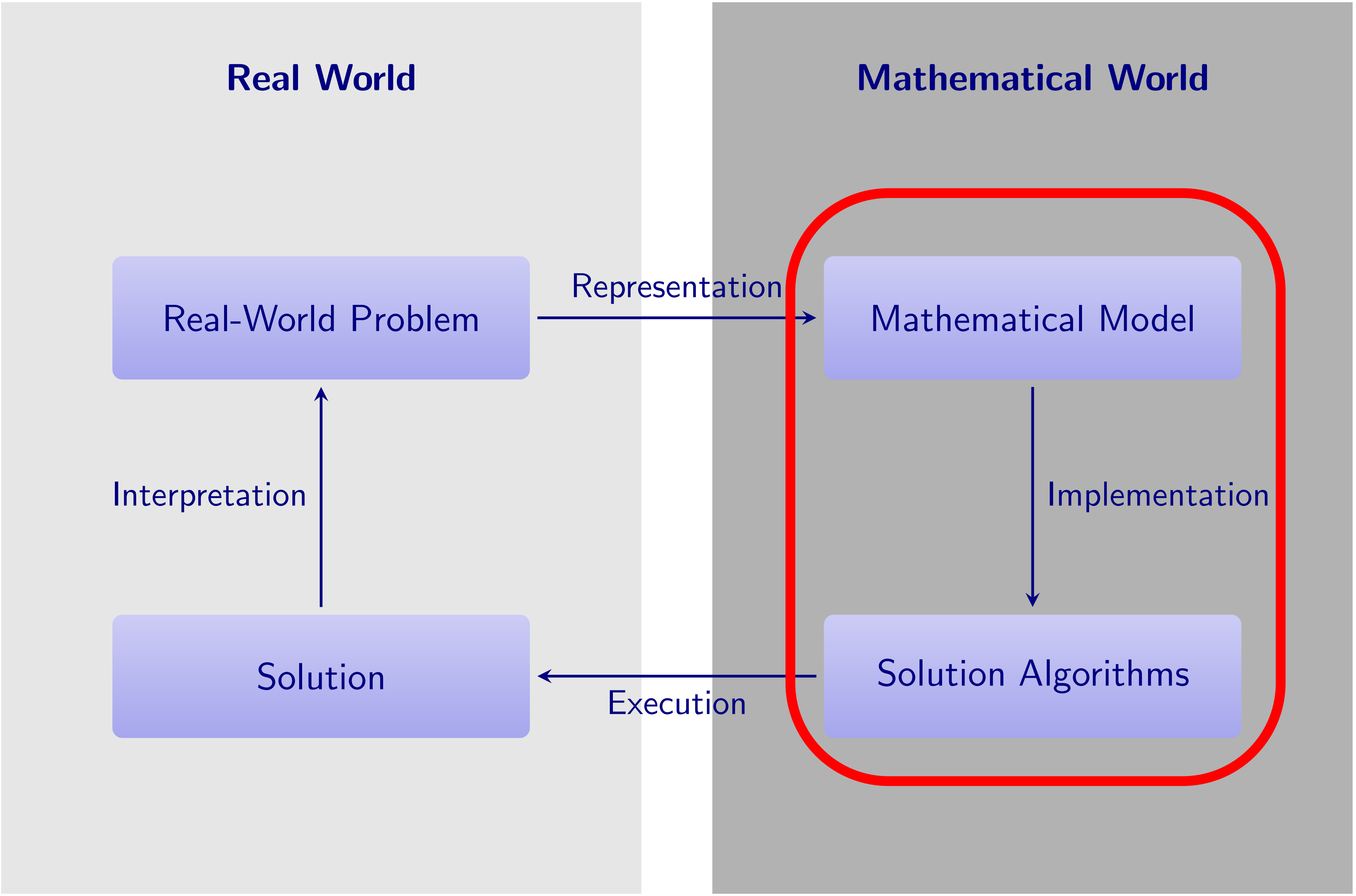
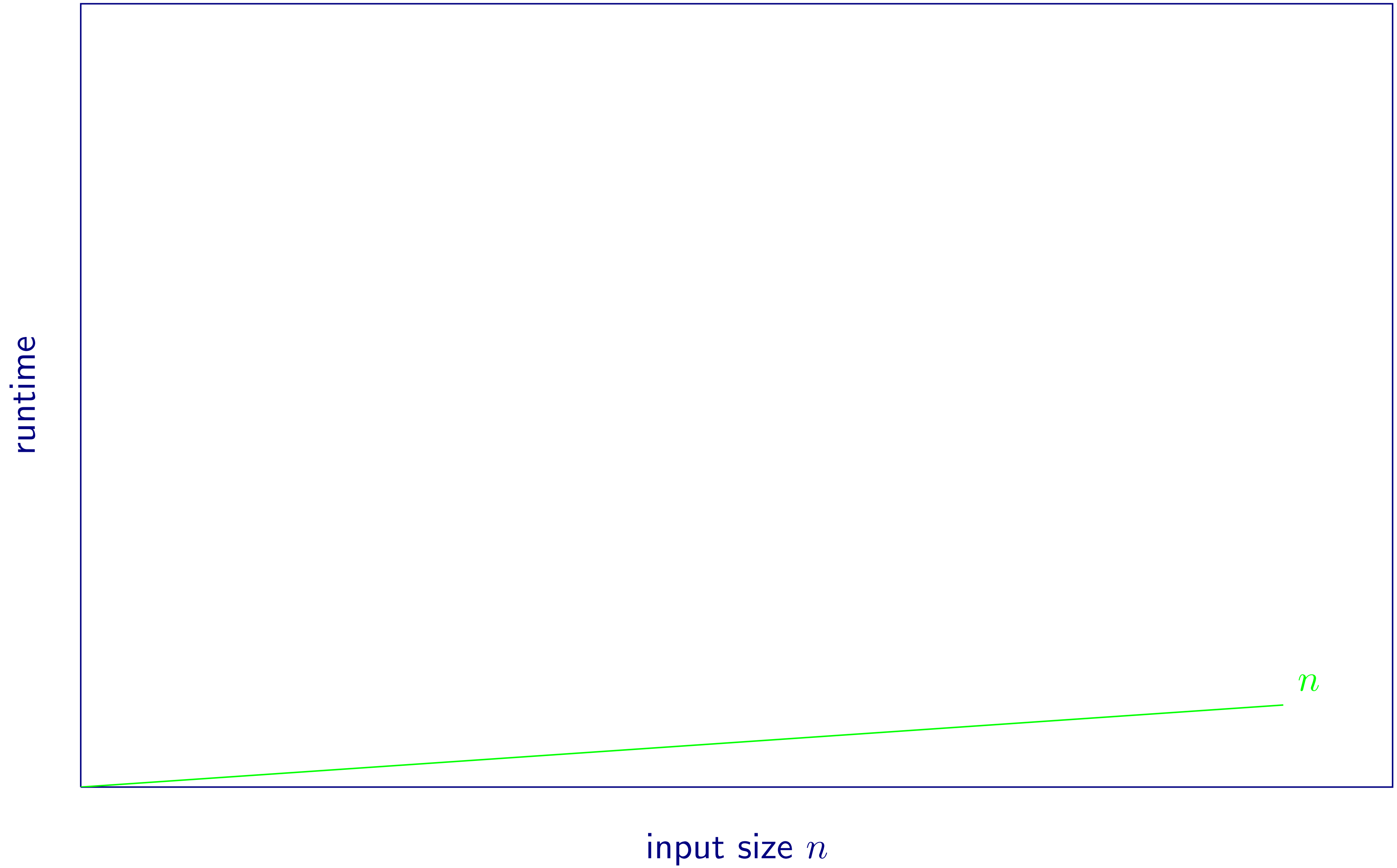
Mathematical Tools for Engineering and Management

