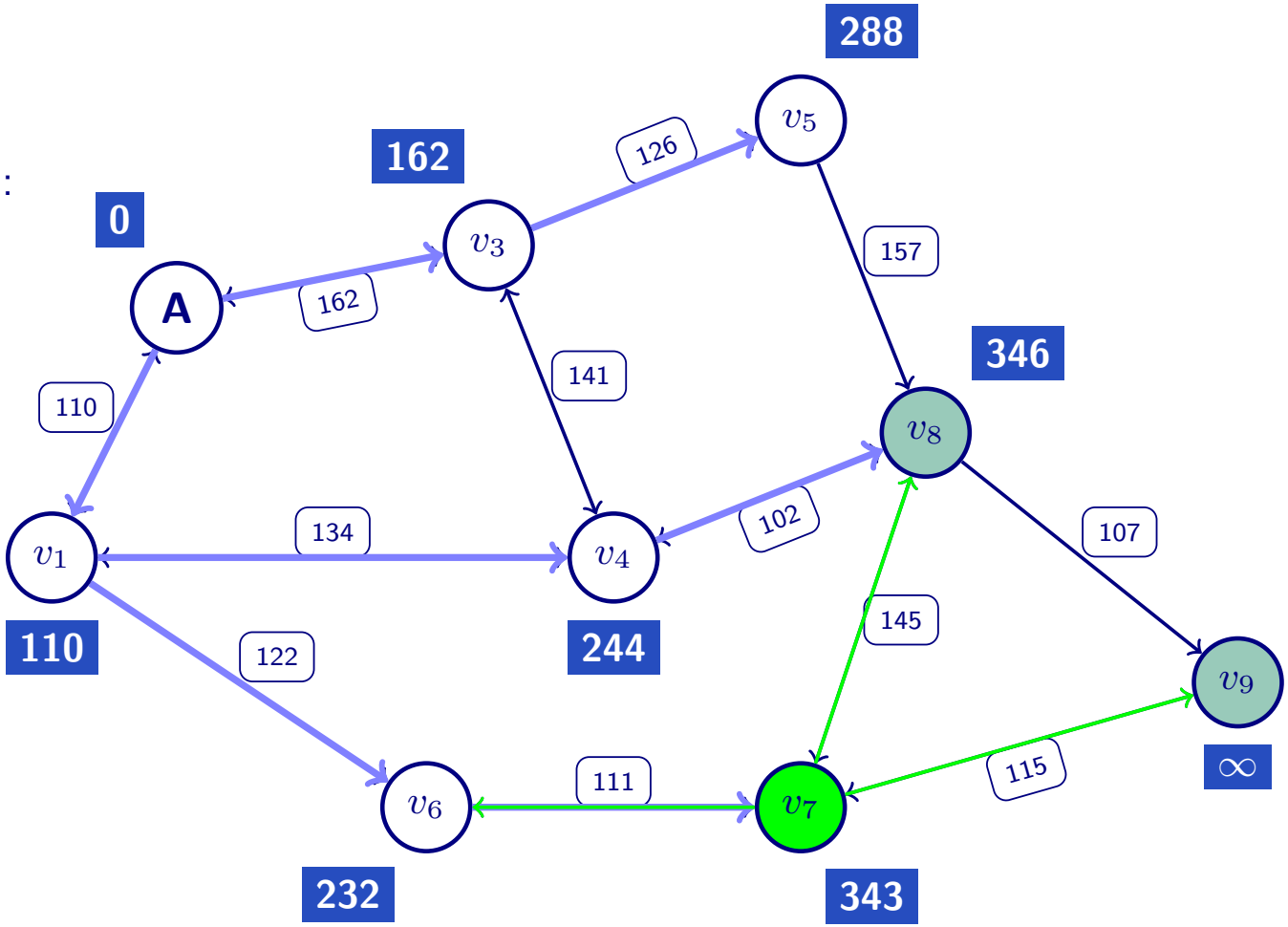

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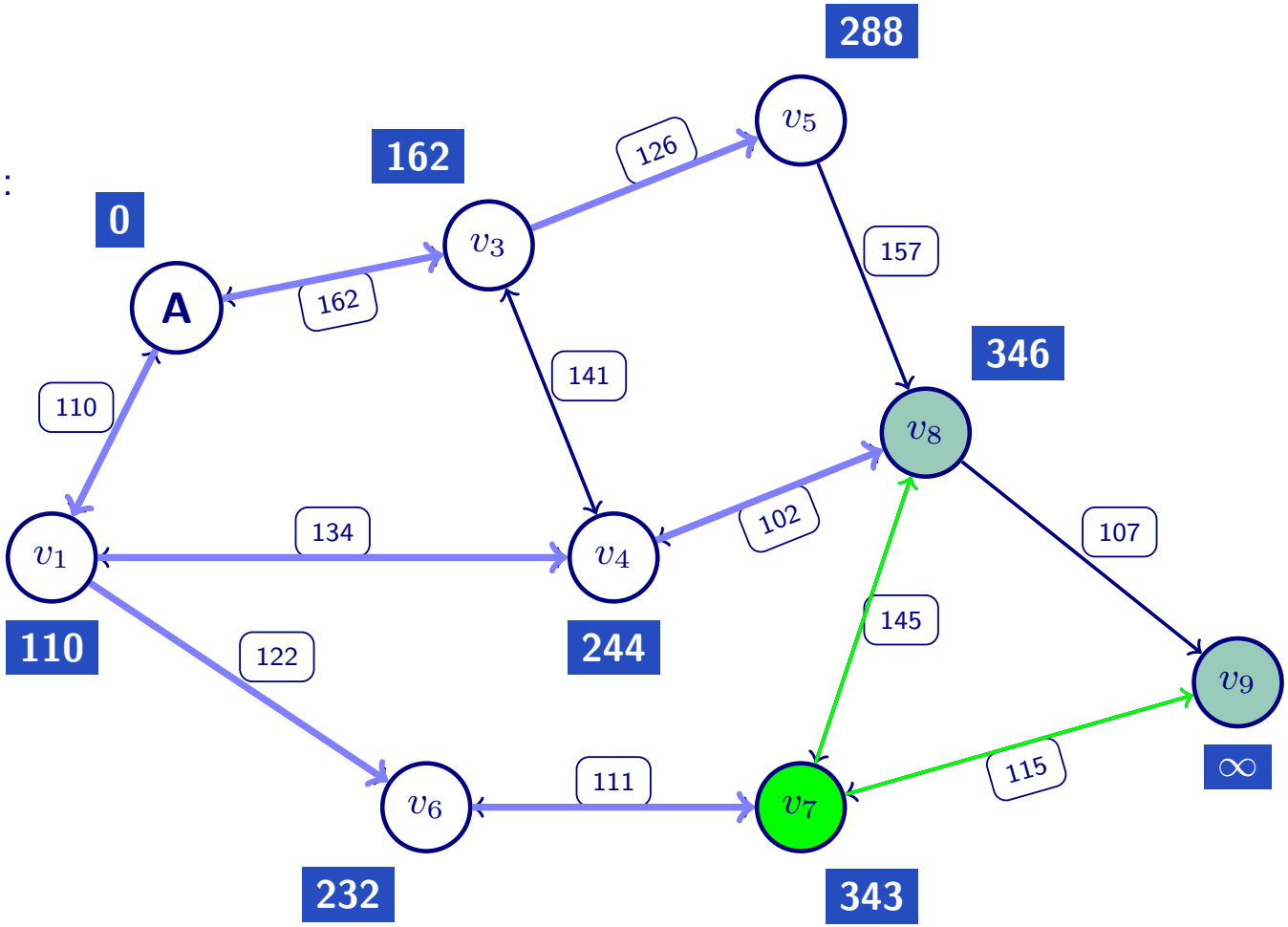

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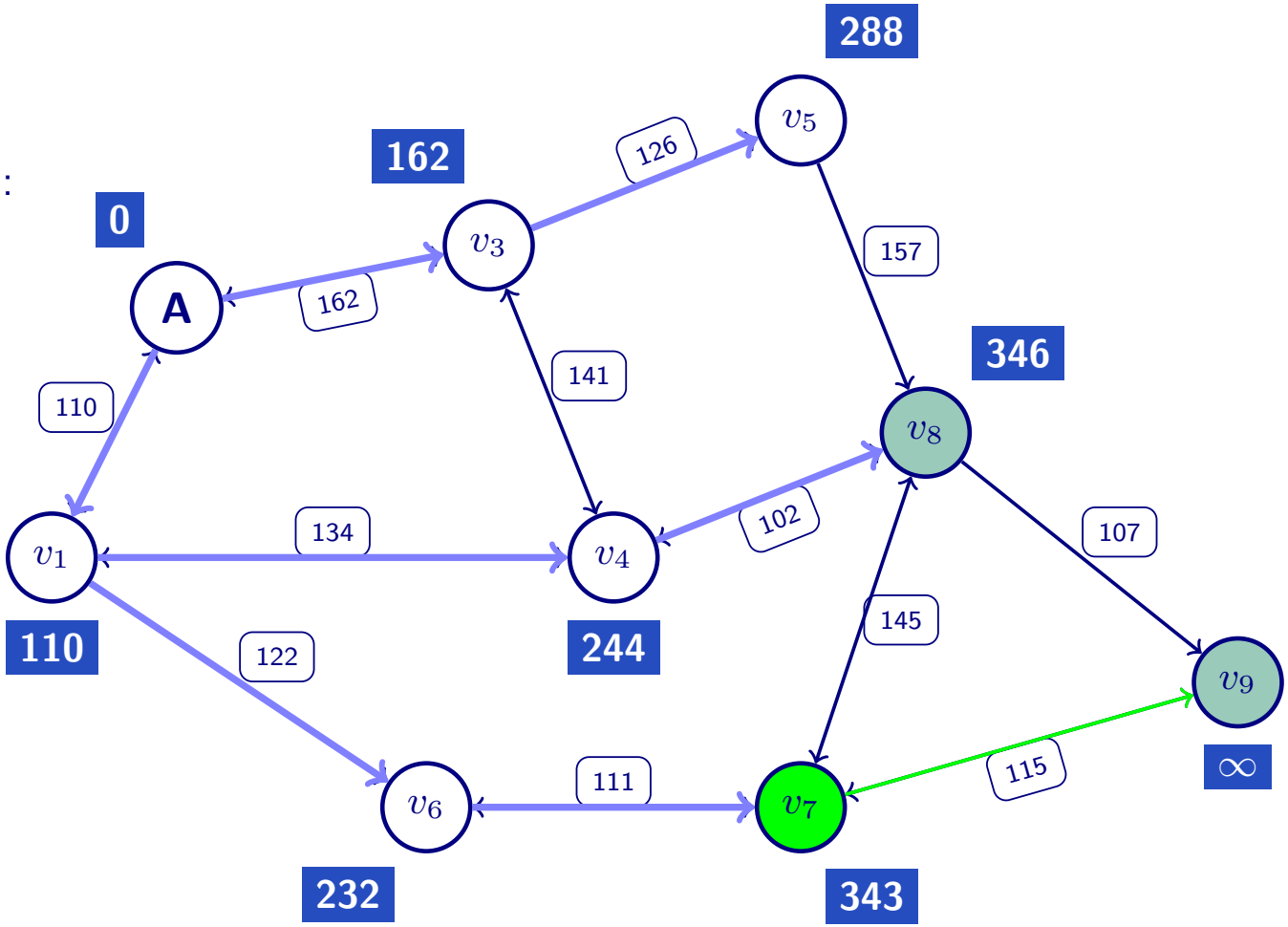

