Mathematical Tools for Engineering and Management

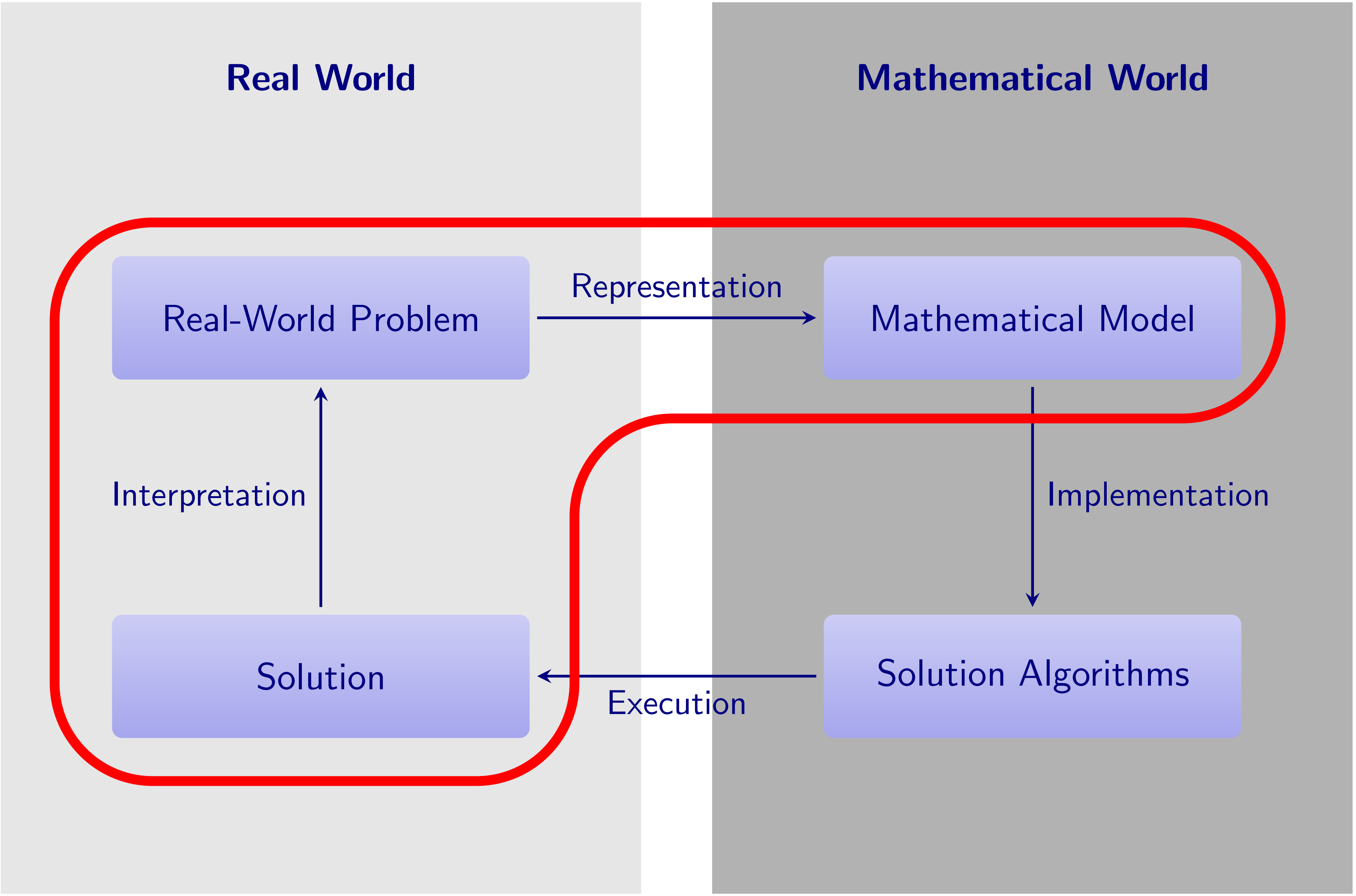
Lecture 12

18 Jan 2012

Overview

- ▷ Models, Data and Algorithms
- ▷ Linear Optimization
- ▷ Mathematical Background: Polyhedra, Simplex-Algorithm
- ▷ Sensitivity Analysis; (Mixed) Integer Programming
- ▷ MIP Modelling
- ▷ MIP Modelling: More Examples; Branch & Bound
- ▷ Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- ▷ TSP-Heuristics
- ▷ Network Flows
- ▷ Shortest Path Problem
- ▷ Complexity Theory
- ▷ Nonlinear Optimization
- ▷ Scheduling (Jan 25)
- ▷ Lot Sizing (Feb 01)
- ▷ Summary (Feb 08)
- ▷ Oral exam (Feb 15)

Nonlinear optimization



Real World

Mathematical World

Real-World Problem

Representation

Mathematical Model

Implementation

Solution Algorithms

Execution

Solution

Interpretation

- Production Planning in Automobile Industry

Product	**Beetle**	**Cabrio**
Revenue	$10000	$20000
Manufacturing	5h	3h
Assembly	4h	7h
Raw material	400kg	400kg

Plant capacity and available raw materials:

- Manufacturing capacity: 50h
- Assembly capacity: 70h
- Raw material: 4500kg

Production Planning revisited

▷ Production Planning in Automobile Industry



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▷ More realistic: Price of Cabrio depending on the demand

Some assumptions

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- Specific values assumed for cabrios: $k := 20000$, $K := 150000$, and $E := -2$
  - ➡ Objective (total revenue): $10000 x_\mathsf{b} + 150000\sqrt{x_\mathsf{c}} - 20000 x_\mathsf{c}$

Nonlinear optimization model

- Model (non-linear program):

$$
\begin{array}{lrcll}
\text{maximize} & 10x_{\mathsf{b}} + 150\sqrt{x_{\mathsf{c}}} - 20x_{\mathsf{c}} & & & \\
\text{subject to} & 4x_{\mathsf{b}} + 4x_{\mathsf{c}} & \leq & 45 & \text{(total raw material)} \\
 & 5x_{\mathsf{b}} + 3x_{\mathsf{c}} & \leq & 50 & \text{(time in manufacturing)} \\
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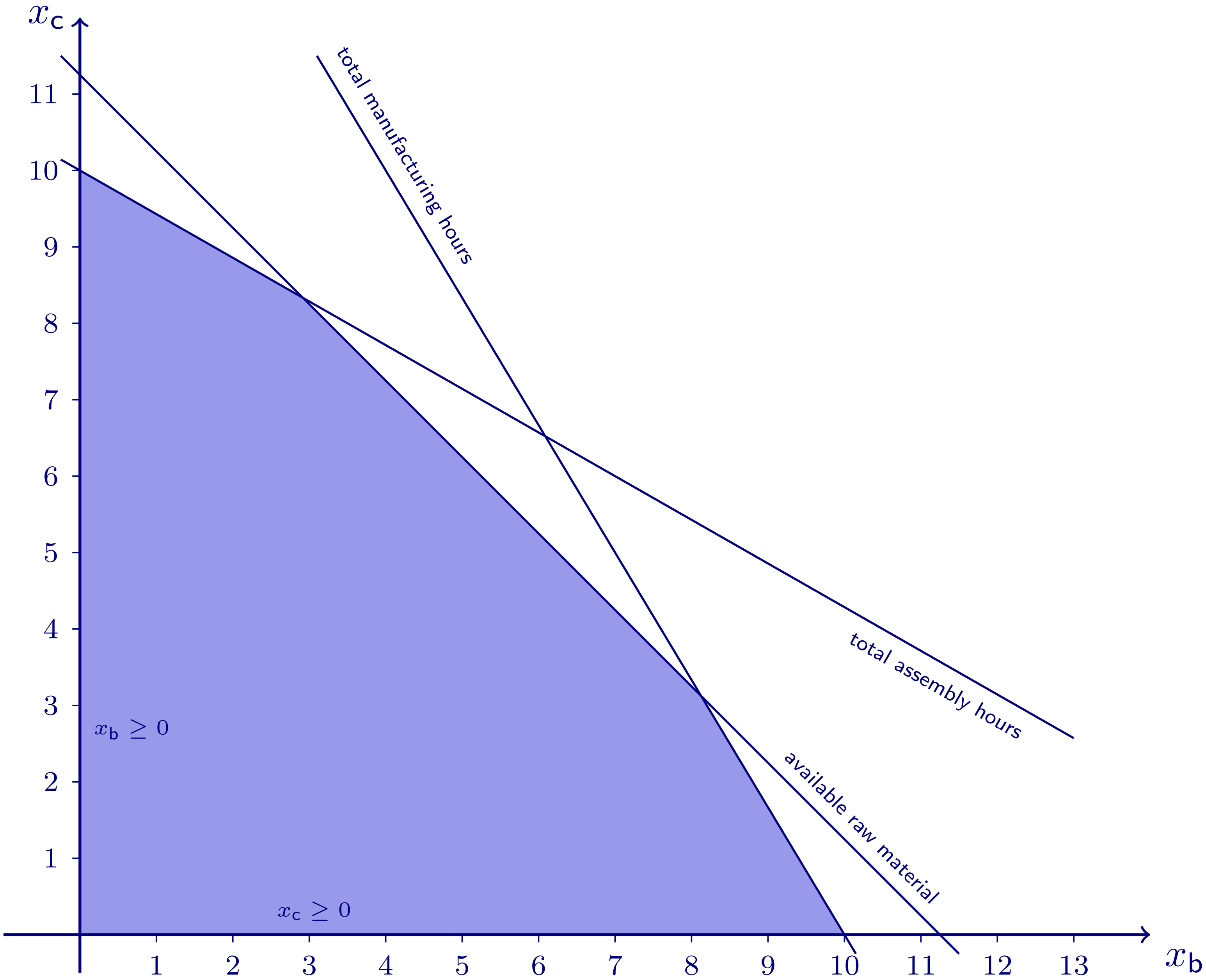
- Optimal solution: $(x_{\mathsf{b}}, x_{\mathsf{c}}) \; = \; (5, 6.25)$
  - ➡ Price for one cabrio at this demand: 60000
  - ➡ Profit for one cabrio: 40000