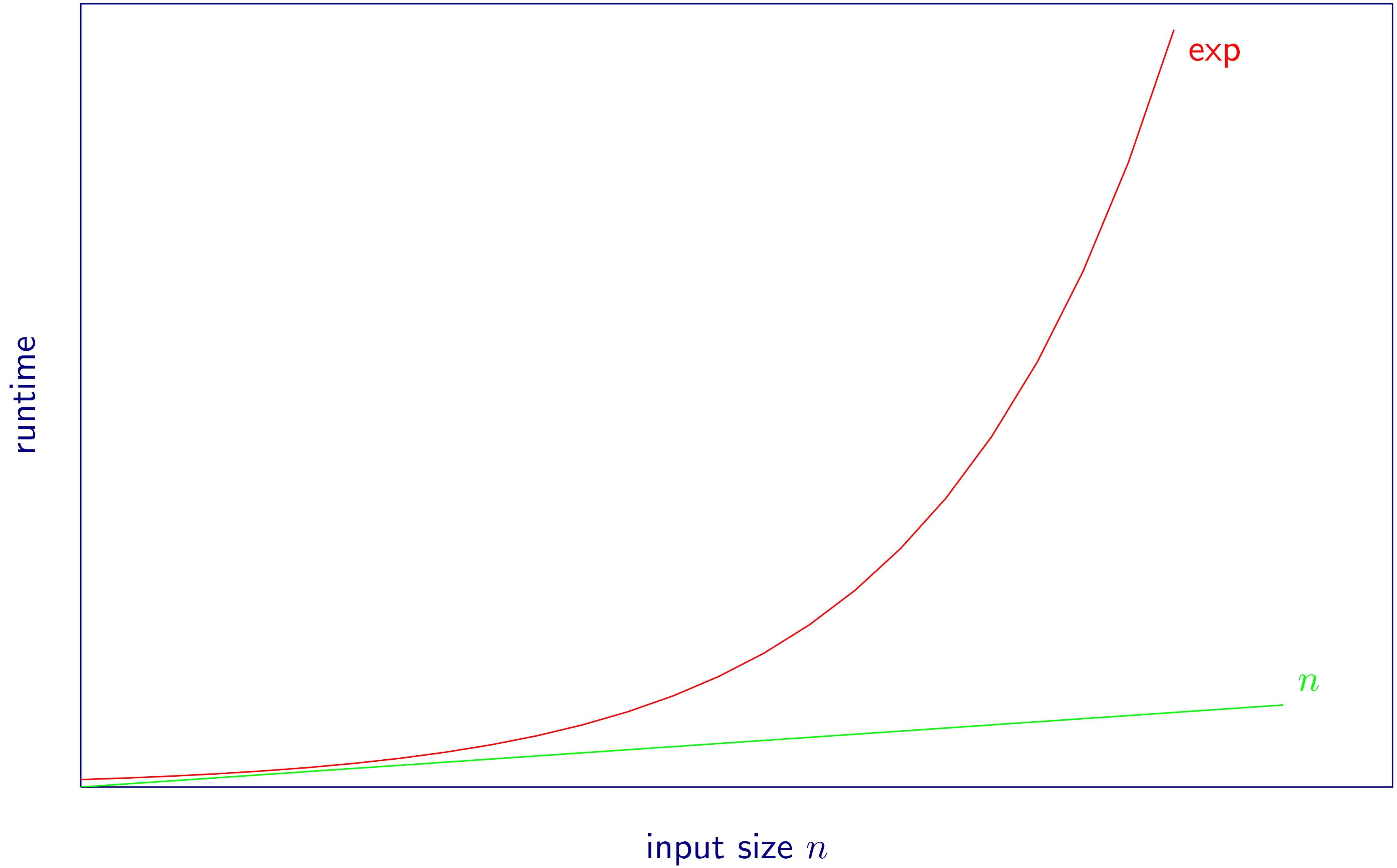
Lecture 11

11 Jan 2012

- Models, Data and Algorithms
- Linear Optimization
- Mathematical Background: Polyhedra, Simplex-Algorithm
- Sensitivity Analysis; (Mixed) Integer Programming
- MIP Modelling
- MIP Modelling: More Examples; Branch & Bound
- Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- TSP-Heuristics
- Network Flows
- Shortest Path Problem
- Complexity Theory
- Nonlinear Optimization, Scheduling
- Lot Sizing, Multicriteria Optimization
- Oral exam


Real World
Mathematical World
Real-World Problem
Representation
Mathematical Model
Interpretation
Implementation
Solution
Execution
Solution Algorithms