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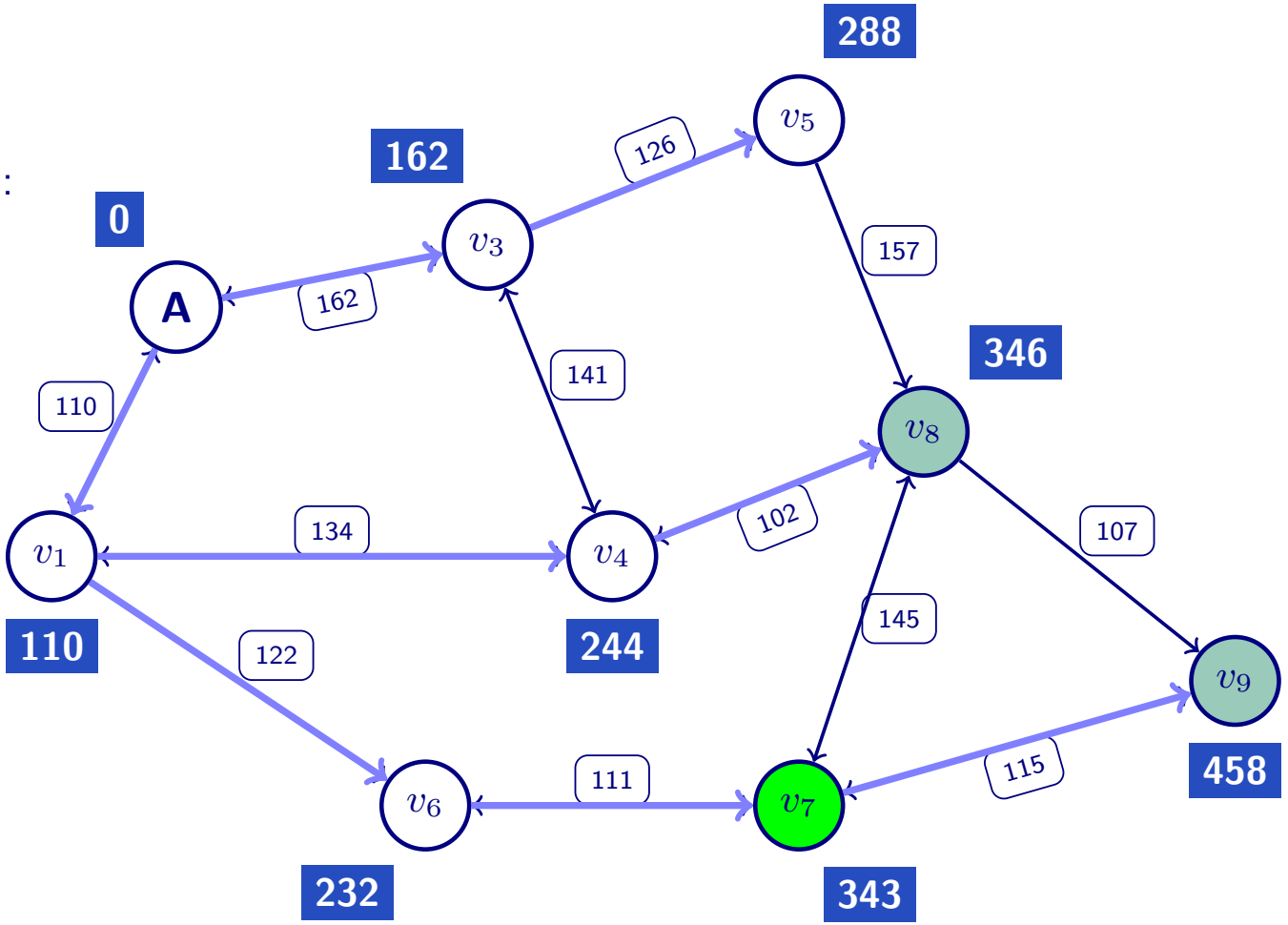

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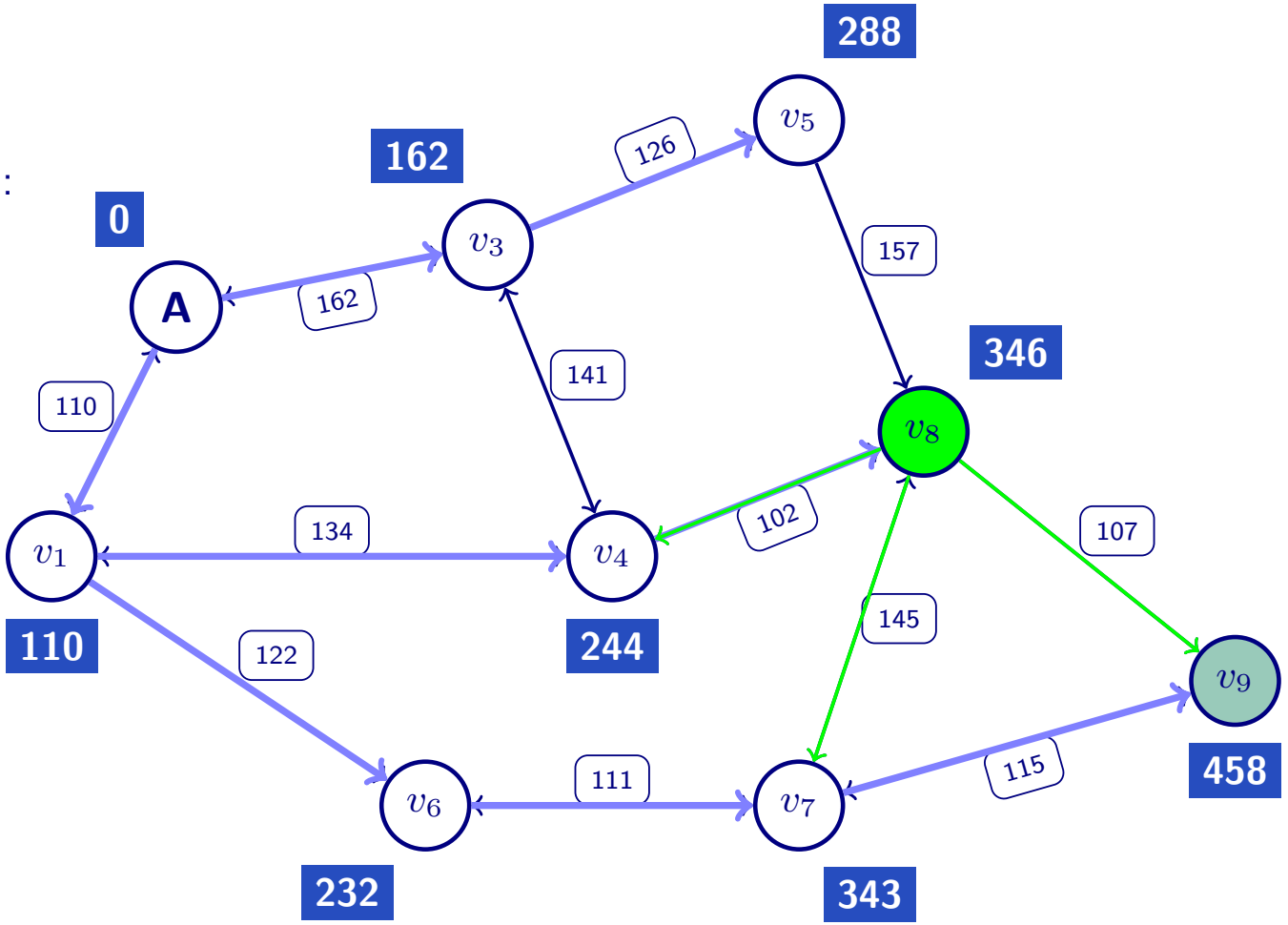

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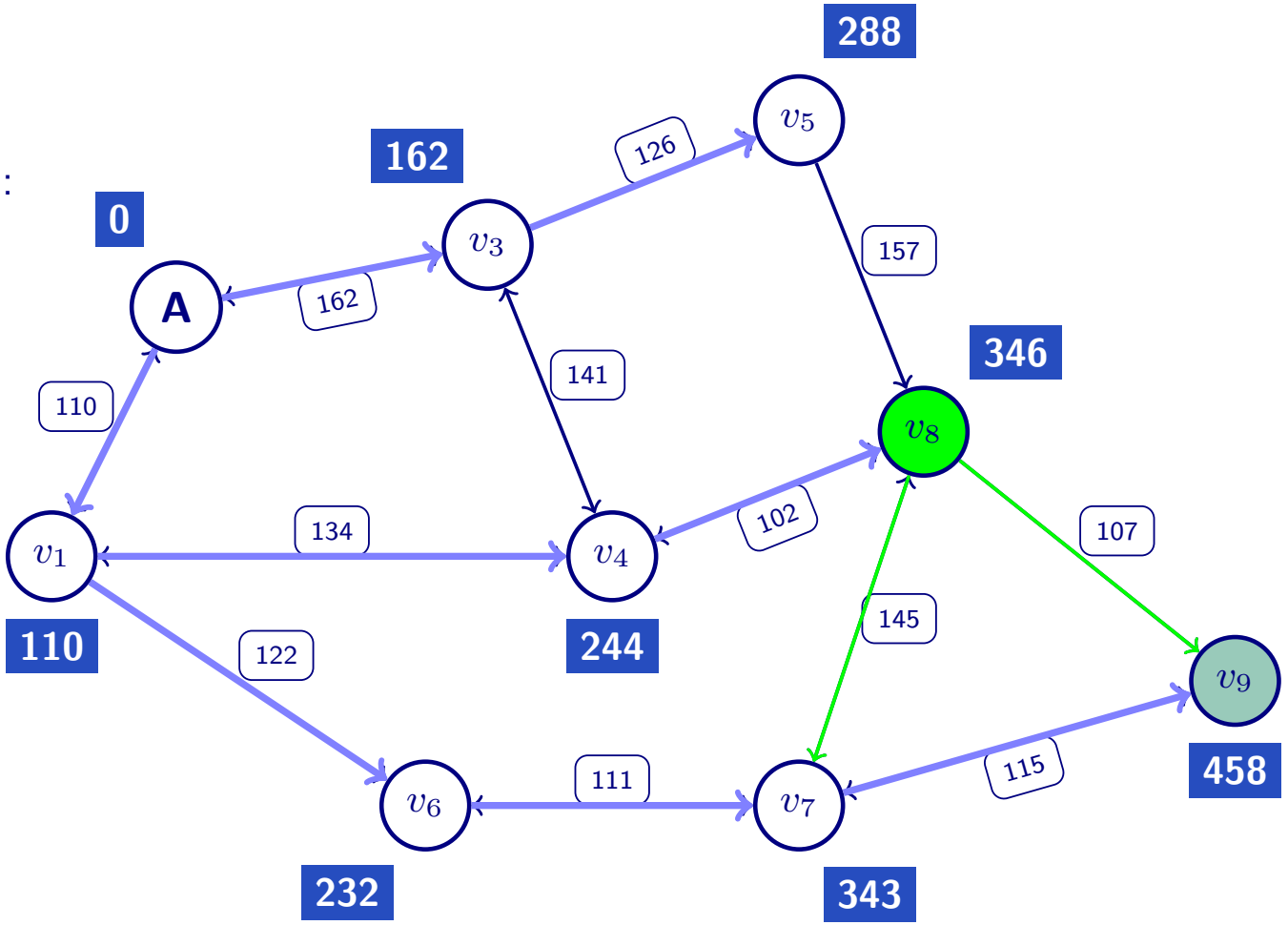