Geometric interpretation



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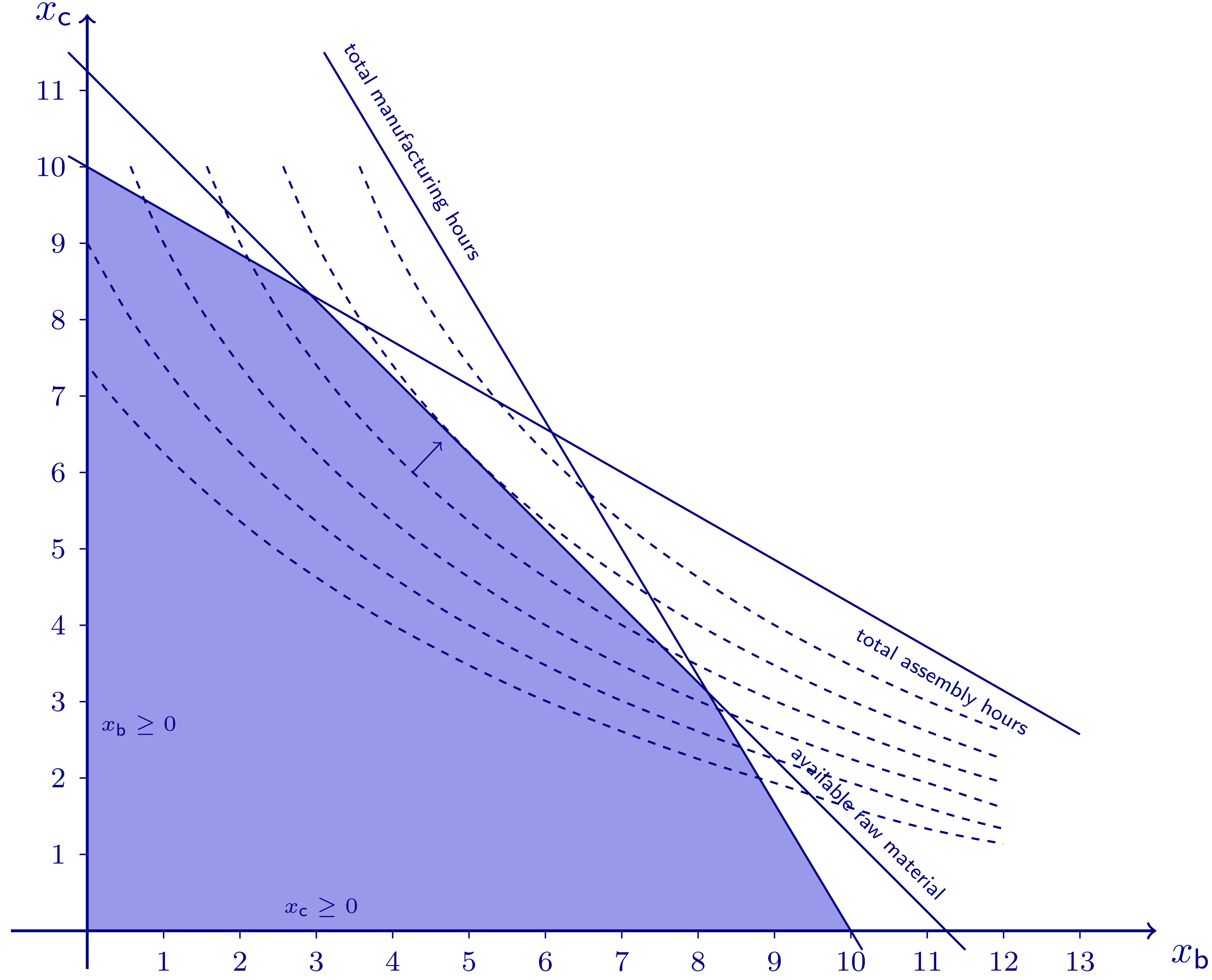

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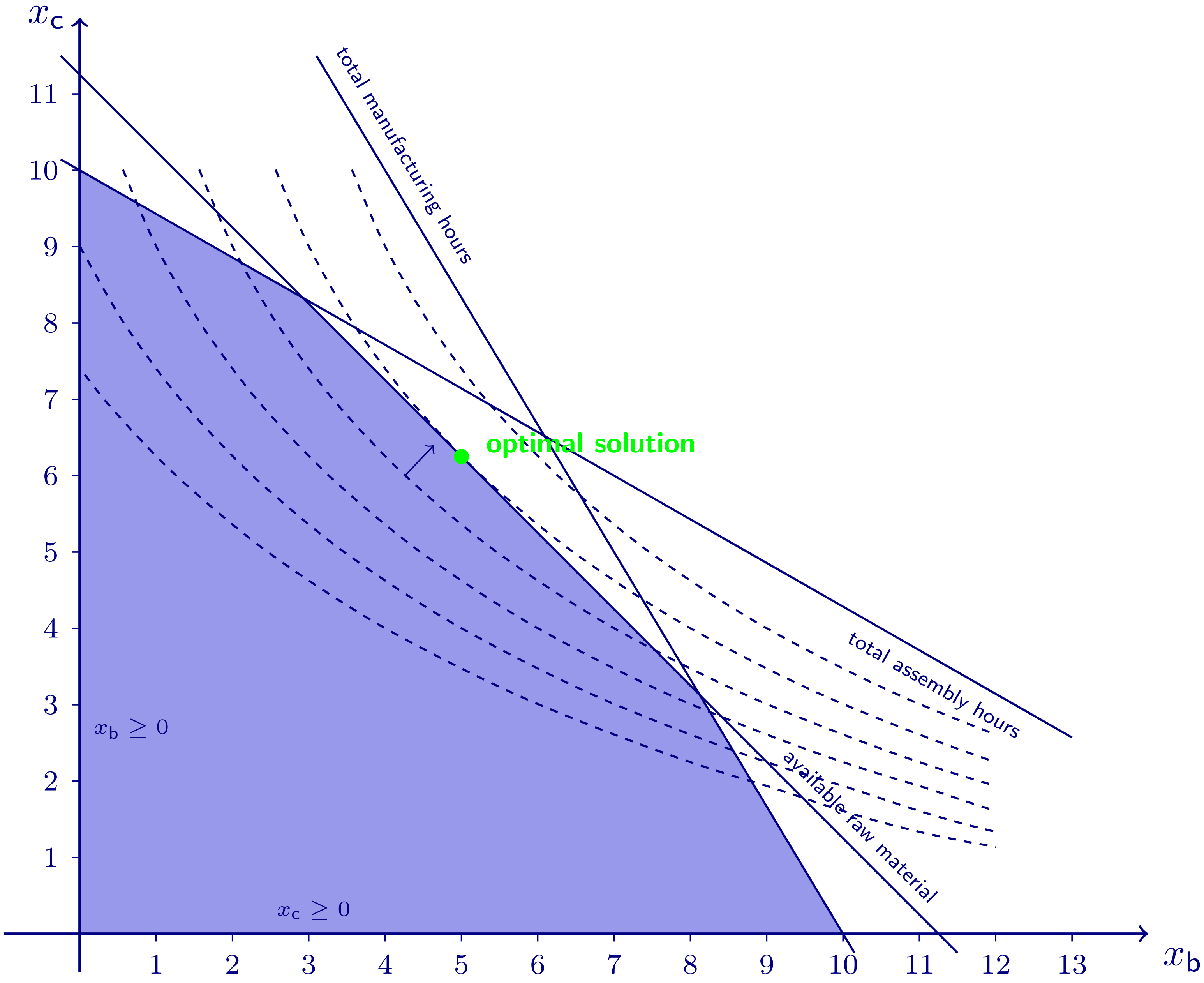

Linear and non-linear expressions

▷ Linear optimization

➡ Linear objective **and** linear constraints

Linear and non-linear expressions

- **Linear optimization**

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➡ Sum of linear terms: $\ldots + a_i \cdot x_i + \ldots$

$$\underset{\textbf{parameter}}{\nearrow} \qquad \underset{\textbf{variable}}{\nwarrow}$$

◁ Linear and non-linear expressions ▷

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  - Products of variables: $x_i \cdot x_j$

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- Higher-order terms of variables: $x_i \cdot x_j \cdot x_k, \quad x_j^5 \cdot x_j$

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- Absolute values or maxima/minima: $|x_i|, \quad \max x_j$