runtime
n
input size n


exp
n
runtime
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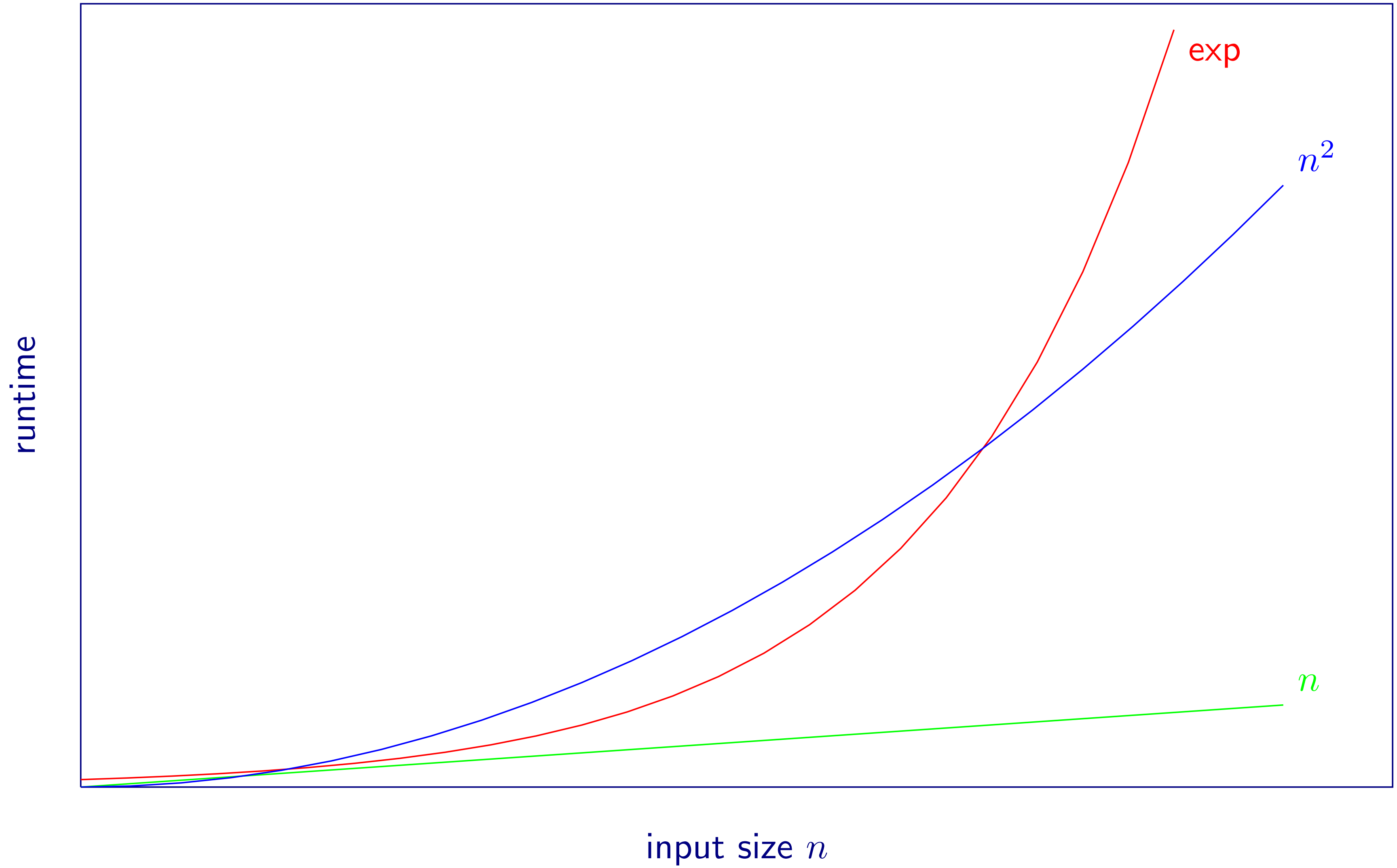