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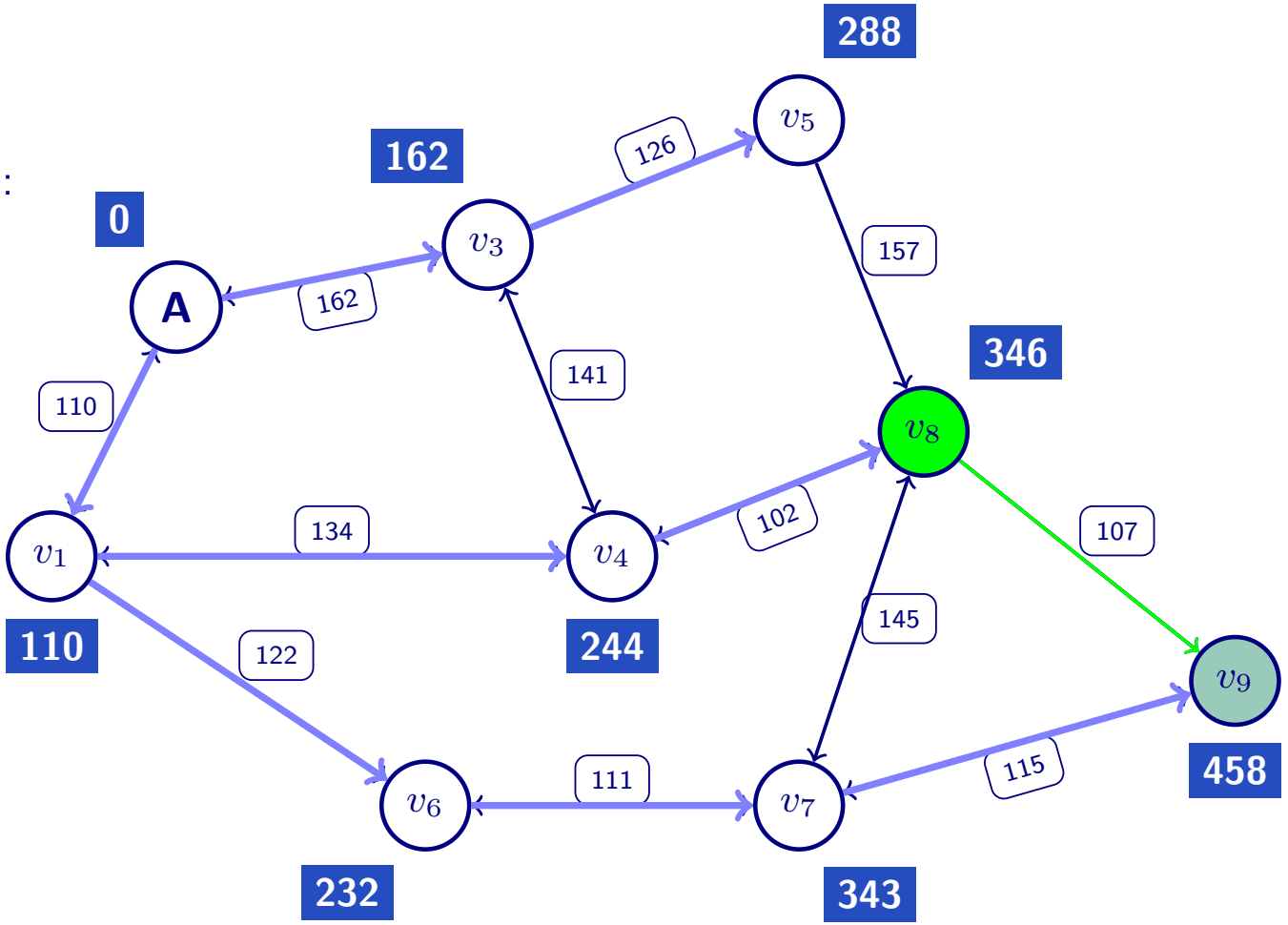

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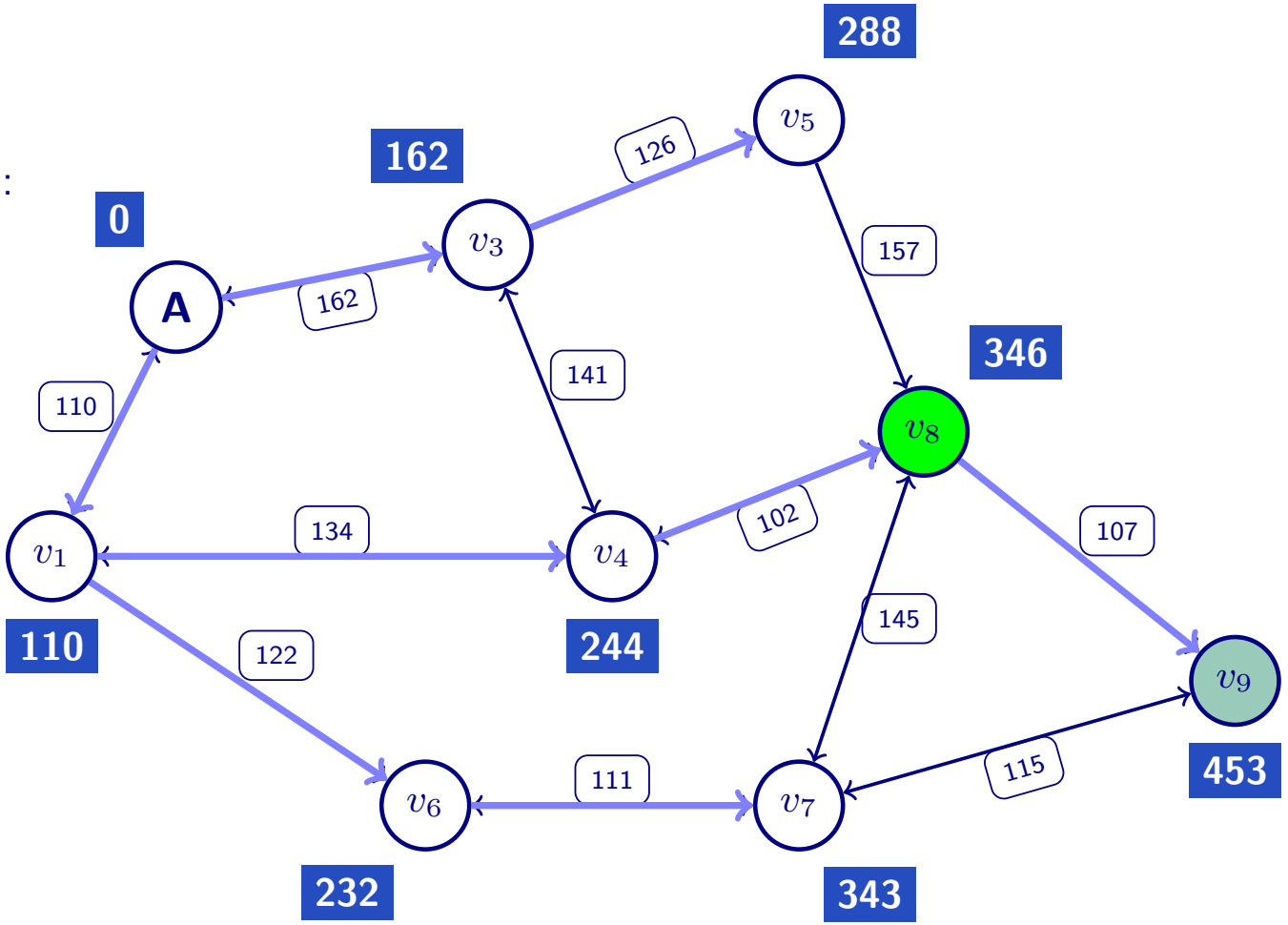

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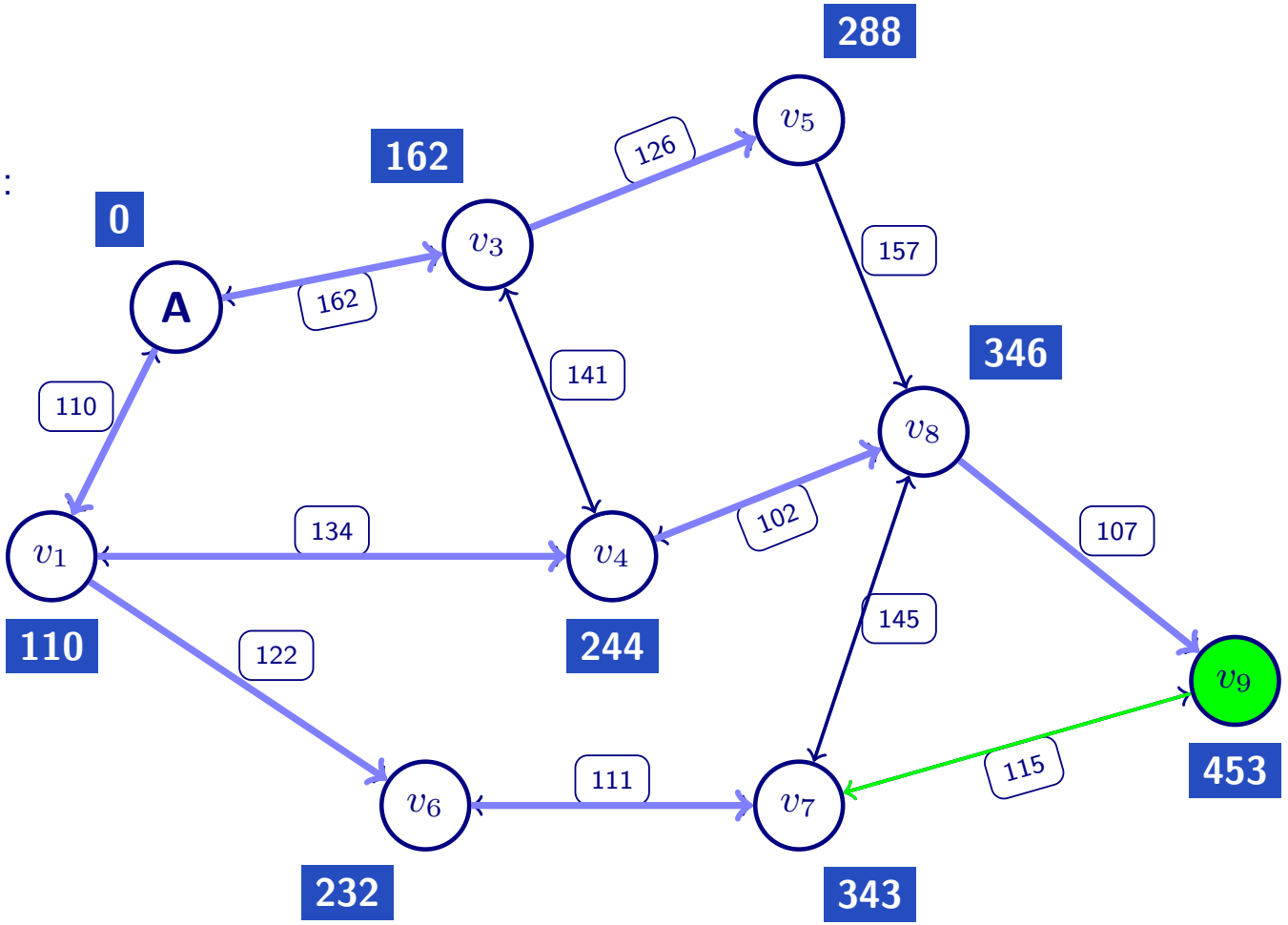