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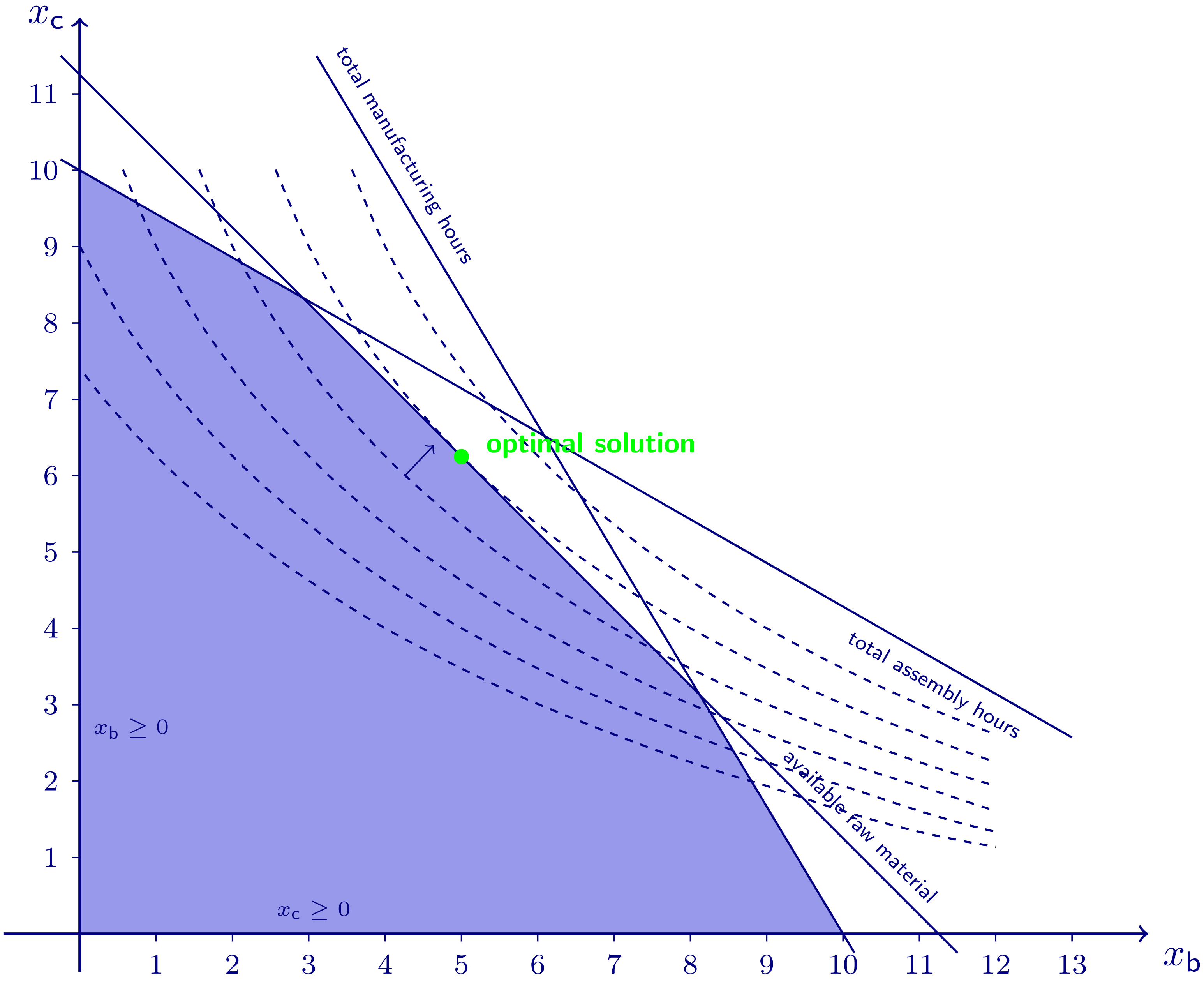
  - Absolute values or maxima/minima: $|x_i|, \quad \max x_j$
  - Terms including elementary functions: $\sin x_i, \quad 2^{x_i \cdot x_j}, \quad \frac{1}{\sqrt{x_i}}, \quad \log(x_i + x_j^{x_k})$

Non-linear objective



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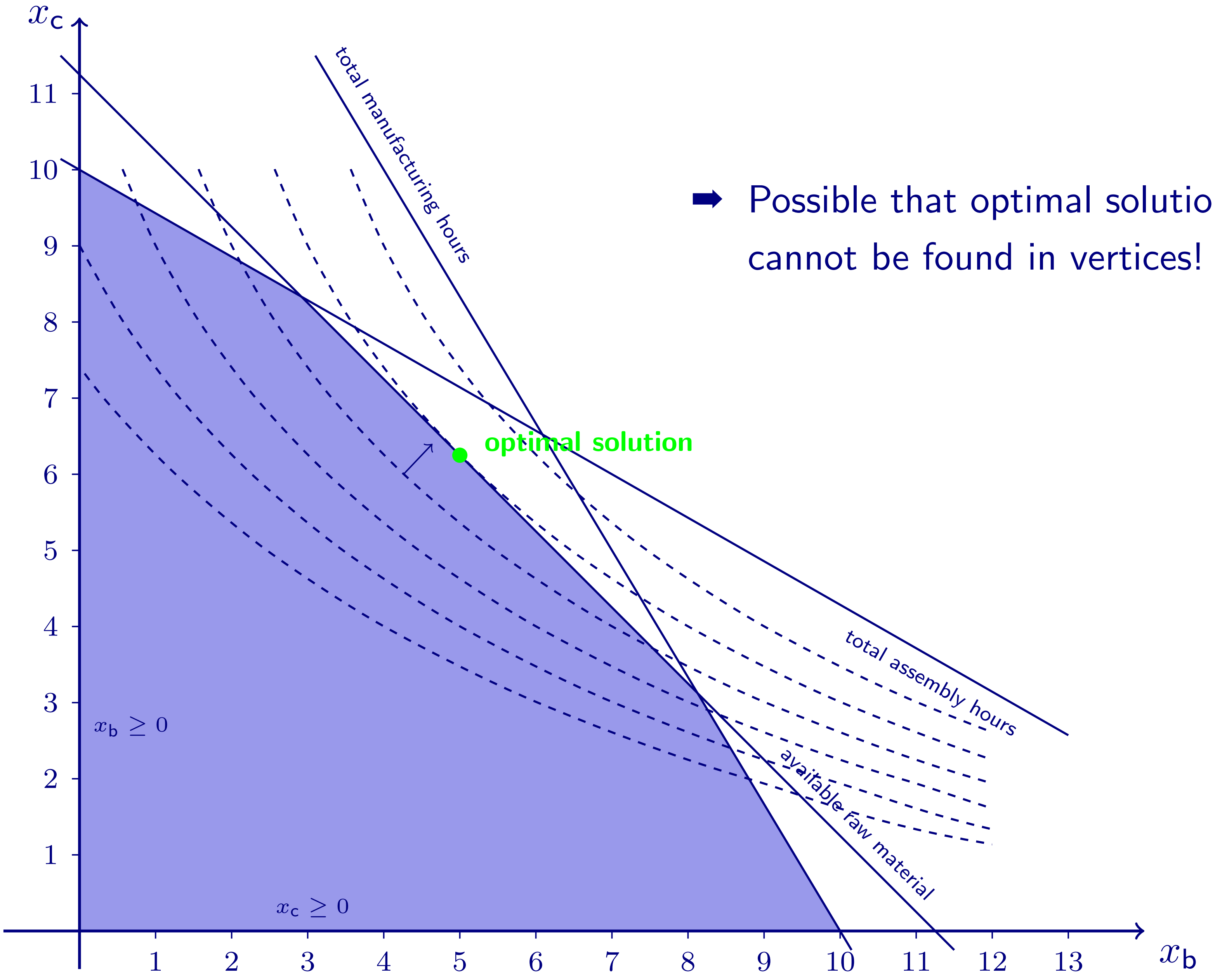

➡ Possible that optimal solutions cannot be found in vertices!

Non-linear objective: economies of scale

- Economy of scale

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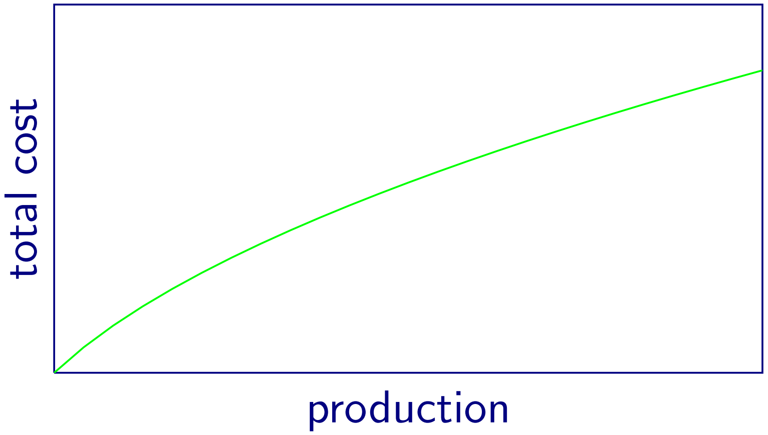
- **Economy of scale**
  - ➡ Production cost per item decrease with number x_i of produced items



Non-linear objective: economies of scale

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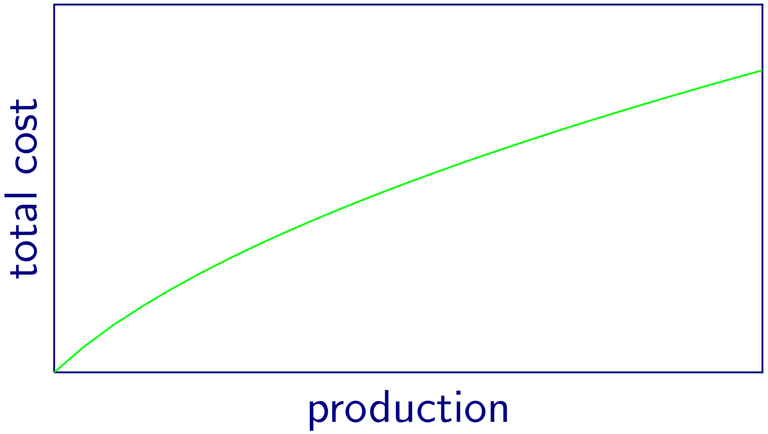
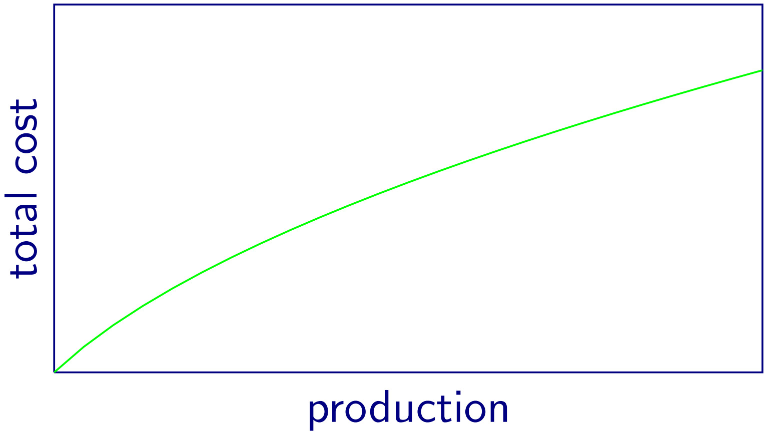