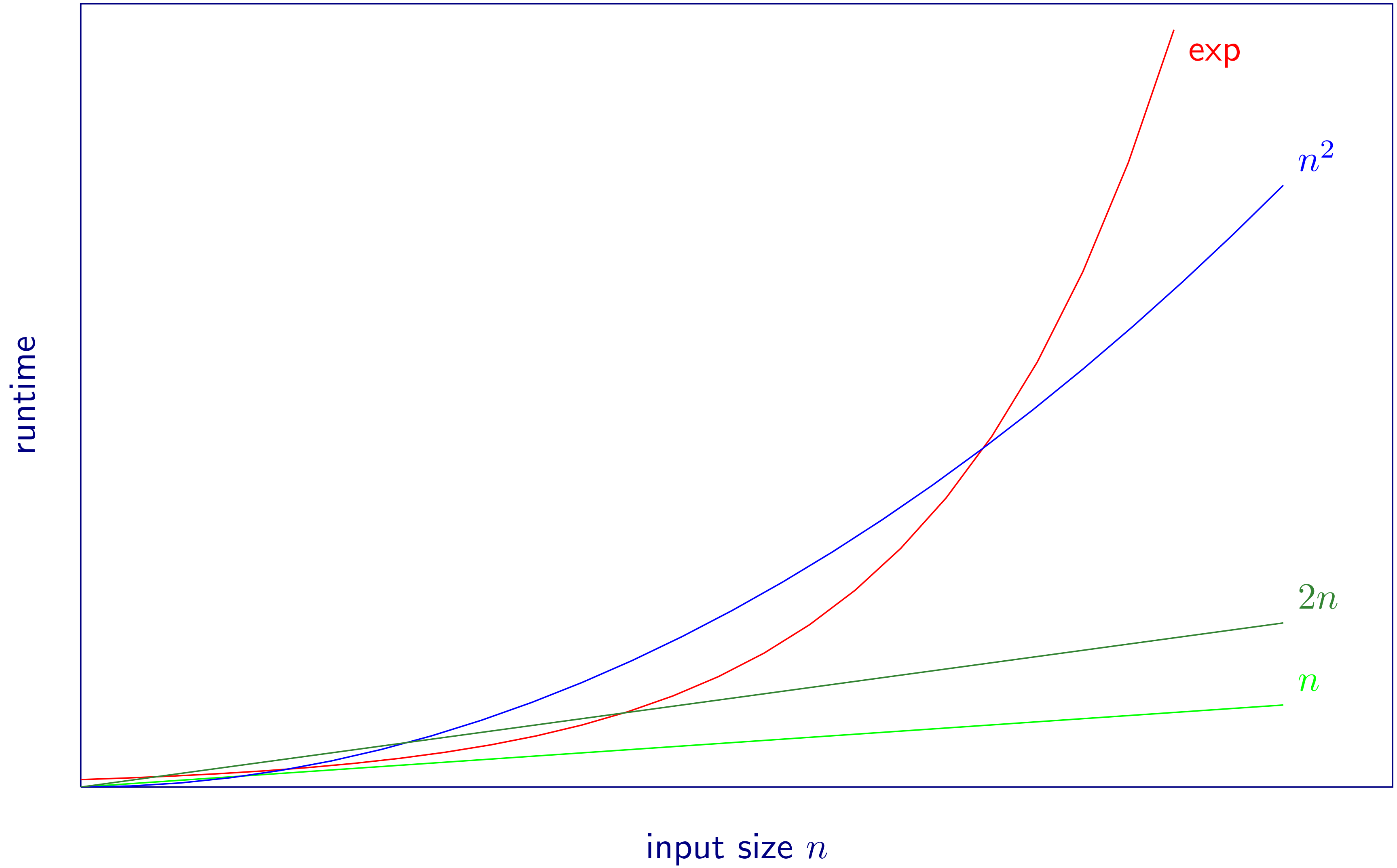
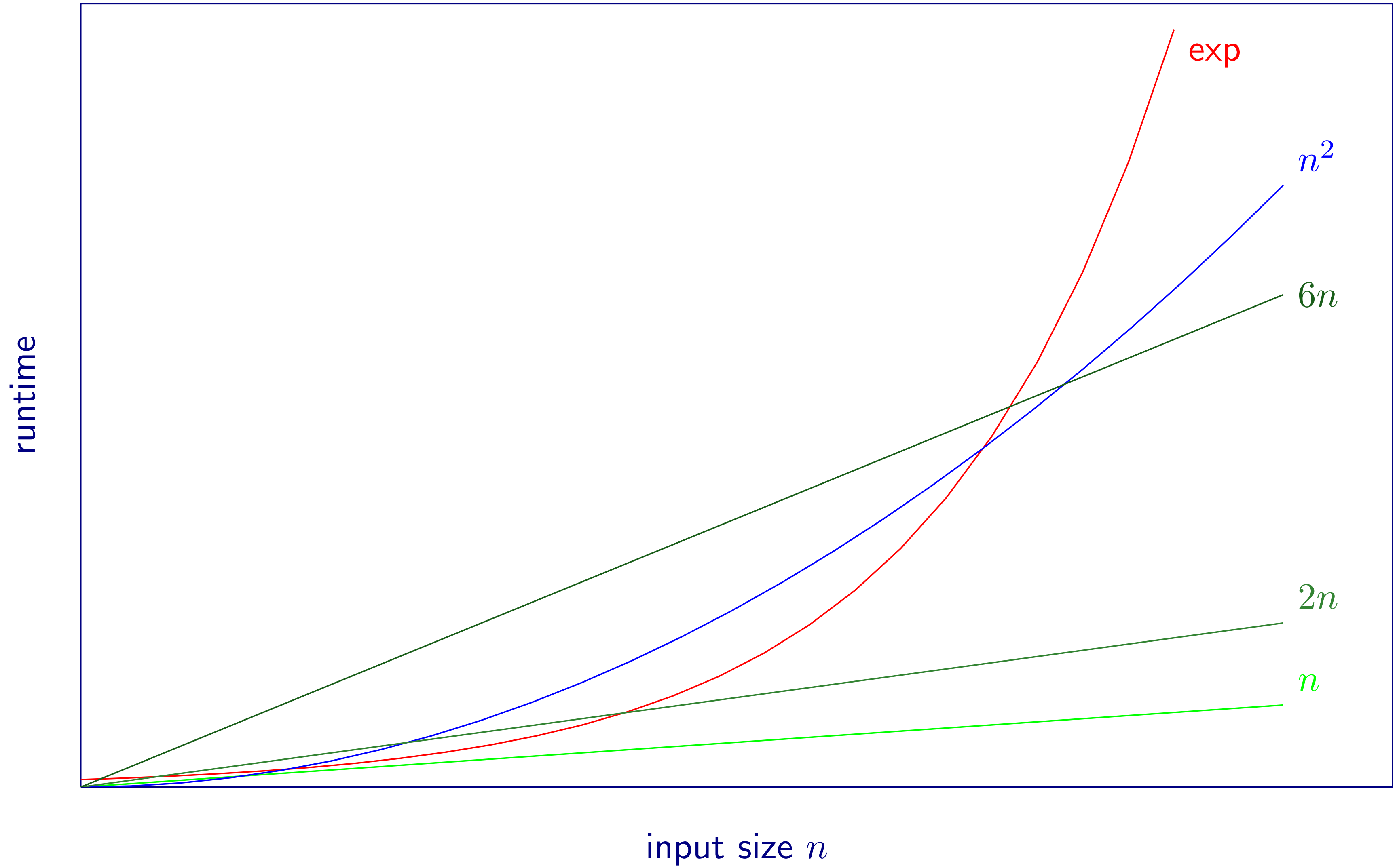
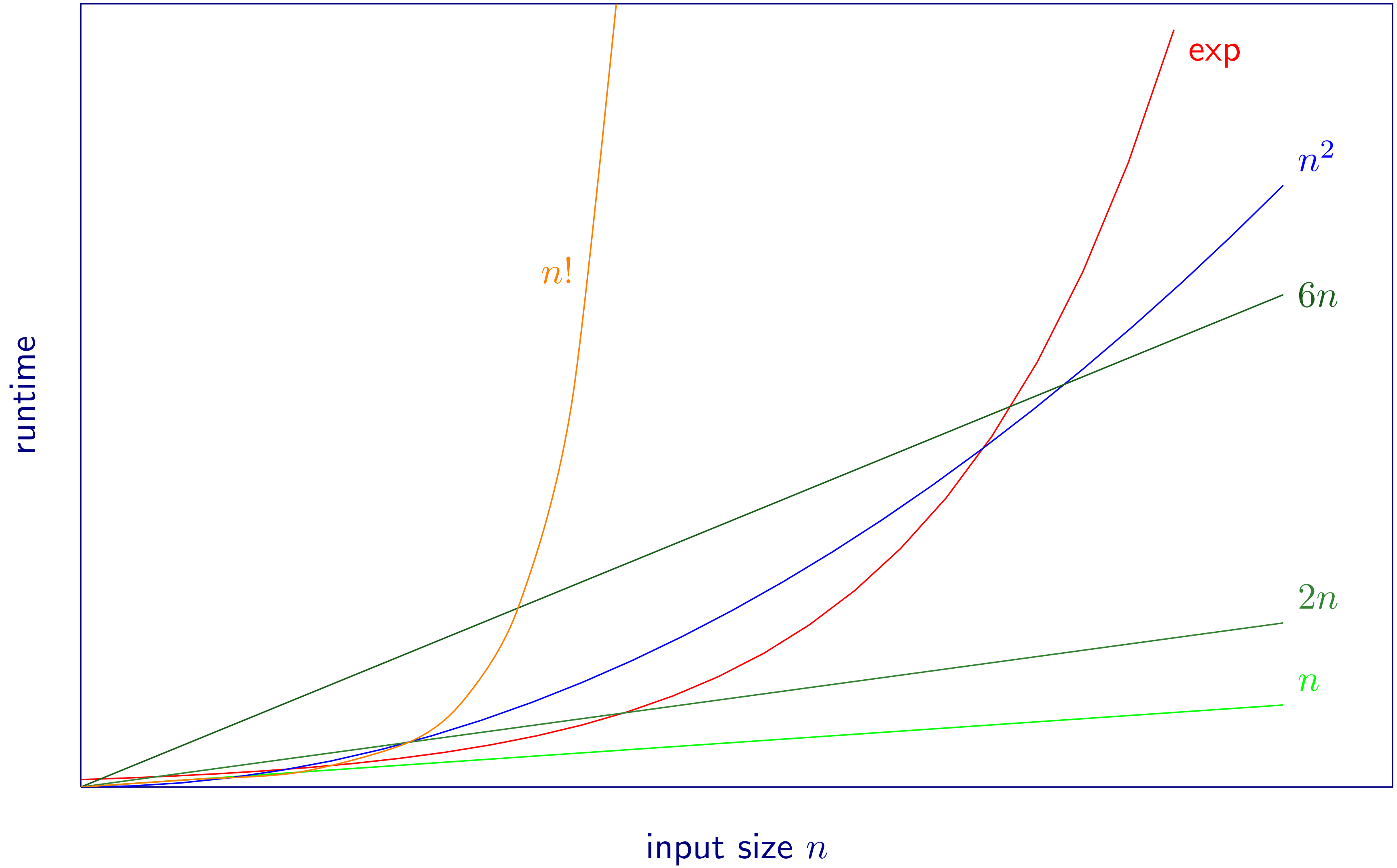