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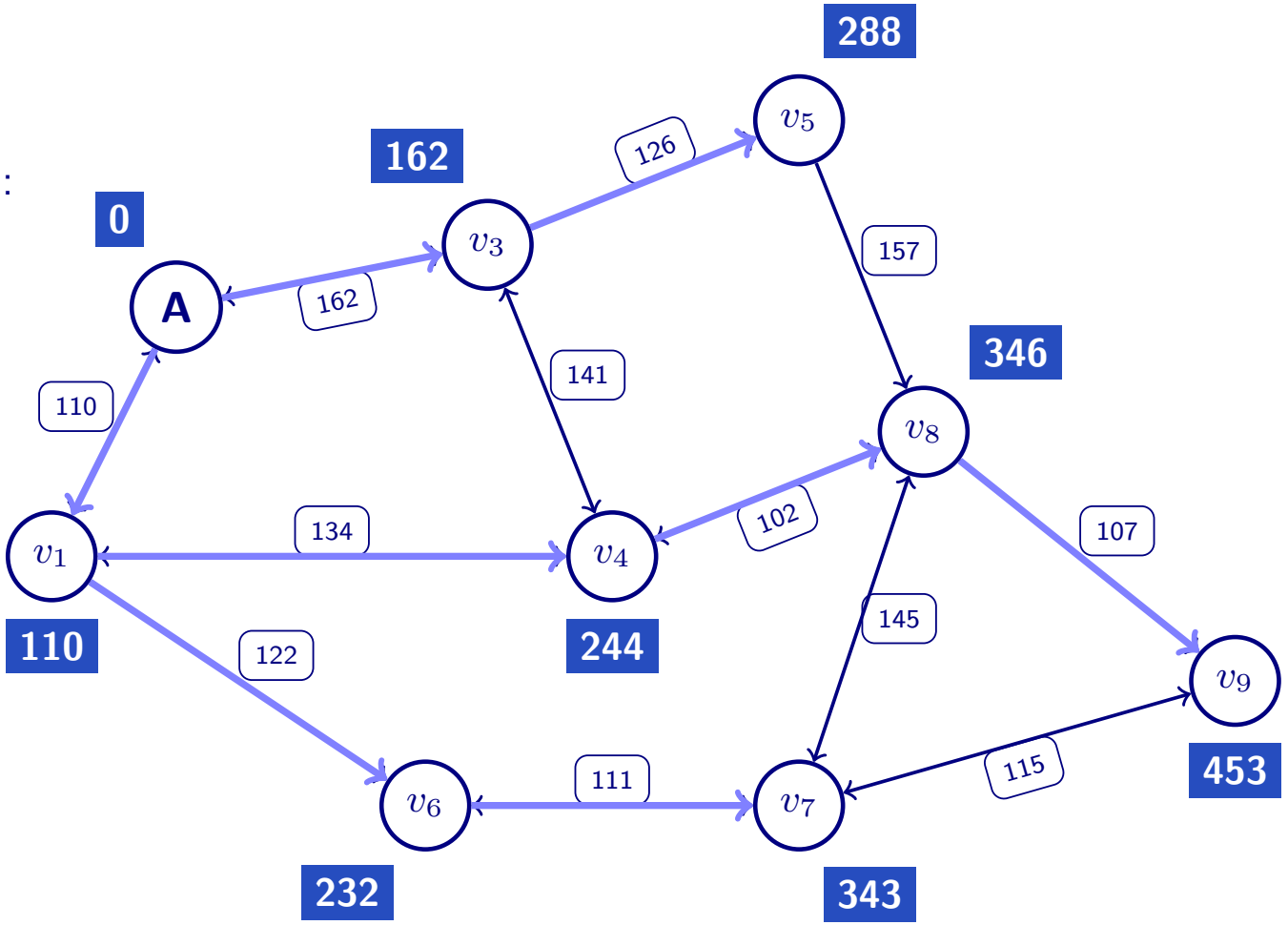

▷ Exchange 100 Mio Can-$ into £

	USD	GBP	CAD	EUR	AUD
USD	1	1.8786	0.9082	1.2953	0.7527
GBP	0.5323	1	0.4834	0.6895	0.4007
CAD	1.1010	2.0685	1	1.4262	0.8288
EUR	0.7720	1.4503	0.7011	1	0.5811
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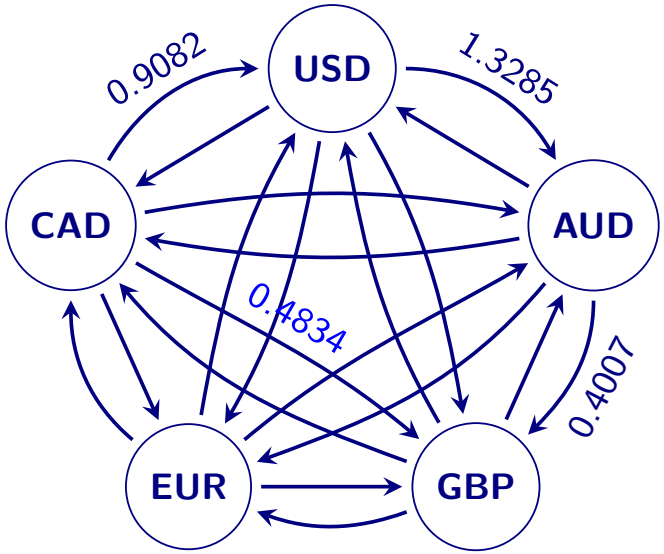

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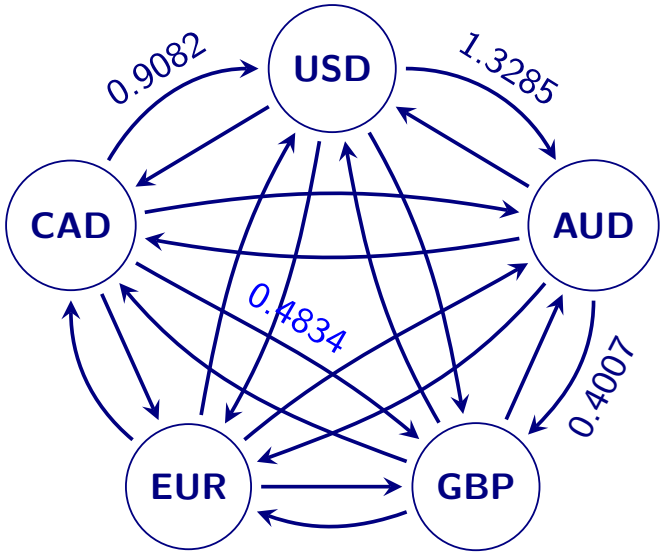

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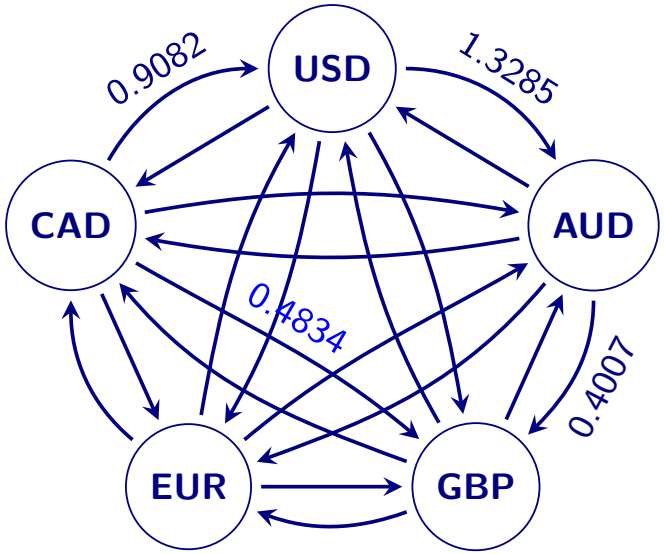

▷ Exchanging **CAD** → **USD** → **AUD** → **GBP**:

➡ Rate: $0.9082 \cdot 1.3285 \cdot 0.4007 = 0.483462 > 0.4834$

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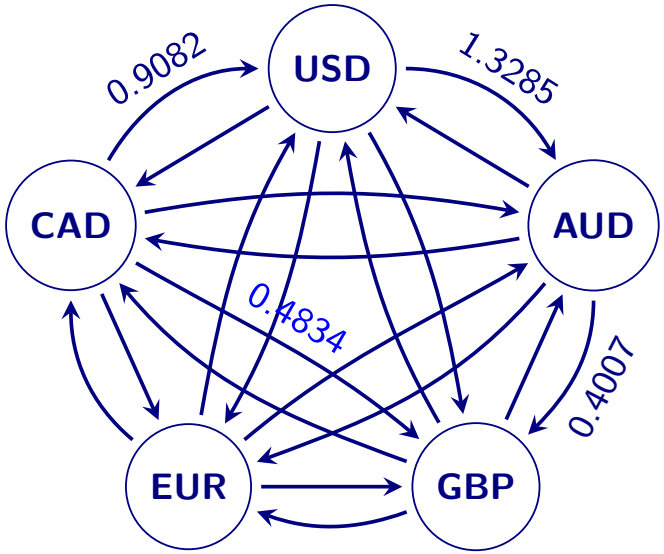

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➡ What is the best possible exchange rate?

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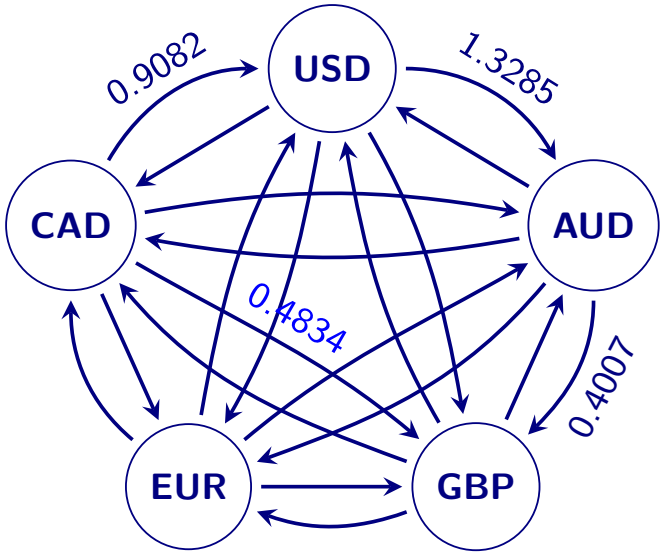

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➡ Find a path from **CAD** to **GBP** with highest possible product!

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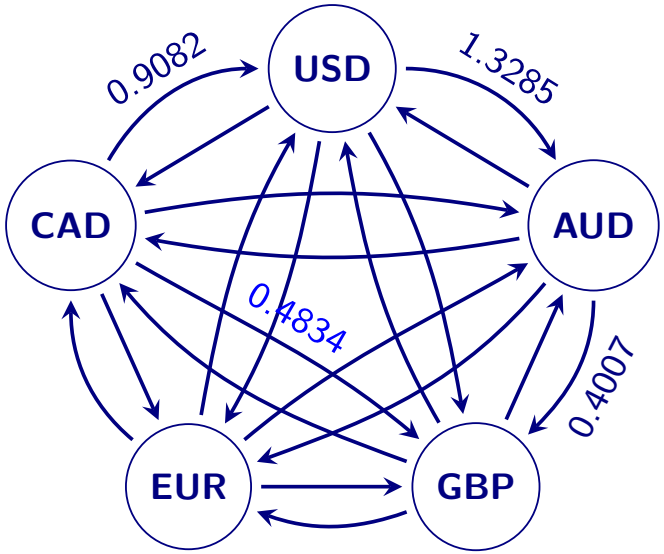

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➡ What is the best possible exchange rate?

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▷ Problem: subsequent steps result in multiplying (rates) instead of adding (lengths)

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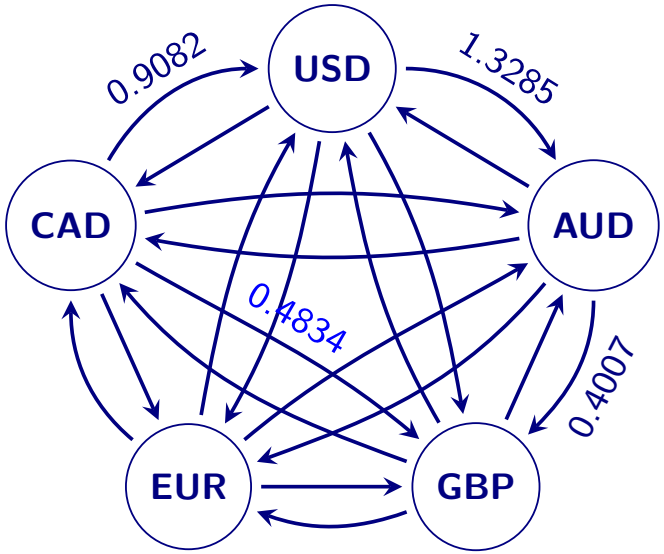

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▷ Problem: subsequent steps result in multiplying (rates) instead of adding (lengths)

➡ Use logarithm: $\log(a \cdot b) = \log a + \log b$ and $\log a < \log b \iff a < b$

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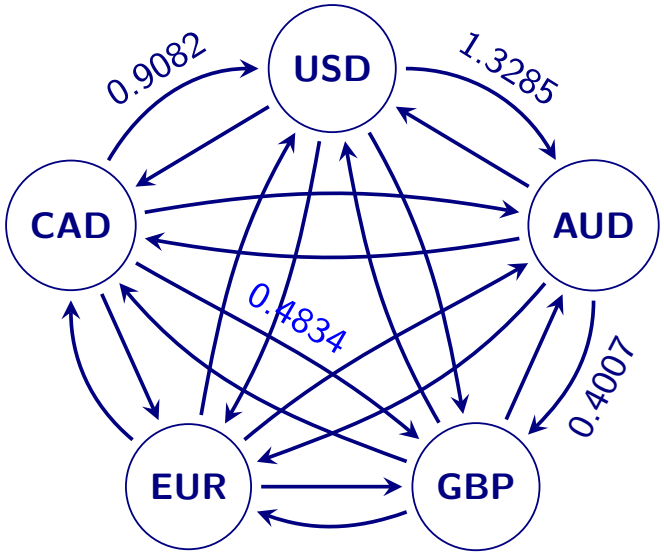

▷ Exchanging **CAD** → **USD** → **AUD** → **GBP**:

➡ Rate: $0.9082 \cdot 1.3285 \cdot 0.4007 = 0.483462 > 0.4834$

➡ What is the best possible exchange rate?