Non-linear objective: economies of scale



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- **Diseconomy of scale**

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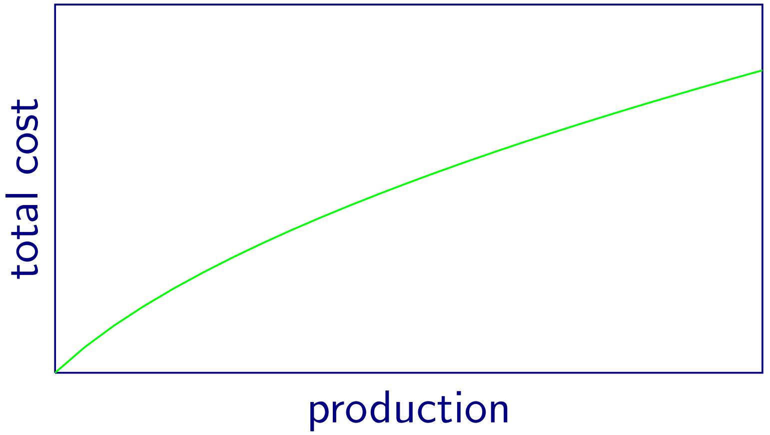

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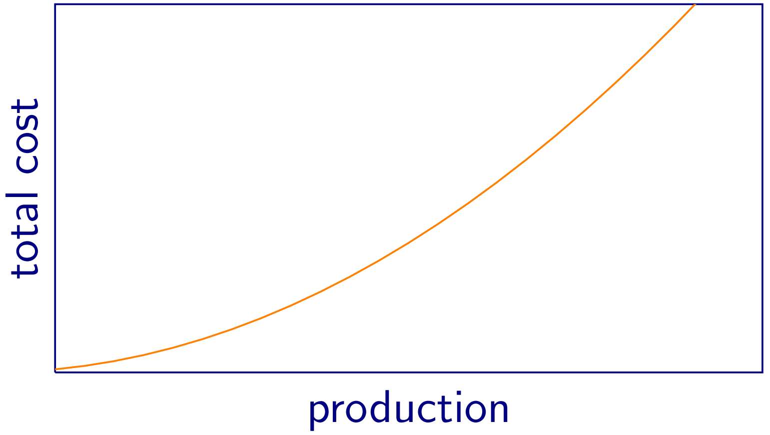

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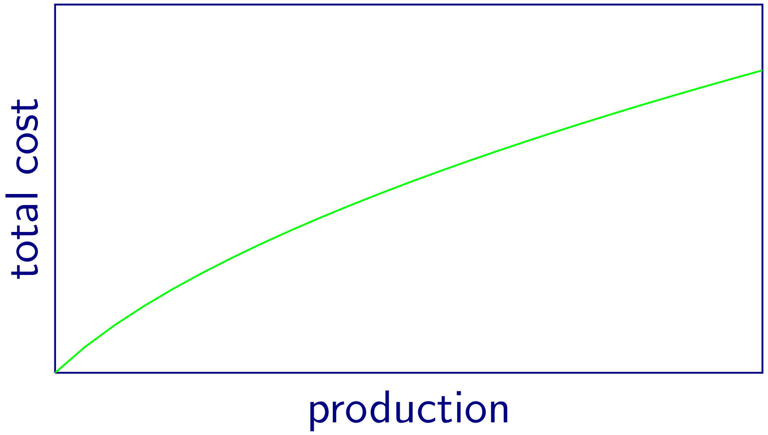

▷ **Diseconomy of scale**

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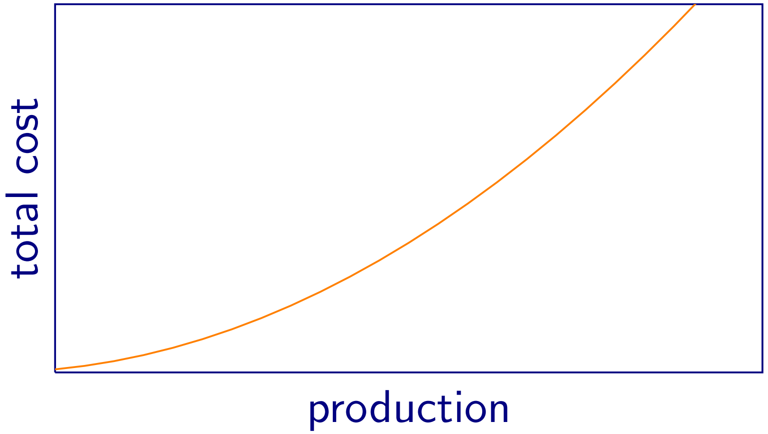
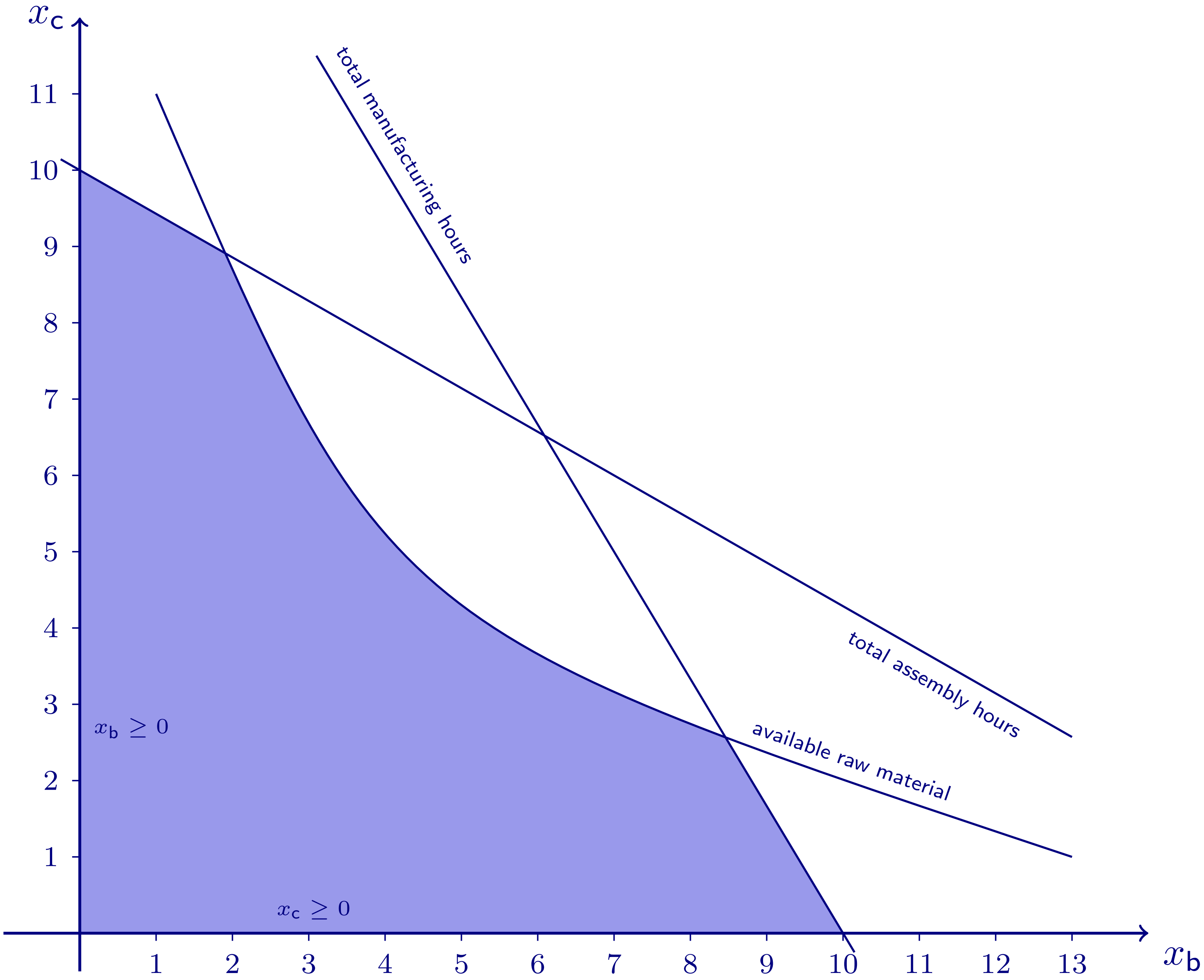
$$\ldots + x_i^2 + \ldots$$