exp
n^2
n
runtime
input size n


exp
n^2
$2n$
n
runtime
input size n


exp
n^2
$6n$
$2n$
n
runtime
input size n



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$n!$
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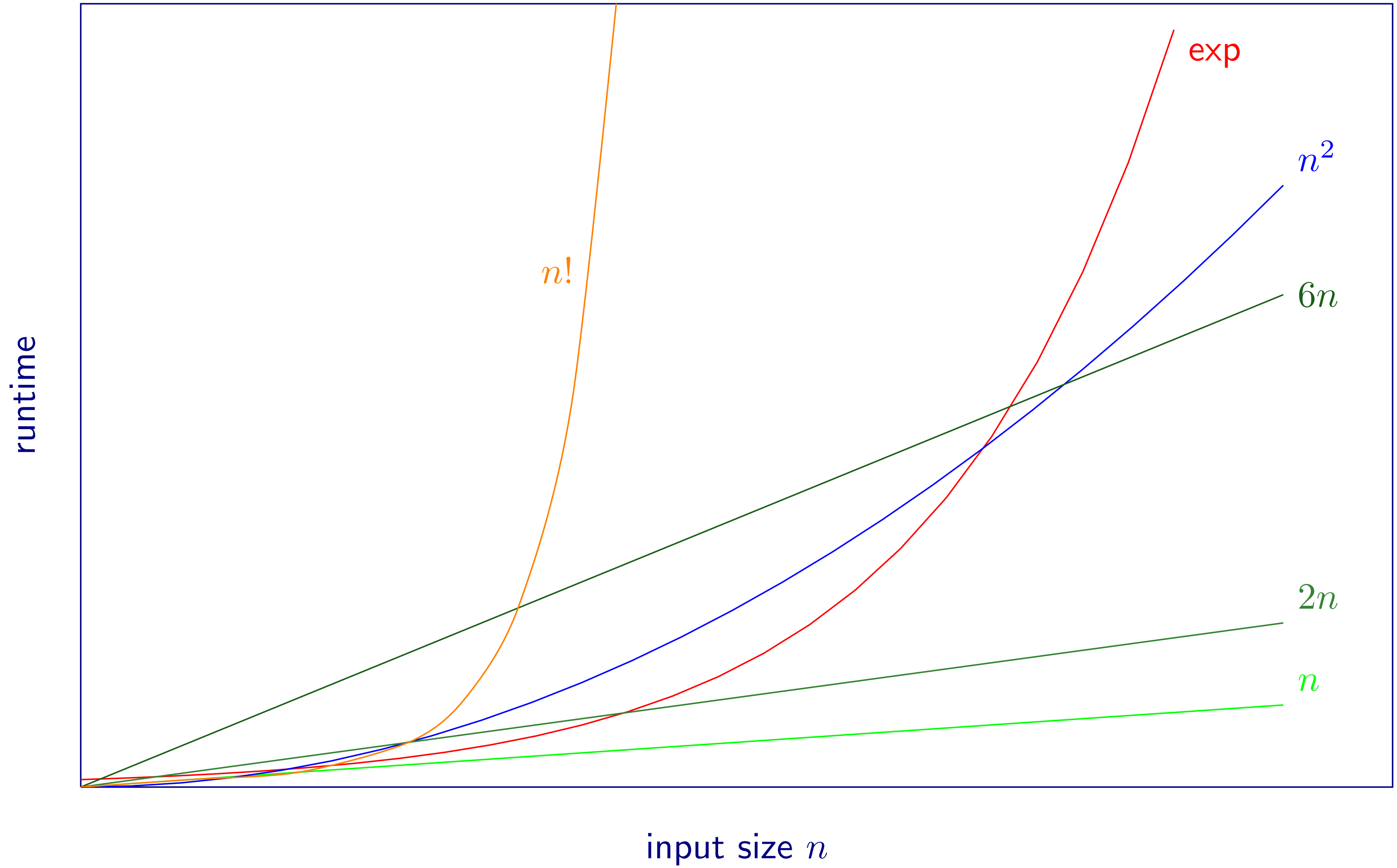

linear — polynomial — exponential

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	Ellipsoid method	Complete enumeration
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	approximation algorithms	

Efficient and exact algorithms

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A mathematical problem for which it is possible to verify feasibility of a given solution in polynomial time, is a member of the complexity class $\mathcal{NP}$.

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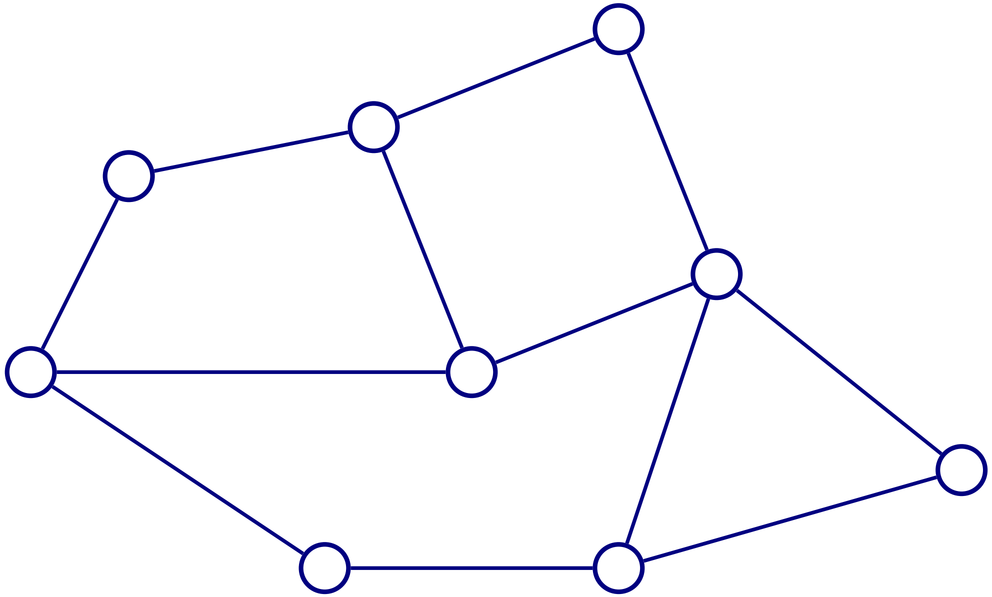