➡ Find a path from **CAD** to **GBP** with highest possible product!

▷ Problem: subsequent steps result in multiplying (rates) instead of adding (lengths)

➡ Use logarithm: $\log(a \cdot b) = \log a + \log b$ and $\log a < \log b \iff a < b$

▷ Plus: search for a longest path (instead of a shortest)

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- ▷ Example: Determine the biggest number in a given list
  - Input: List $x_1, \ldots, x_n$ of numbers
  - Algorithm: Set the current biggest number $x_{\max} := -\infty$

    For all i from 1 to n...

    ...if $x_i > x_{\max}$ then set $x_{\max} := x_i$

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  - Algorithm: Set the current biggest number $x_{\max} := -\infty$ (constant time)

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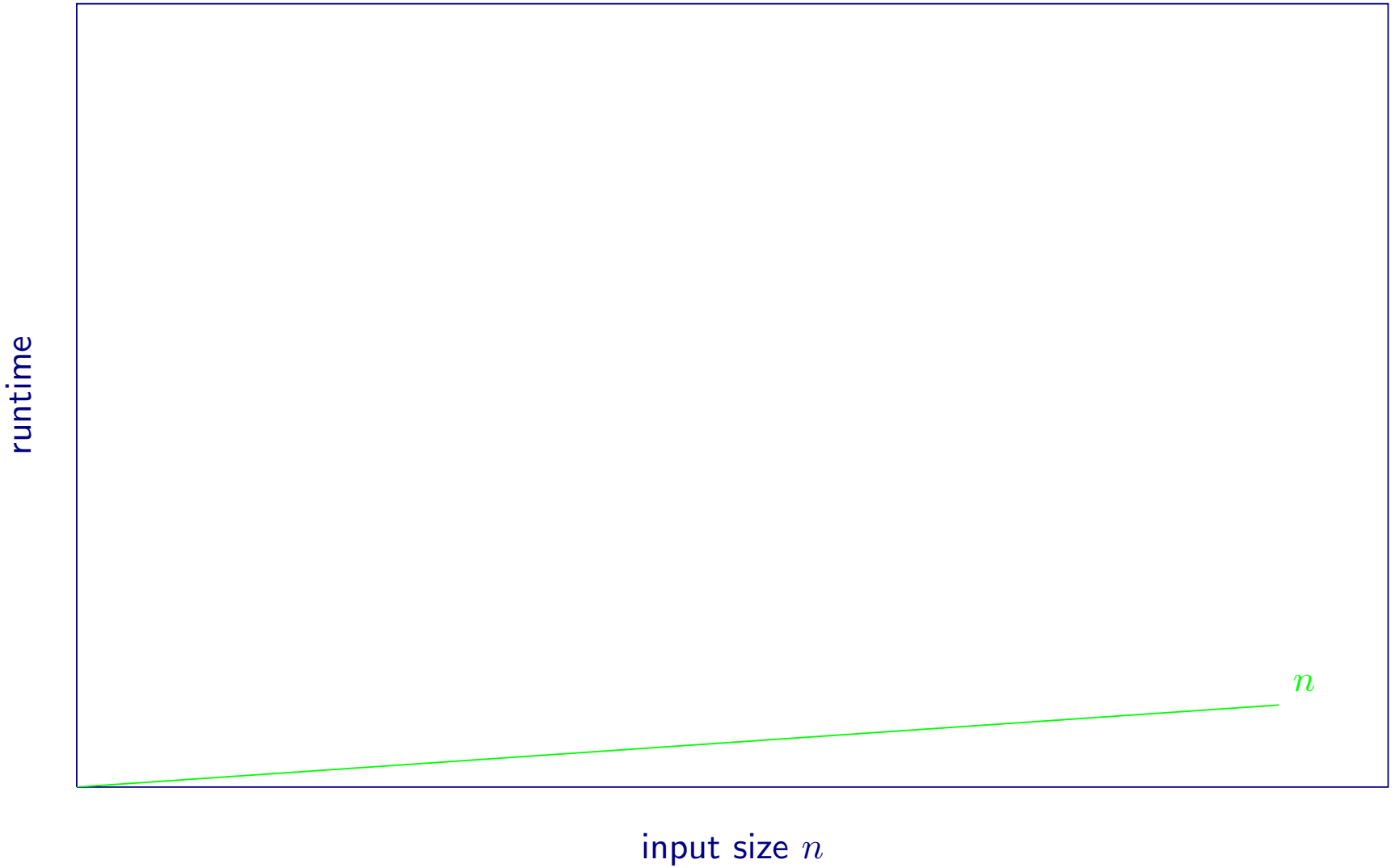
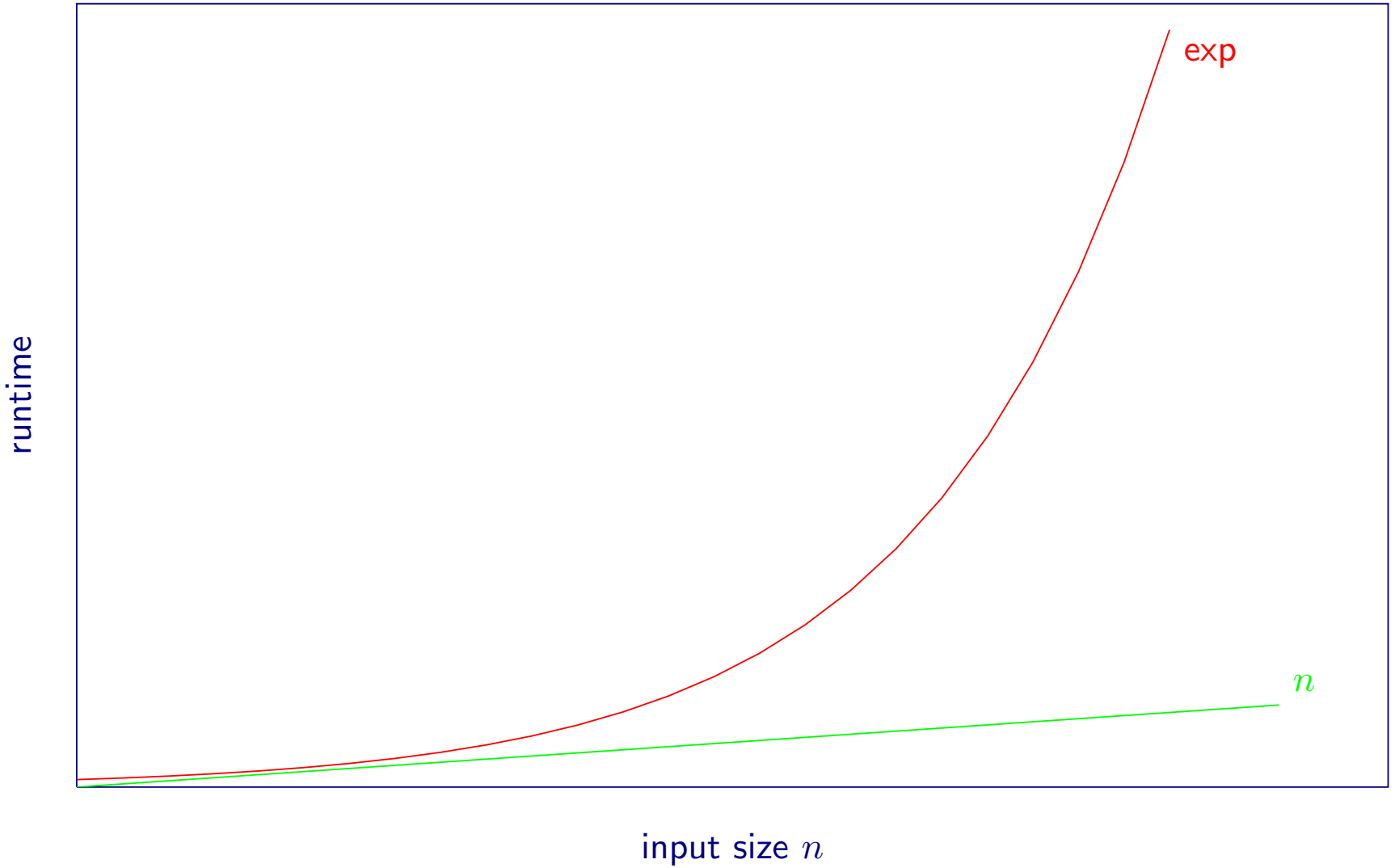
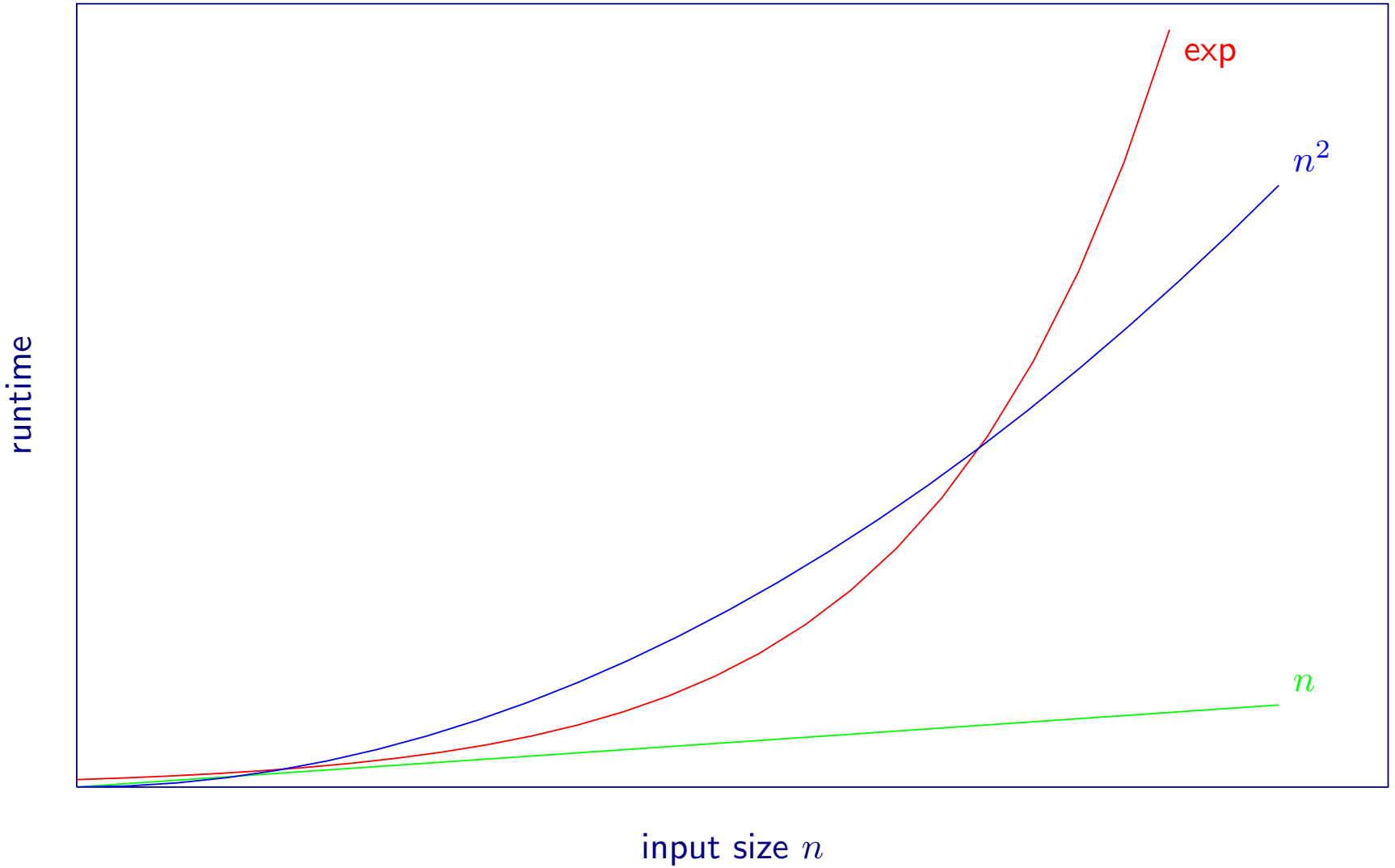
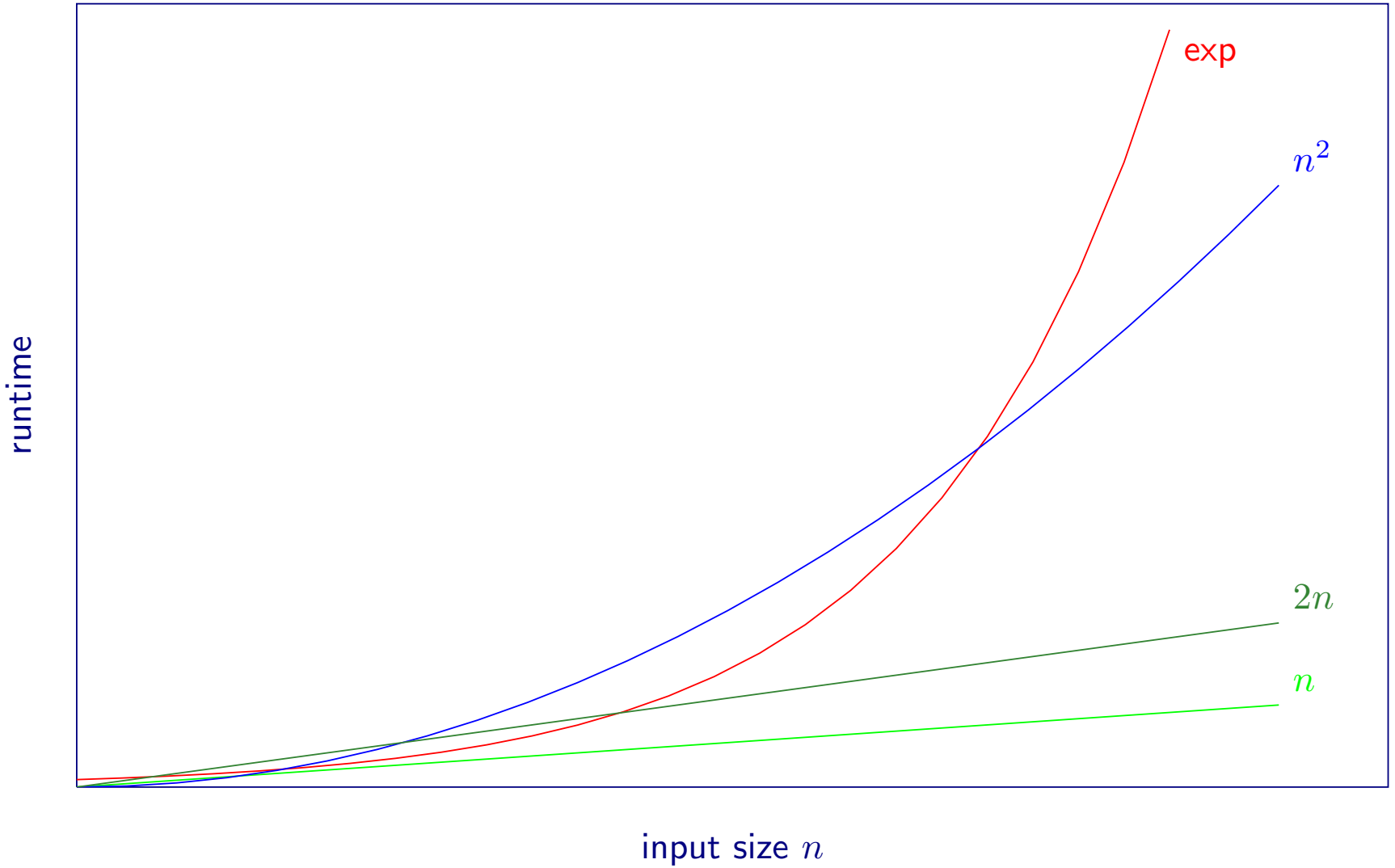
  - ➡ Input size: n numbers
  - ➡ Runtime: $n+1$ "basic steps" (order of n, linear runtime)