$$\ldots + x_i \log x_i + \ldots$$



Non-linear constraints



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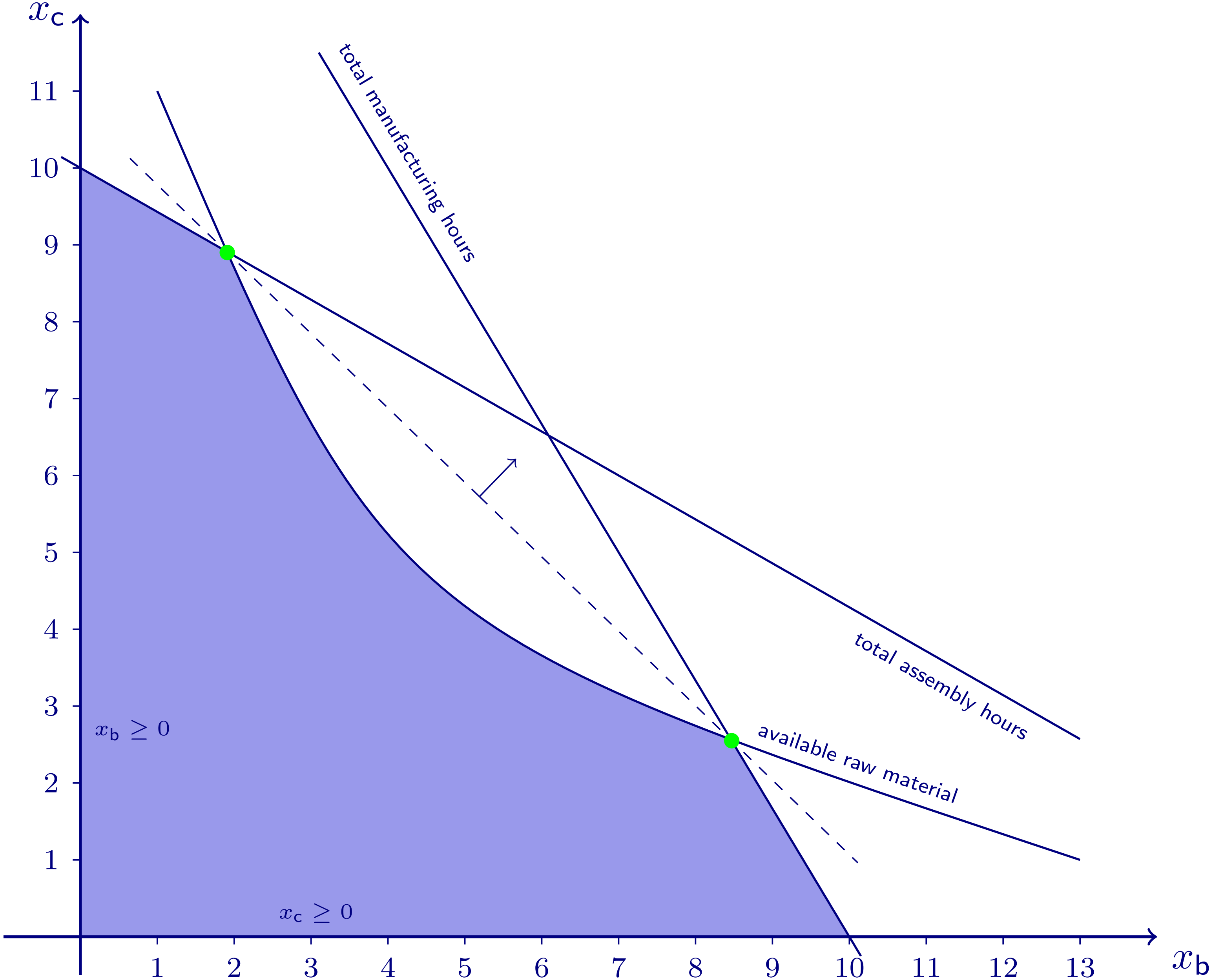

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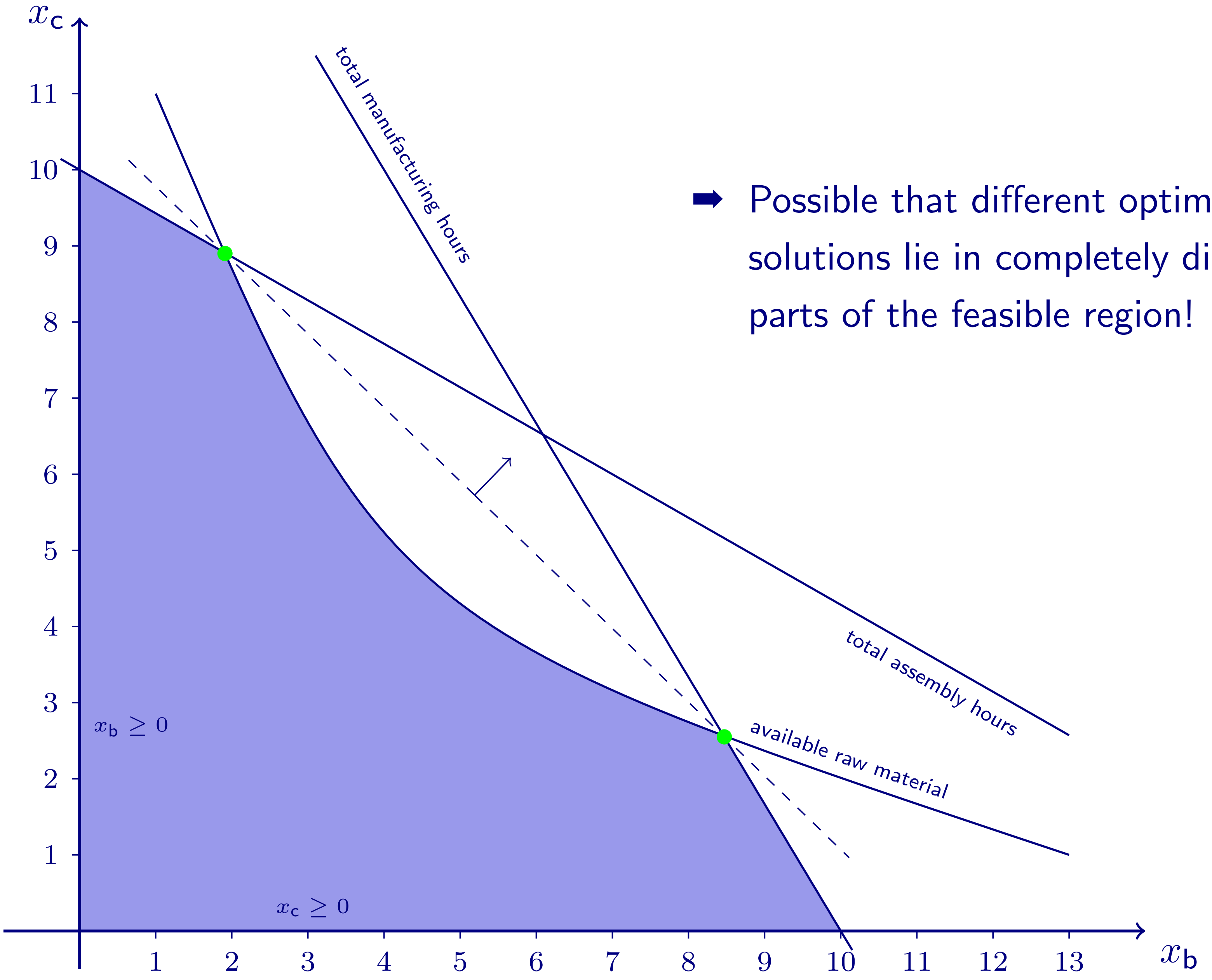

- ➡ Possible that different optimal solutions lie in completely different parts of the feasible region!

Example: facility location

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- Task: Find an optimal location for a central unit connecting every given location!

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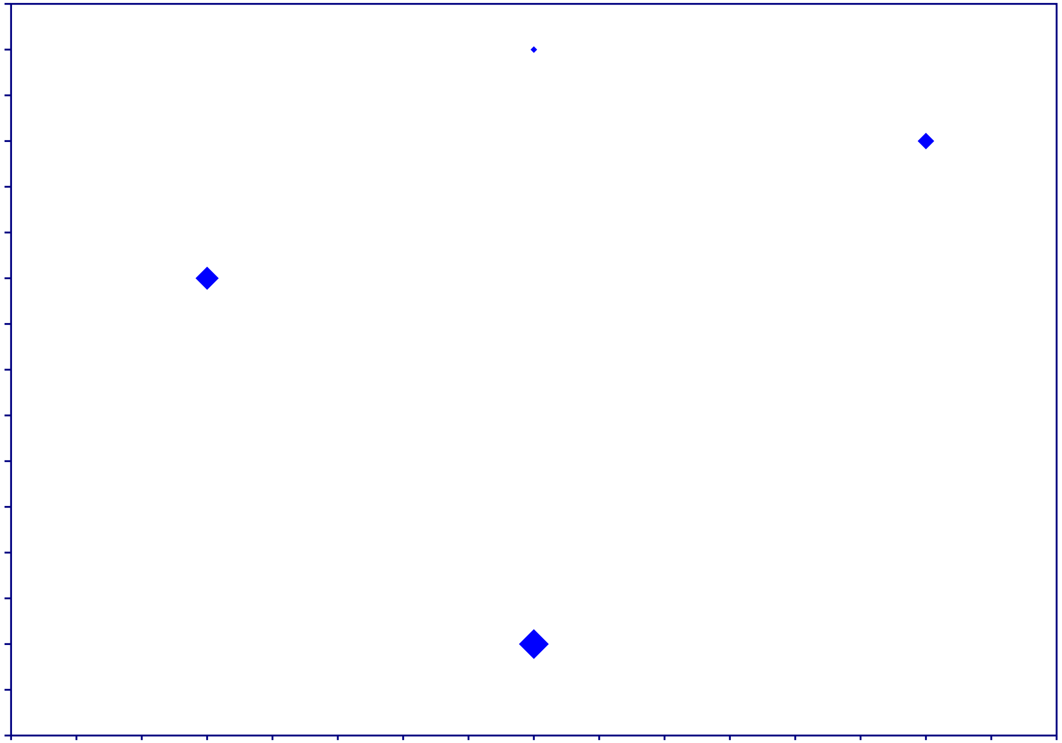