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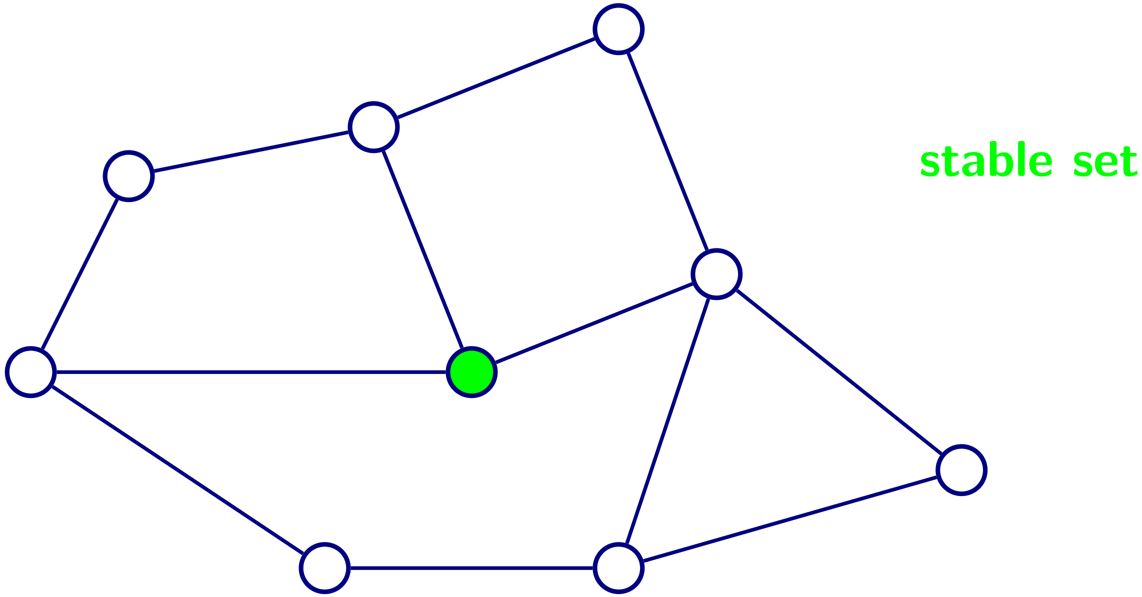

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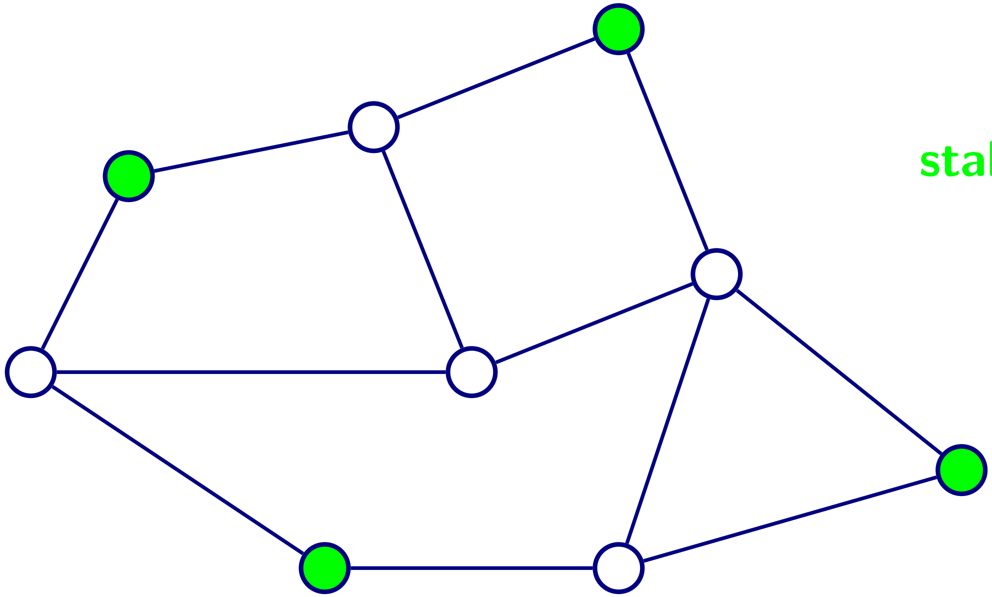

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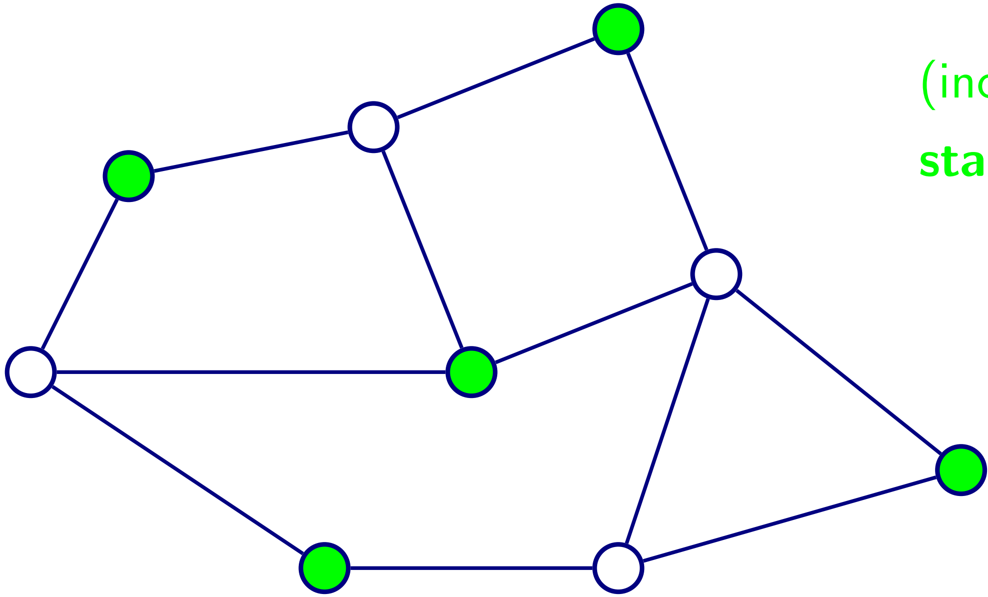