runtime
n
input size n


exp
n
runtime
input size n


exp
n^2
n
runtime
input size n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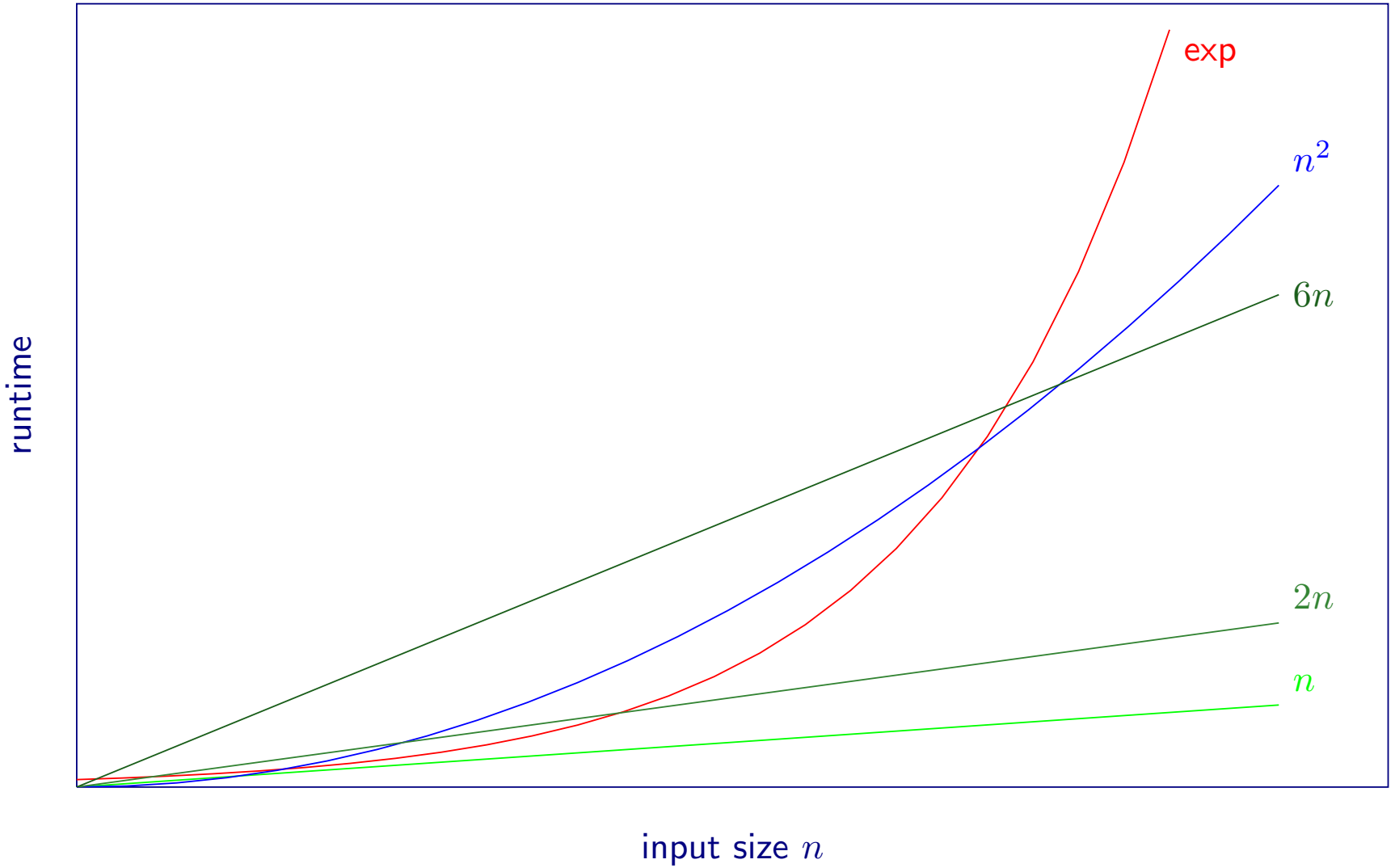
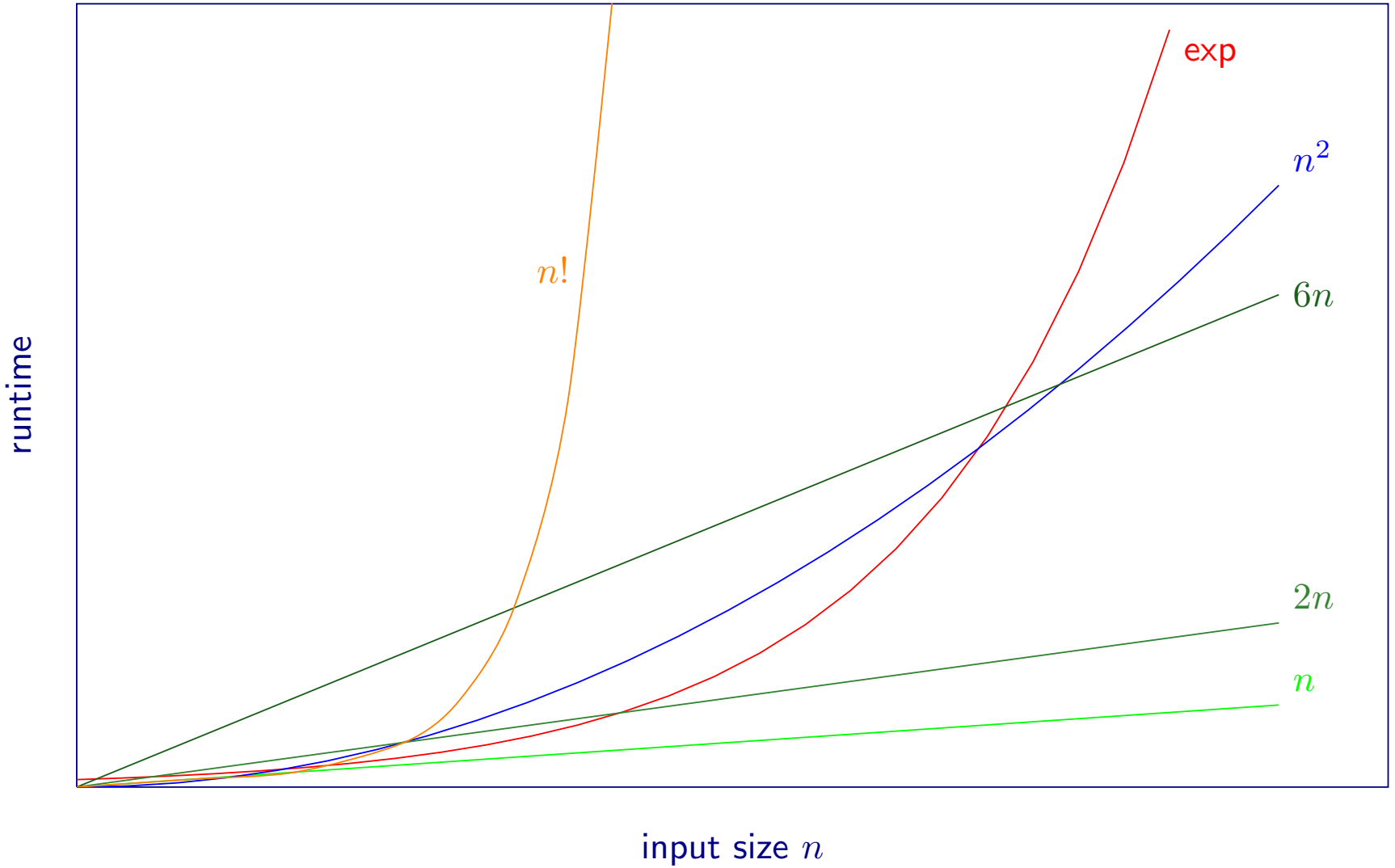
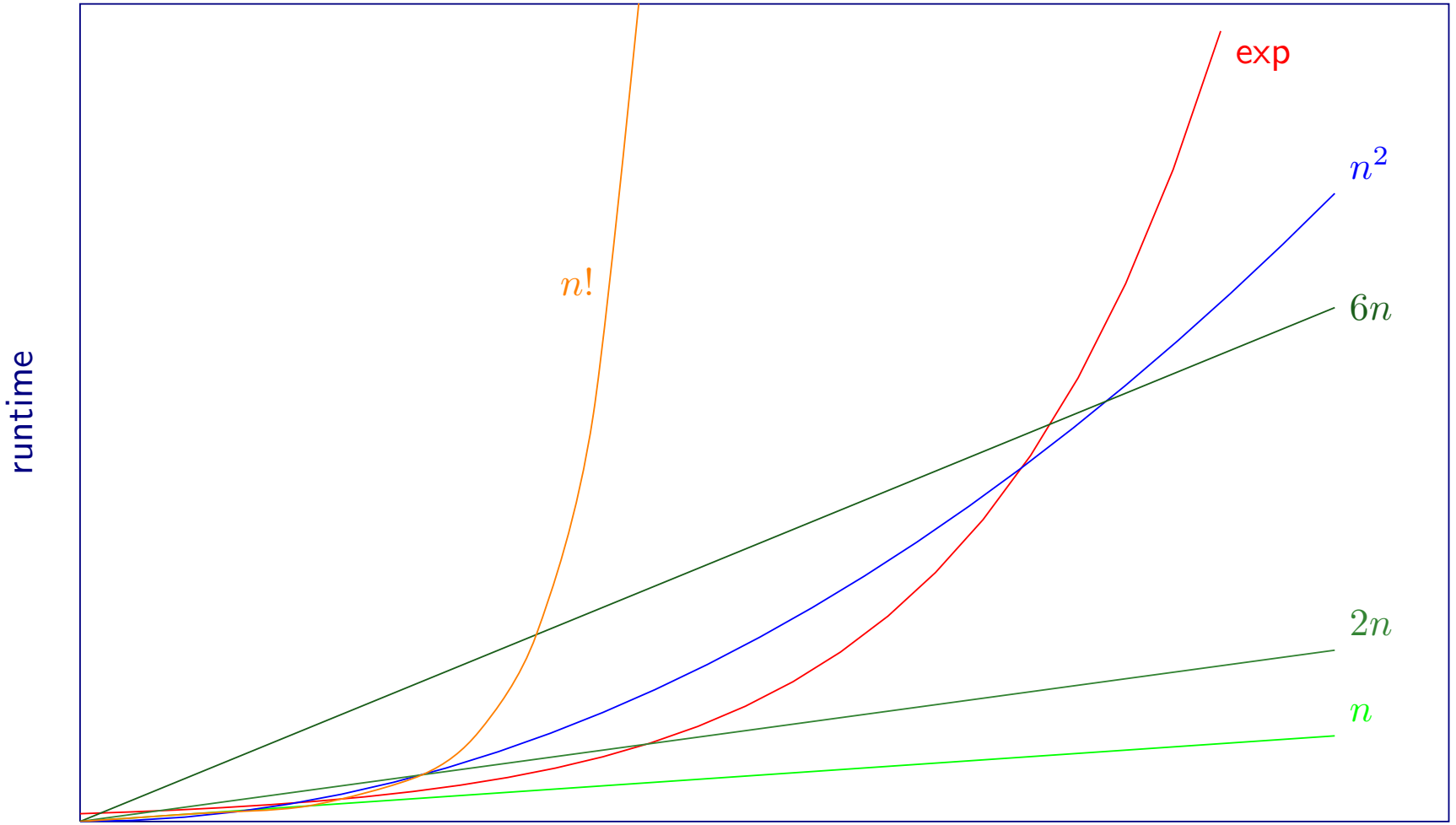
$2n$
n
runtime
input size n