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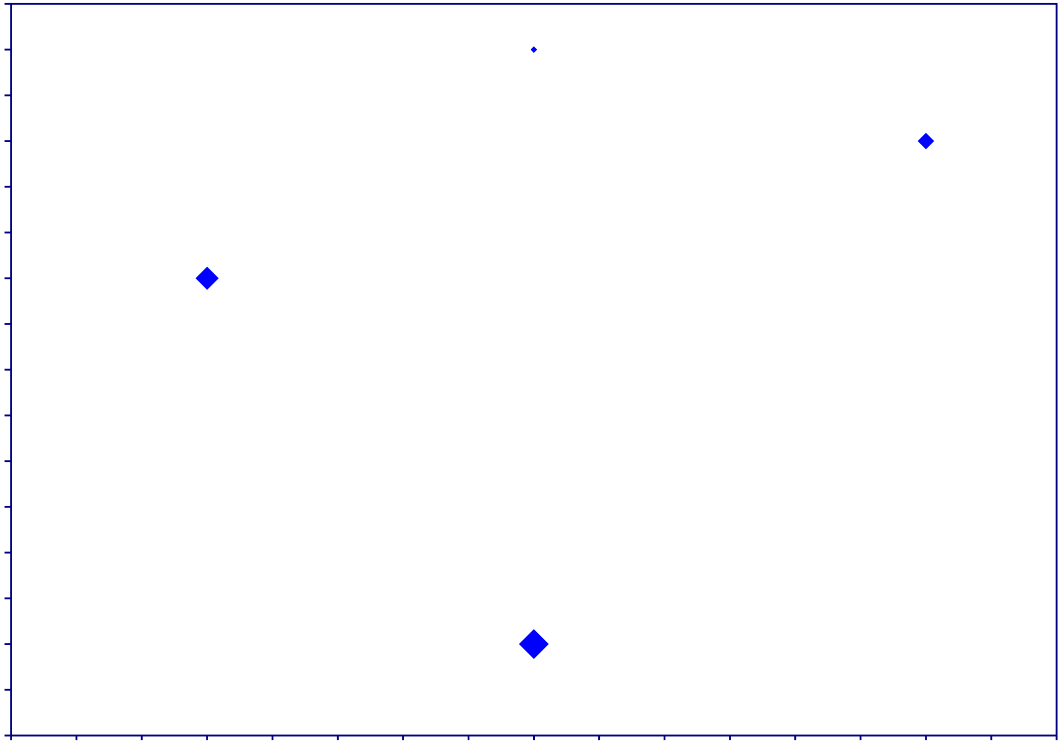

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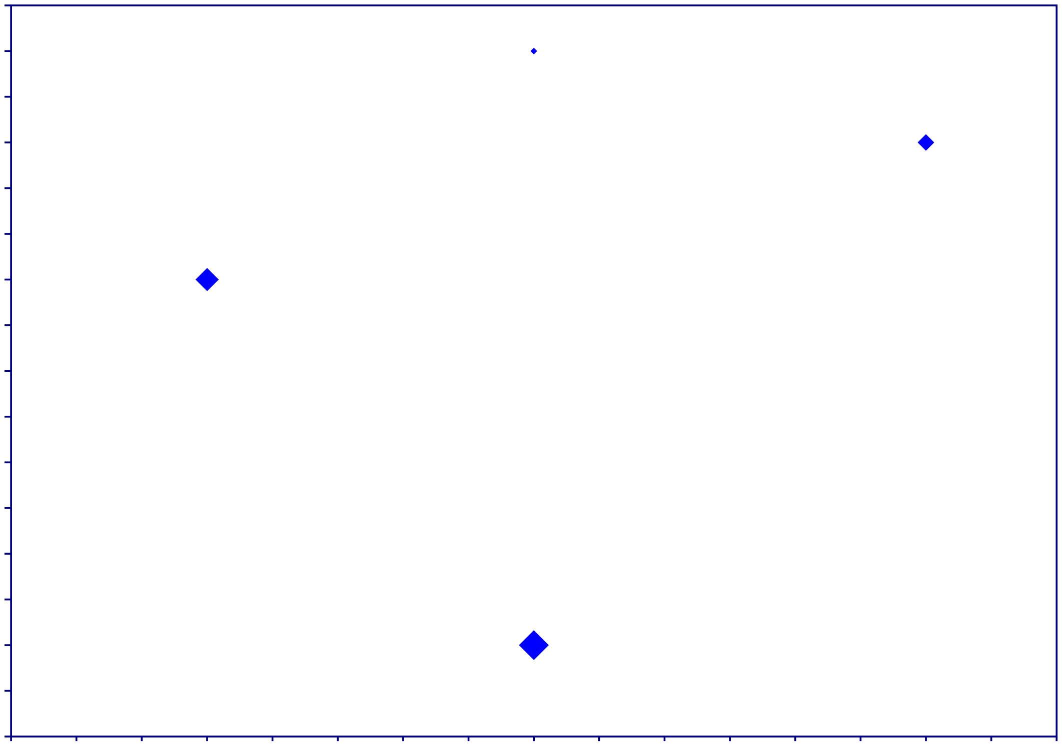

V Variables: x, y ➡ coordinates of central unit

Objective: minimize sum of connection costs to all given locations

Facility location – model

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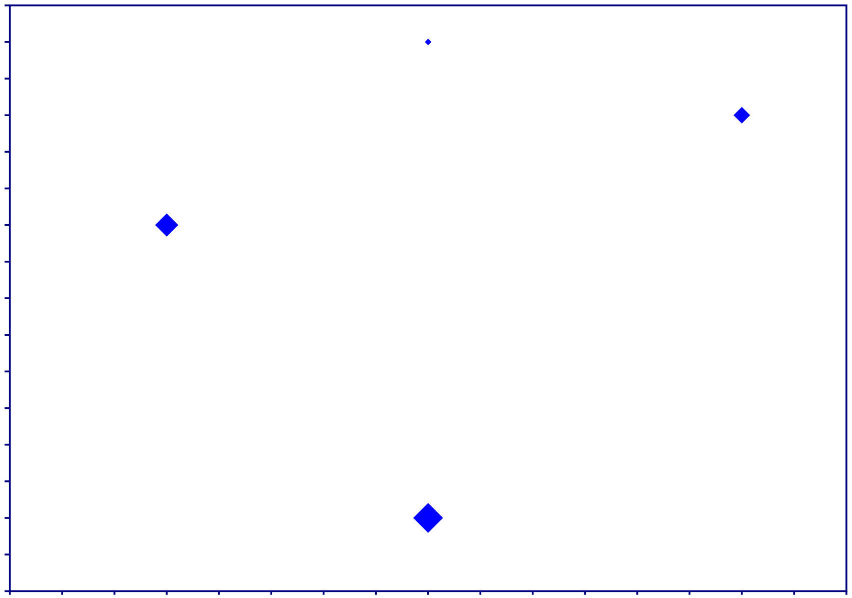

Facility location – model

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➡ Distance to **A**: $\sqrt{(x-8)^2+(y-2)^2}$

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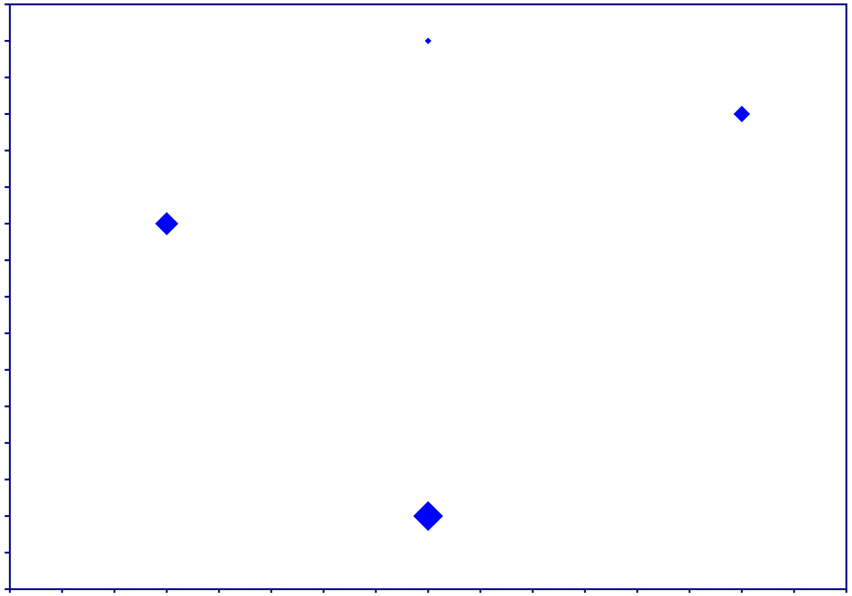