(inclusion-maximal)
stable set

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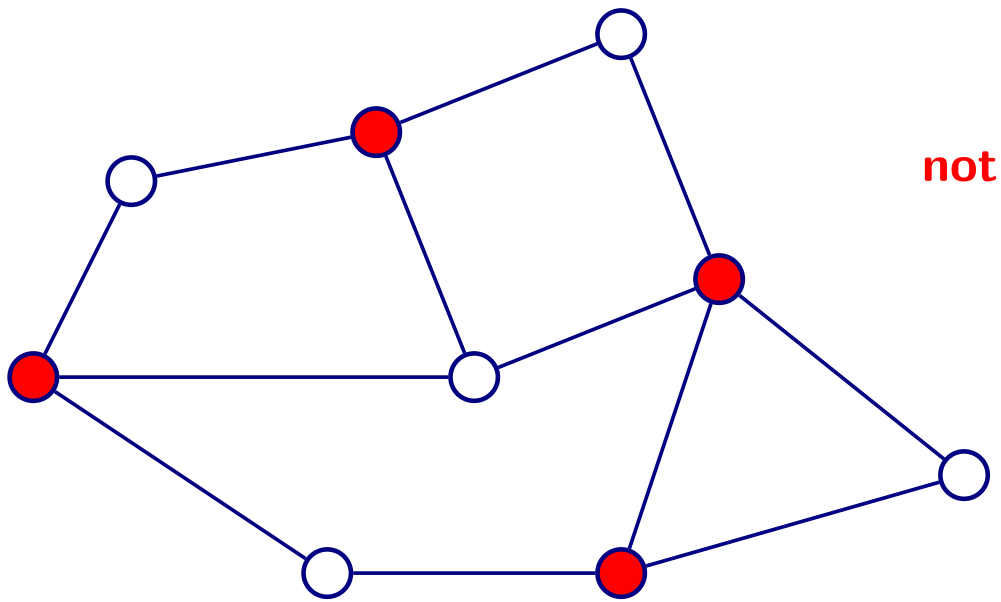

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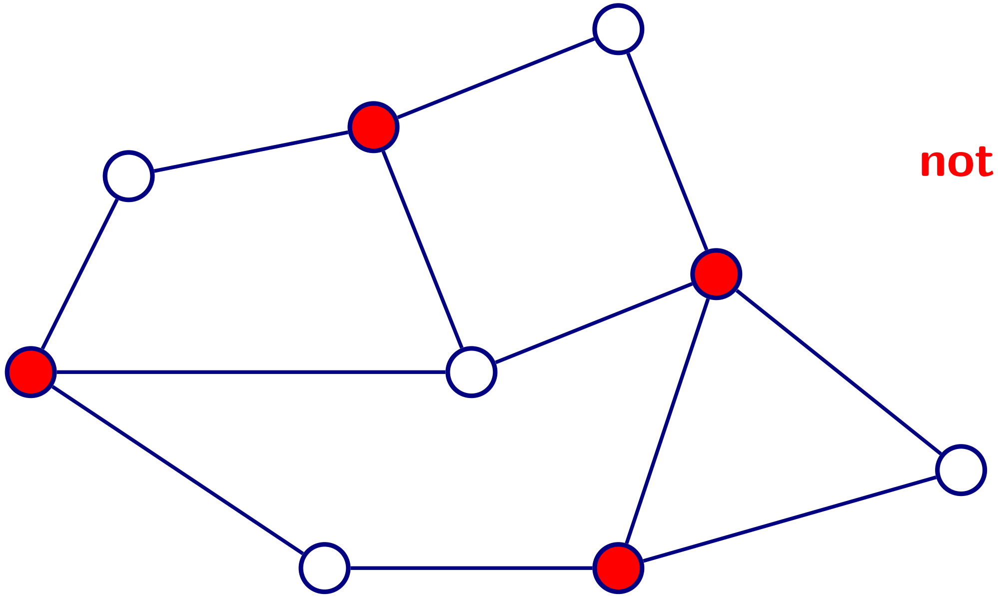

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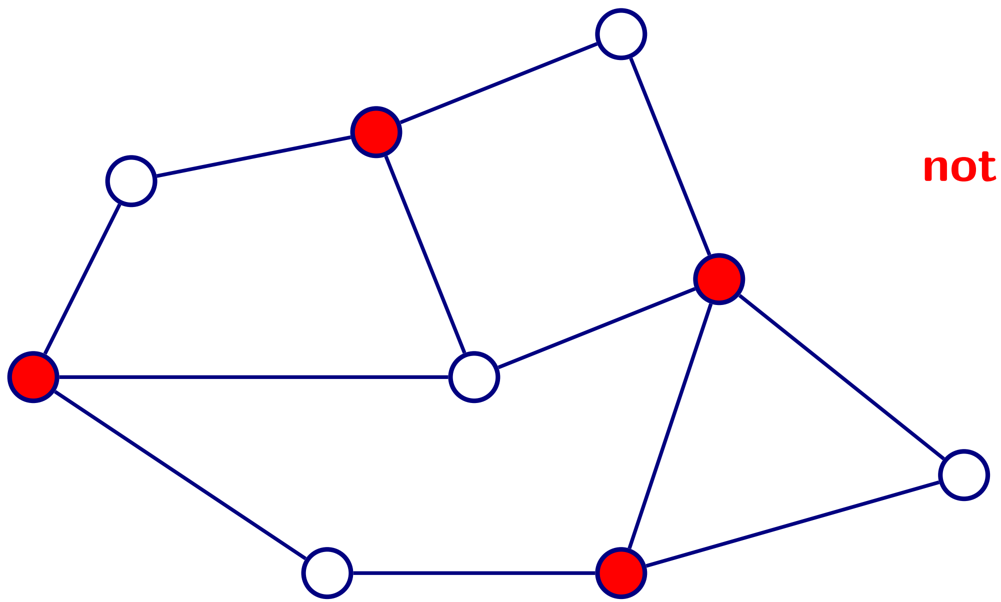

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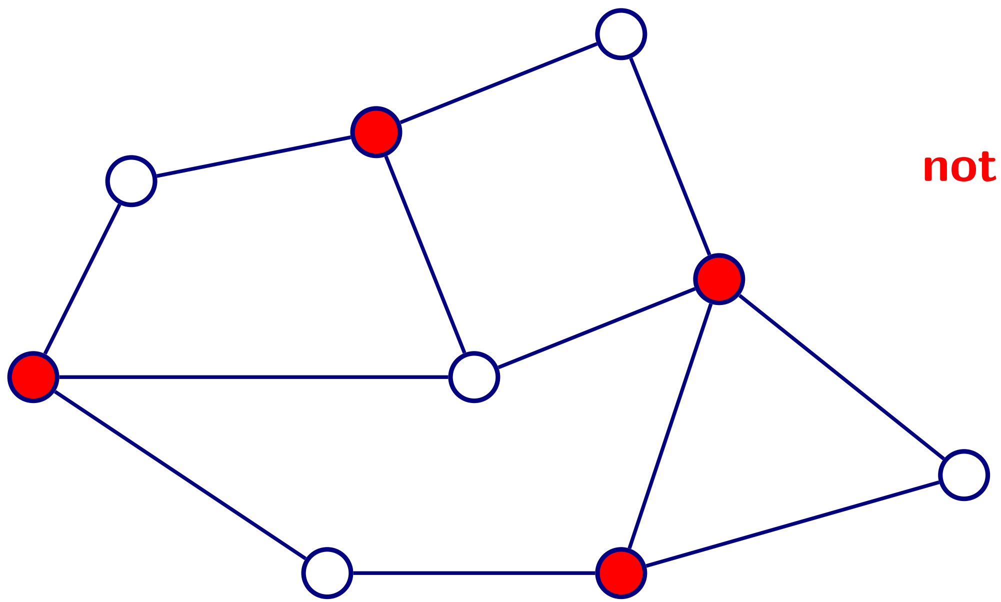