exp
n^2
$6n$
$2n$
n
runtime
input size n


exp
n^2
$n!$
$6n$
$2n$
n
runtime
input size n



linear — polynomial — exponential

▷ Input: directed graph with n nodes V

▷ Algorithm:

Set remaining nodes $R := V$, and $d(\mathrm{A}) := 0$, $d(v) := \infty$ for all nodes $v \neq \mathrm{A}$, $\mathrm{pred}(\mathrm{A}) := \mathrm{A}$

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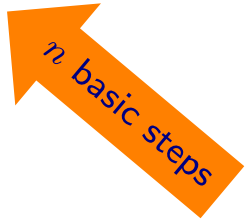

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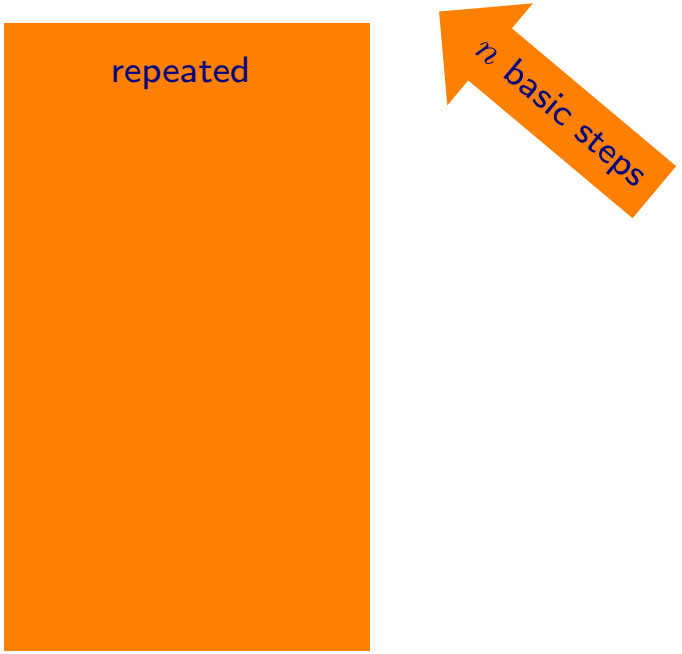

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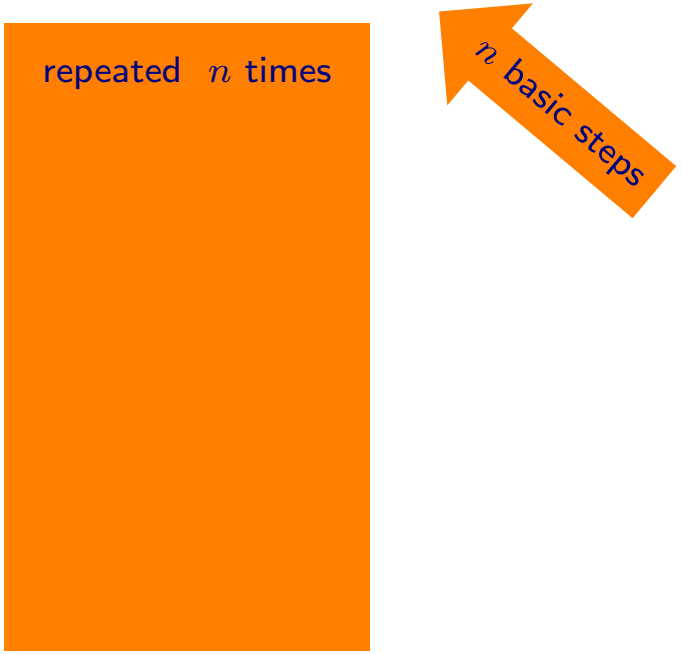

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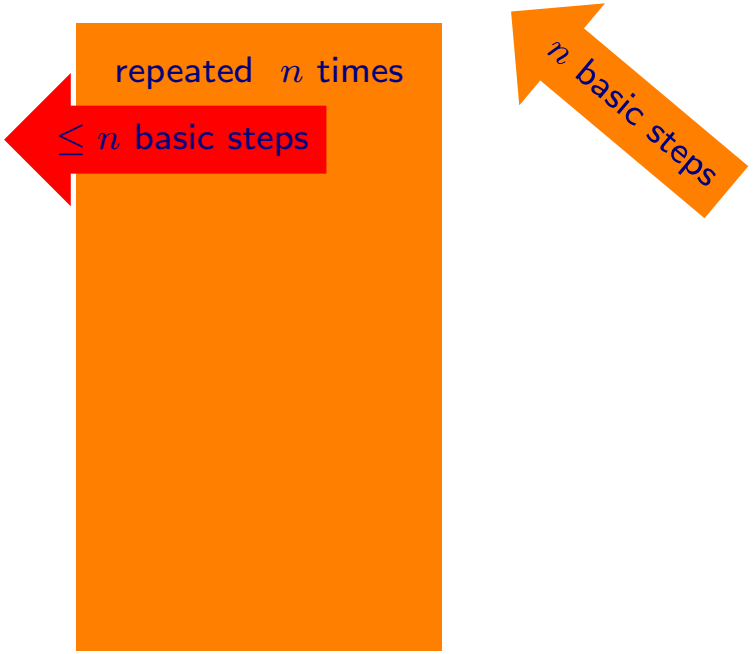

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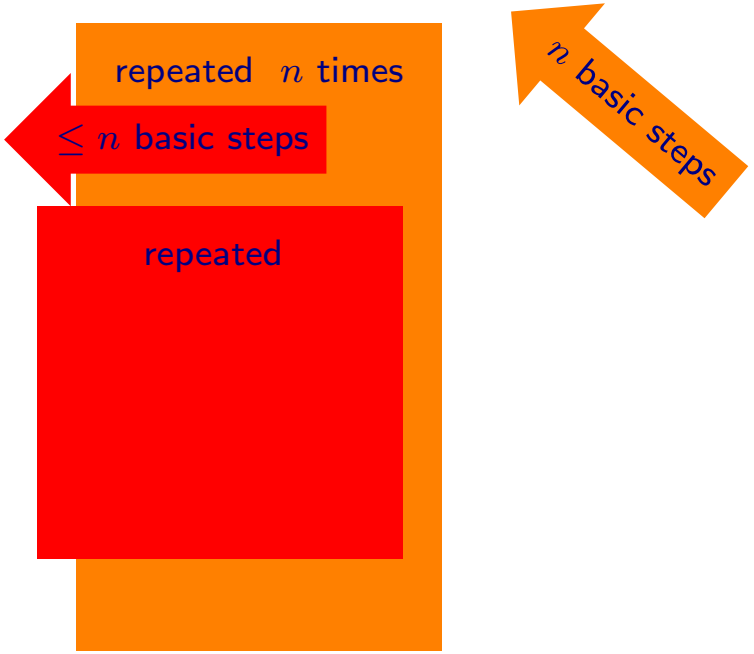

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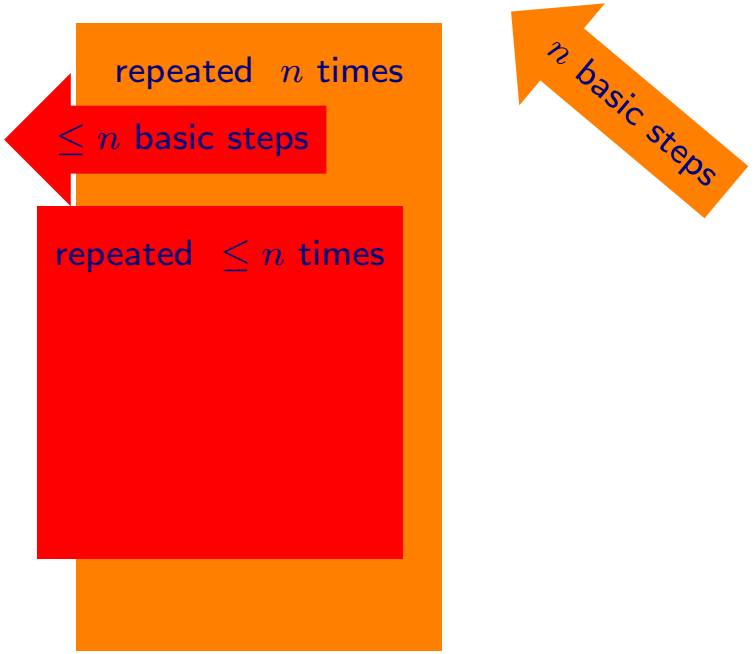