Facility location – model

V

Variables: x, y ➡ coordinates of central unit

➡ Distance to **A**: $\sqrt{(x-8)^2 + (y-2)^2}$

➡ Cost for all connections to **A**: $9 \cdot \sqrt{(x-8)^2 + (y-2)^2}$

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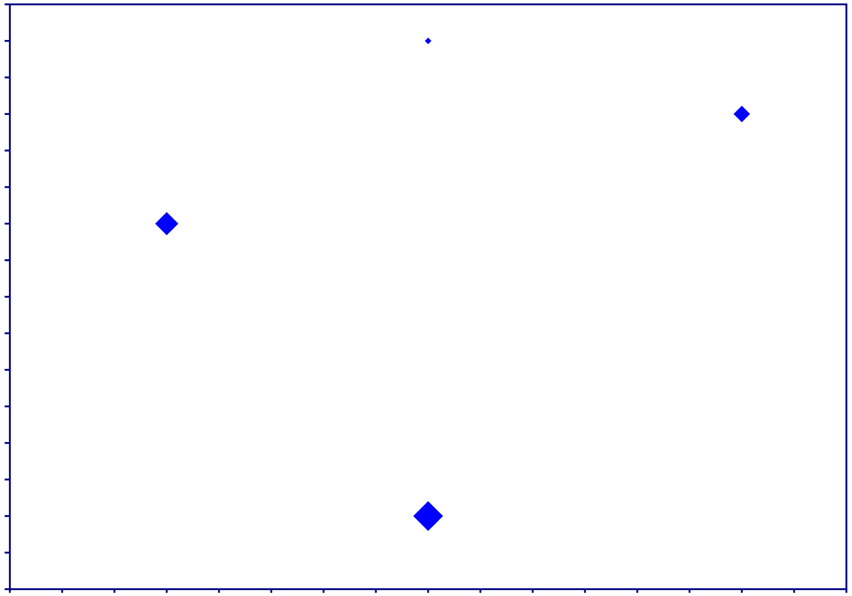

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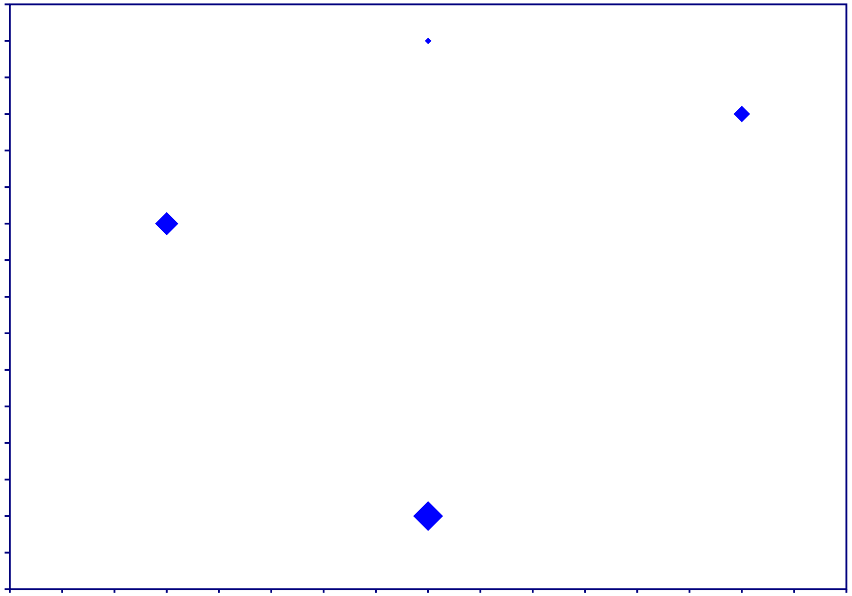

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- ➡ Cost for all connections to **A**: $9 \cdot \sqrt{(x-8)^2 + (y-2)^2}$
- ➡ Analogous for **B**, **C**, and **D**
- ➡ **Objective**:

$$\min \quad 9\sqrt{(x-8)^2 + (y-2)^2} + 7\sqrt{(x-3)^2 + (y-10)^2} + 2\sqrt{(x-8)^2 + (y-15)^2} + 5\sqrt{(x-14)^2 + (y-13)^2}$$

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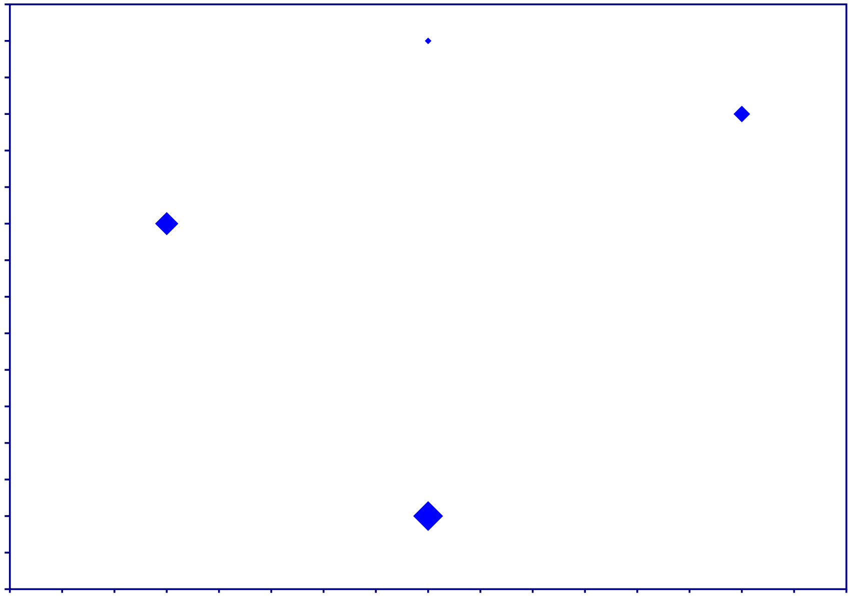

Facility location – solution

V Variables: x, y ➡ coordinates of central unit

Objective:
$$\min \quad 9\sqrt{(x-8)^2+(y-2)^2} + 7\sqrt{(x-3)^2+(y-10)^2} + 2\sqrt{(x-8)^2+(y-15)^2} + 5\sqrt{(x-14)^2+(y-13)^2}$$

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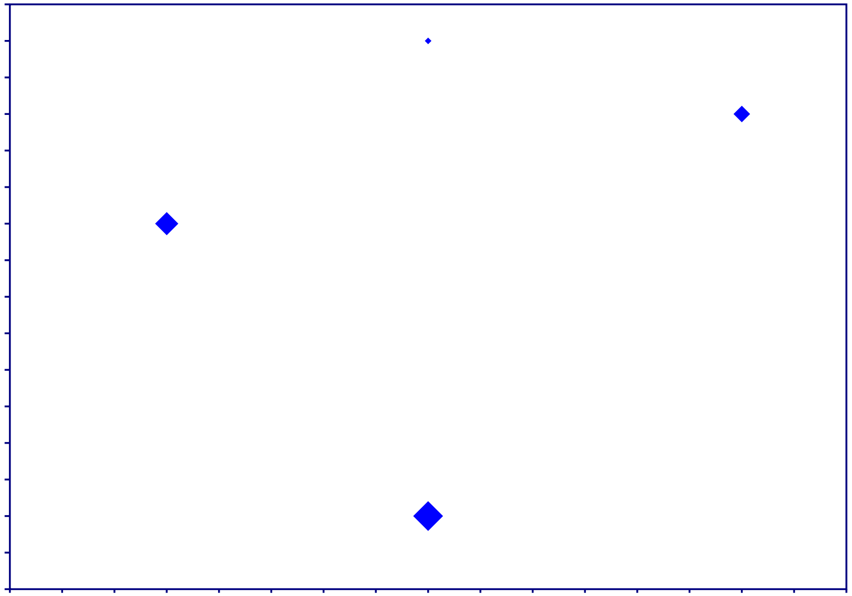

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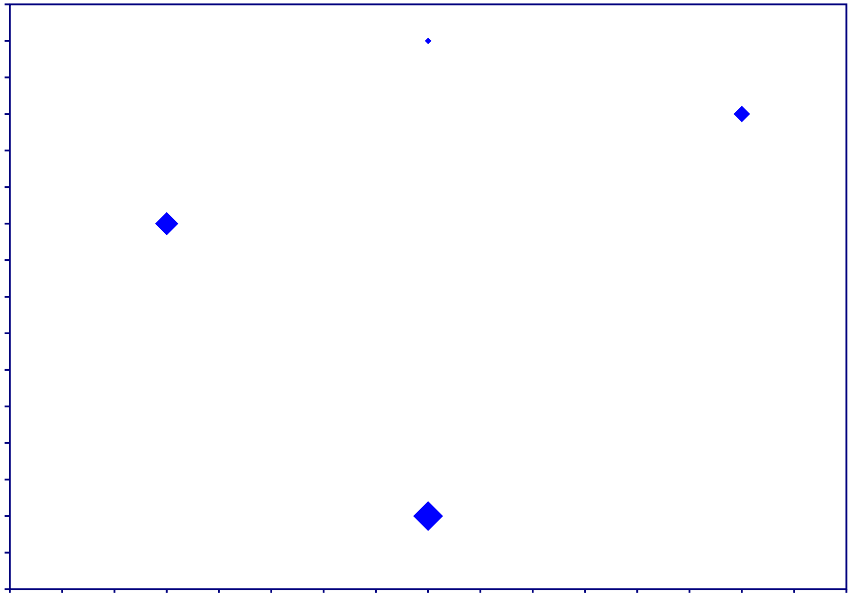