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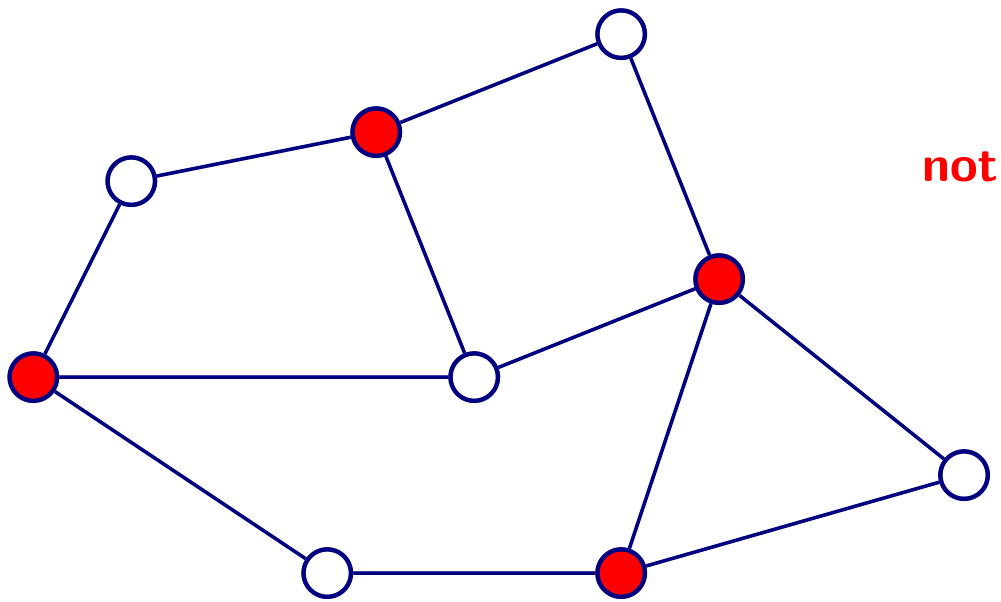

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  ➡ Stable Set Problem is in $\mathcal{NP}$

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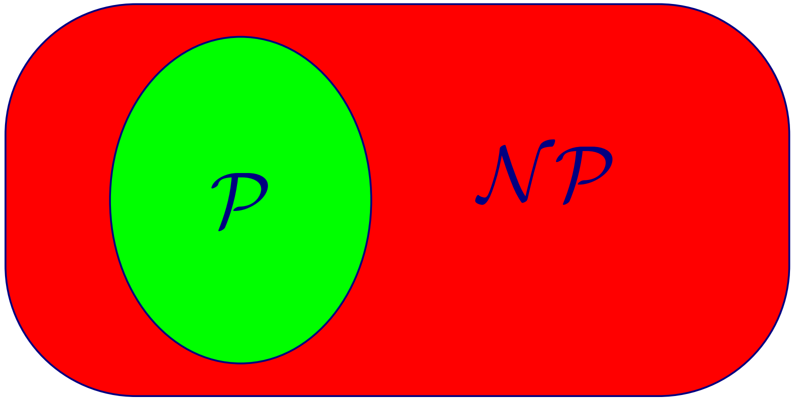

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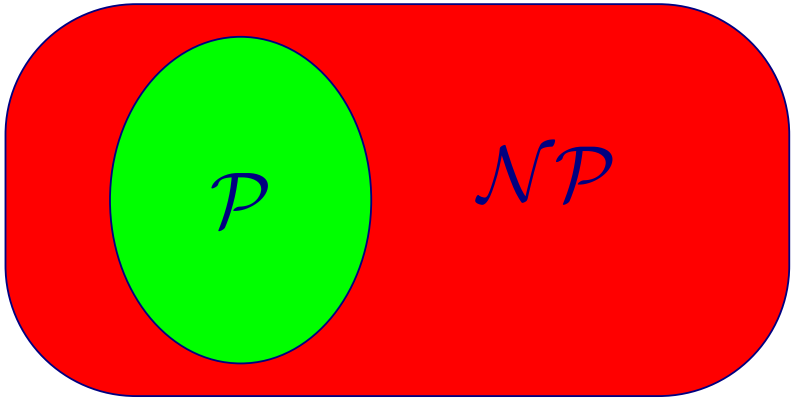

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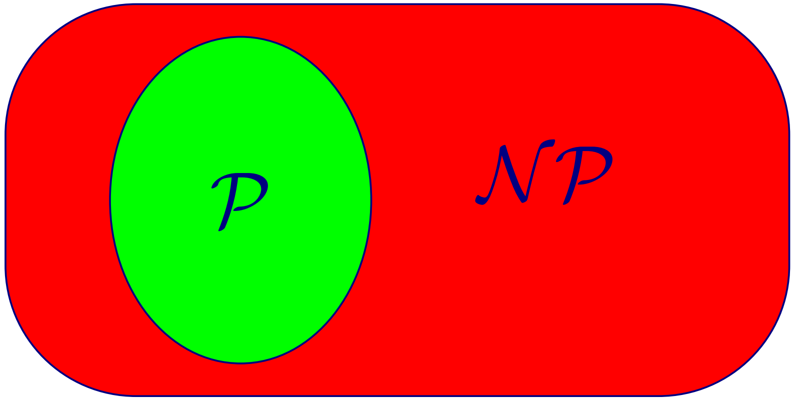

➡ Question: is $\mathcal{P} = \mathcal{NP}$? ➡ $1,000,000 for the answer to this question!
see `http://www.claymath.org/millennium/P_vs_NP/`

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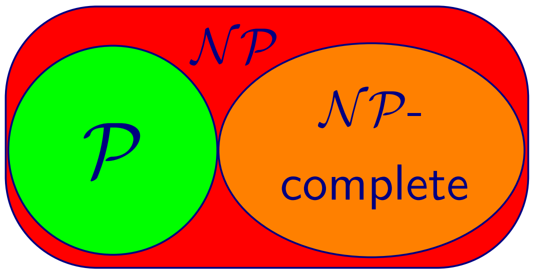

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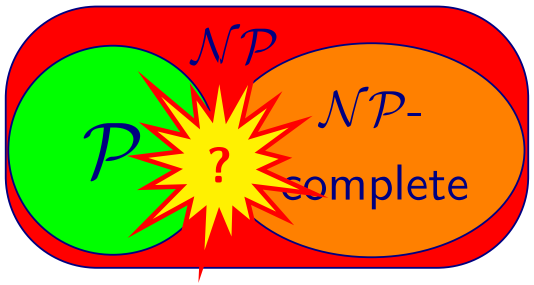

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