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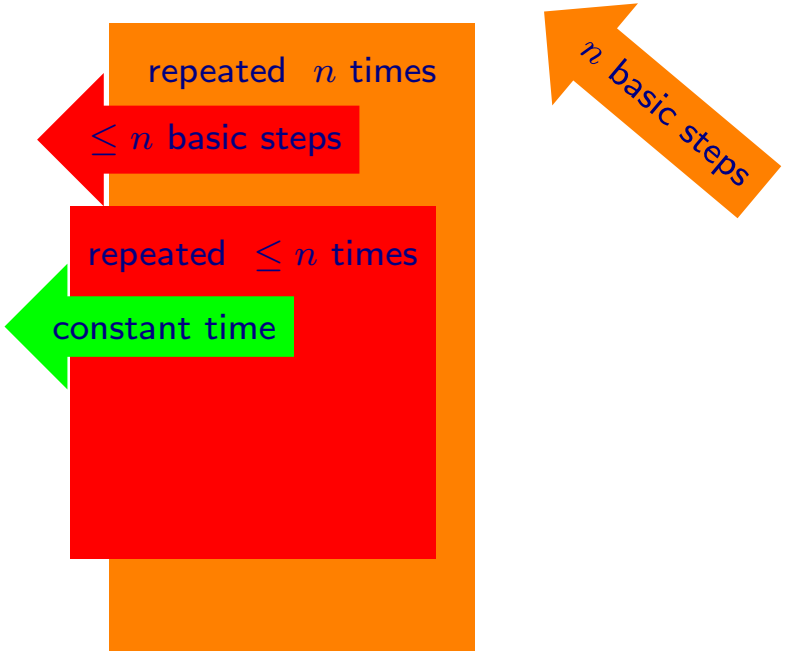

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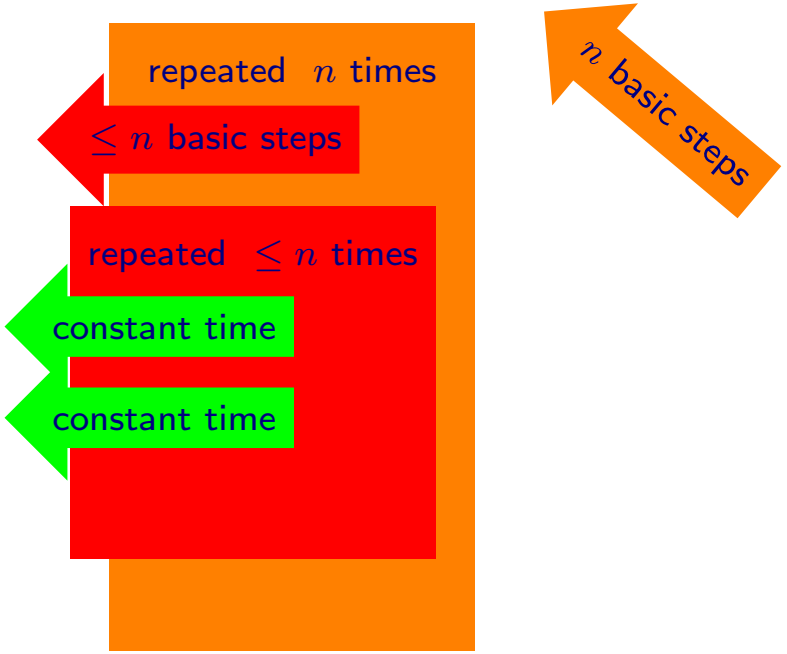

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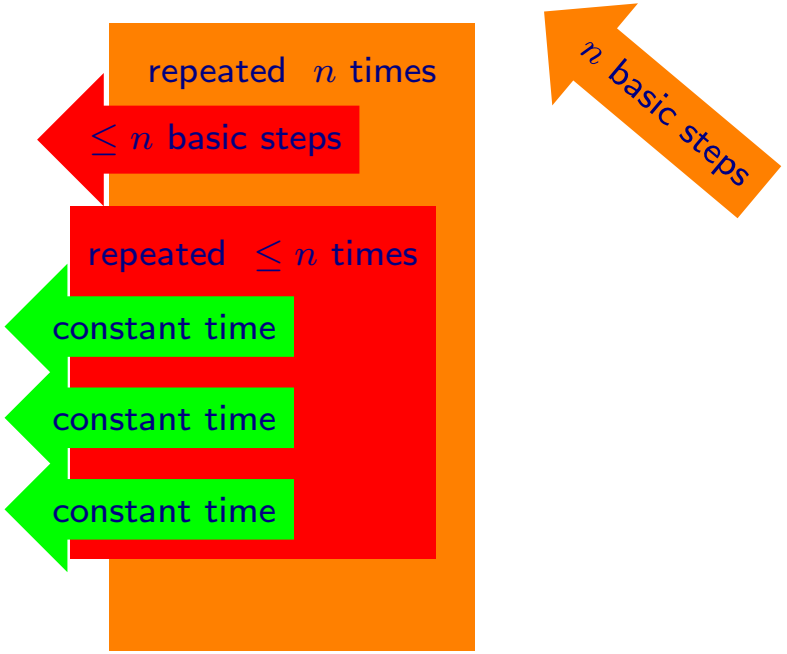

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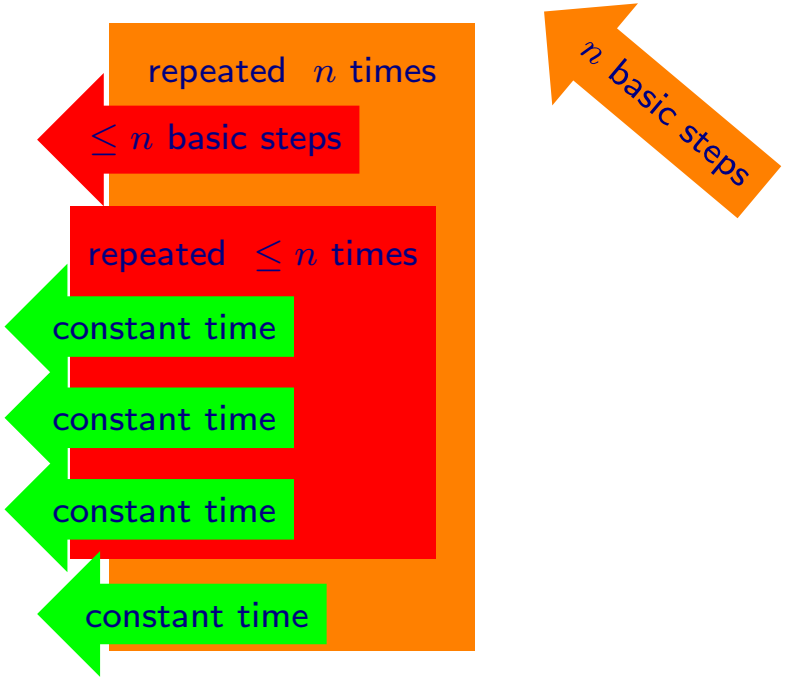

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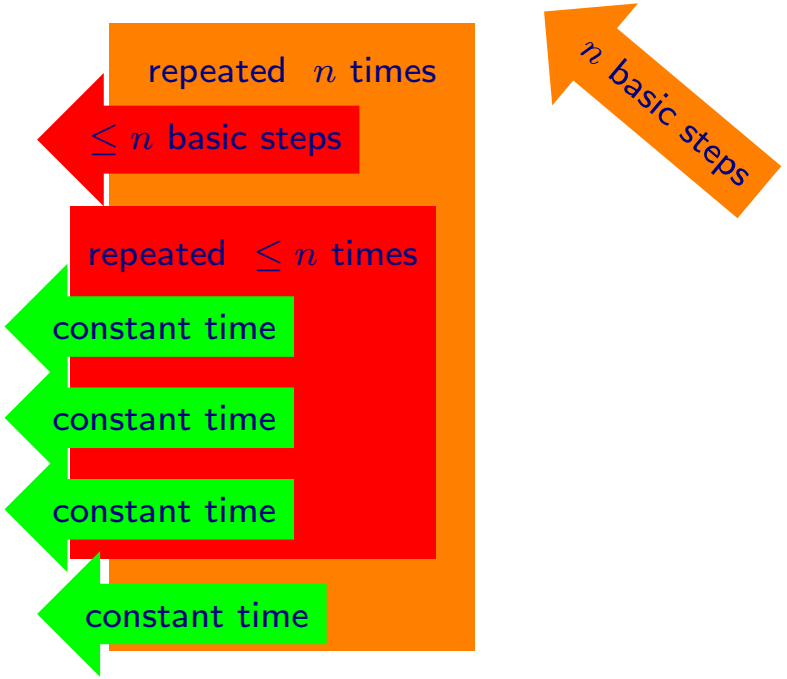

➡ Altogether: at most $n \cdot (n + 3n + 1) = 4n^2 + n$ "basic steps"

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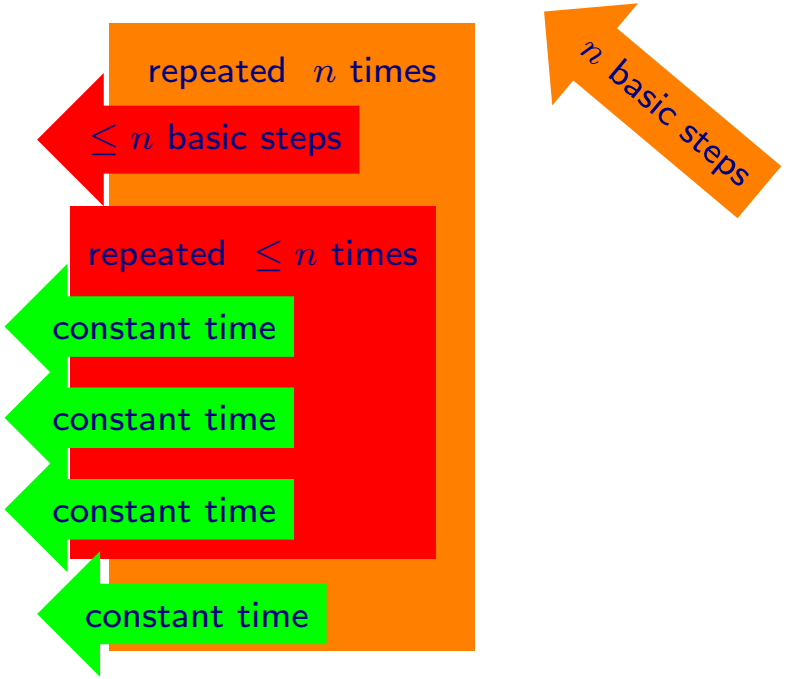

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➡ Polynomial runtime ($\mathcal{O}(n^2)$)

- ▷ Models, Data and Algorithms
- ▷ Linear Optimization
- ▷ Mathematical Background: Polyhedra, Simplex-Algorithm
- ▷ Sensitivity Analysis; (Mixed) Integer Programming
- ▷ MIP Modelling
- ▷ MIP Modelling: More Examples; Branch & Bound
- ▷ Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- ▷ TSP-Heuristics
- ▷ Network Flows
- ▷ Shortest Path Problem
- ▷ Complexity Theory
- ▷ Nonlinear Optimization, Scheduling
- ▷ Lot Sizing, Multicriteria Optimization
- ▷ Oral exam