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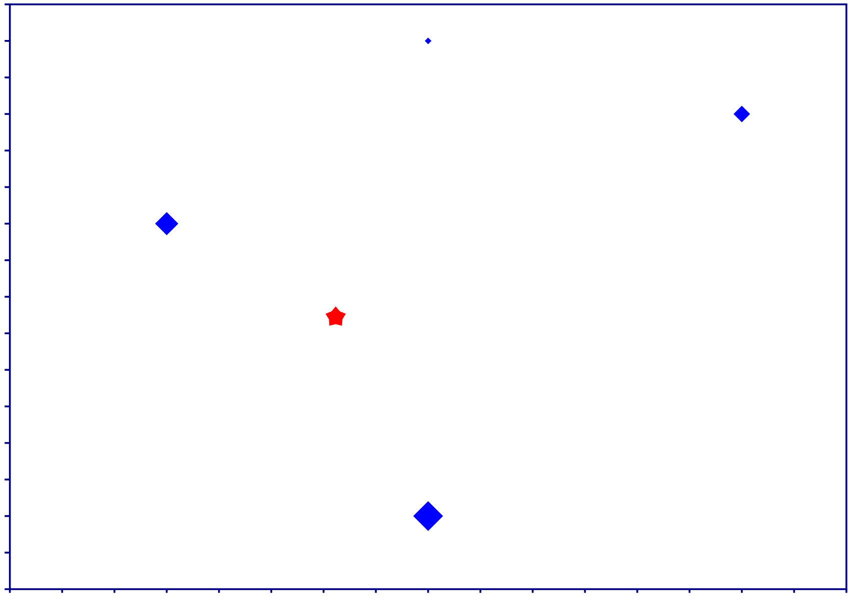

Facility location – additional constraints

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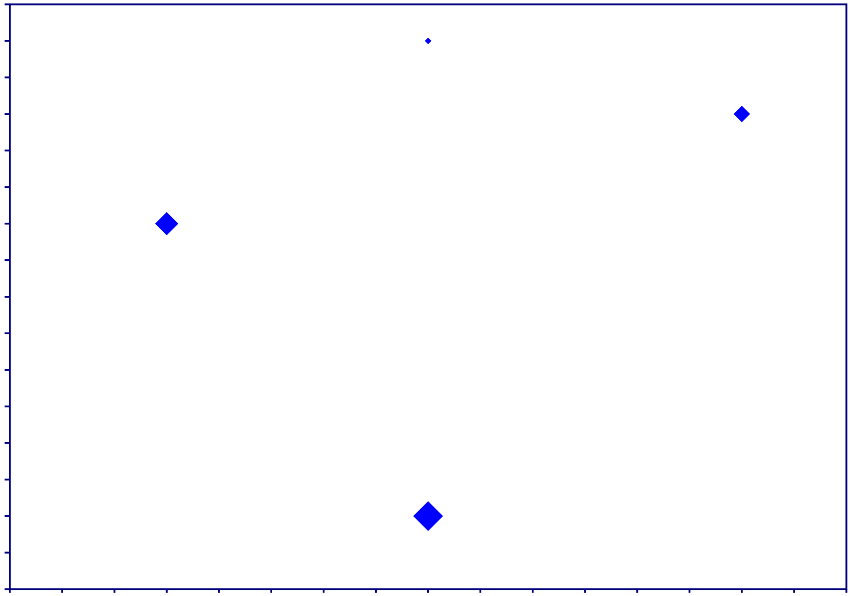

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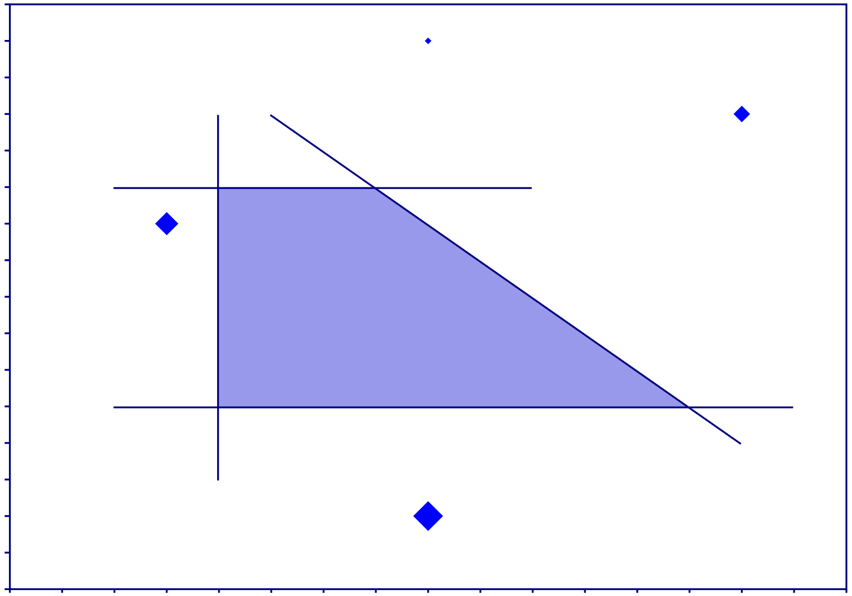

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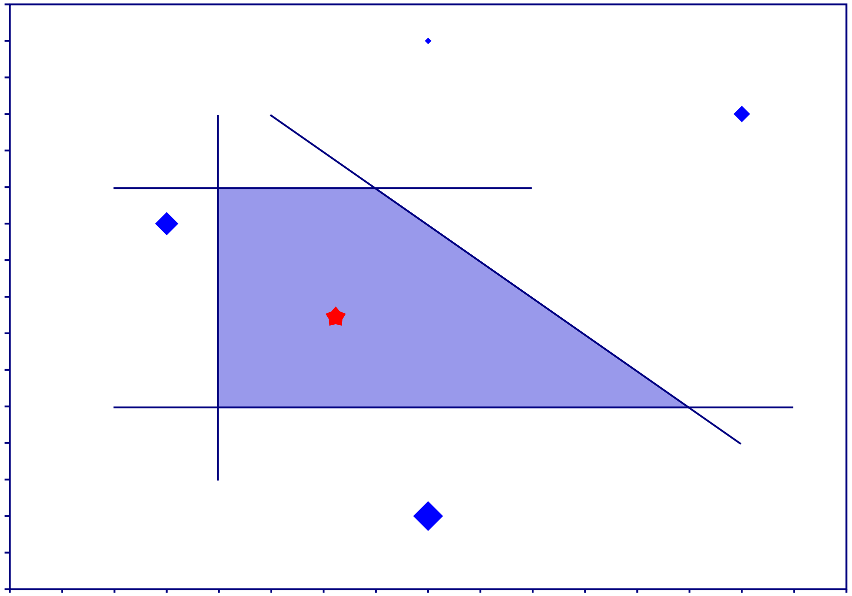

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➡ How to find an optimal solution...?

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  - ➡ Level sets are straight lines
    (in higher dimension: hyperplanes)
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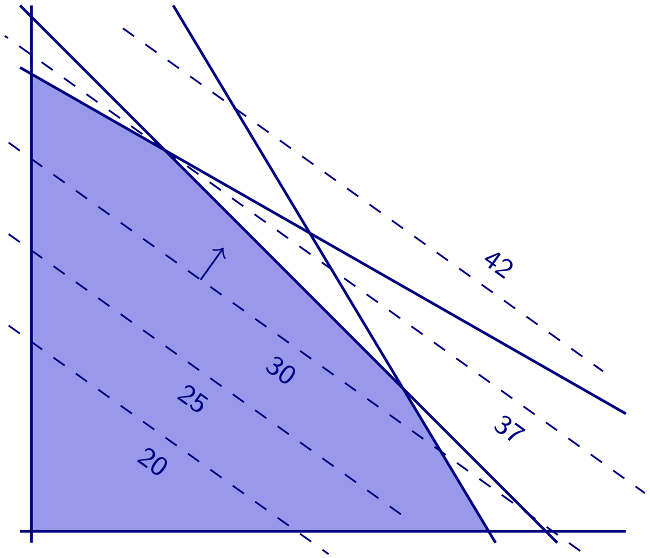

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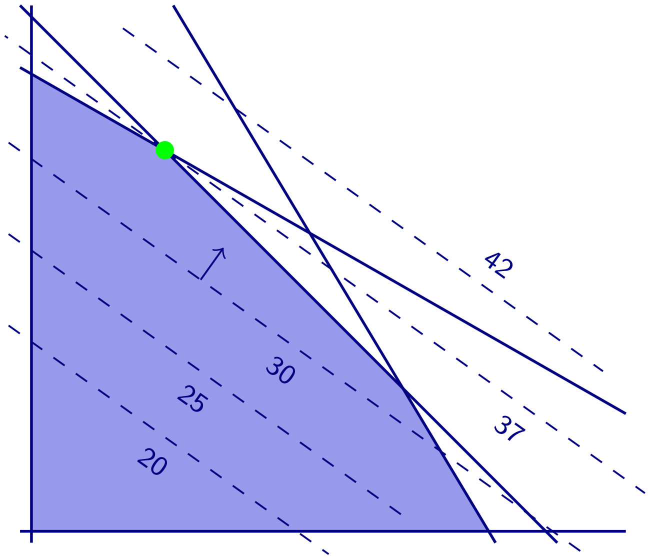

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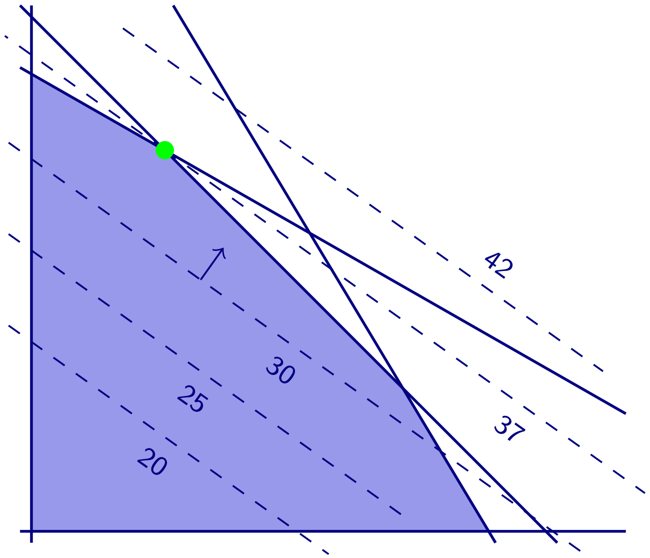

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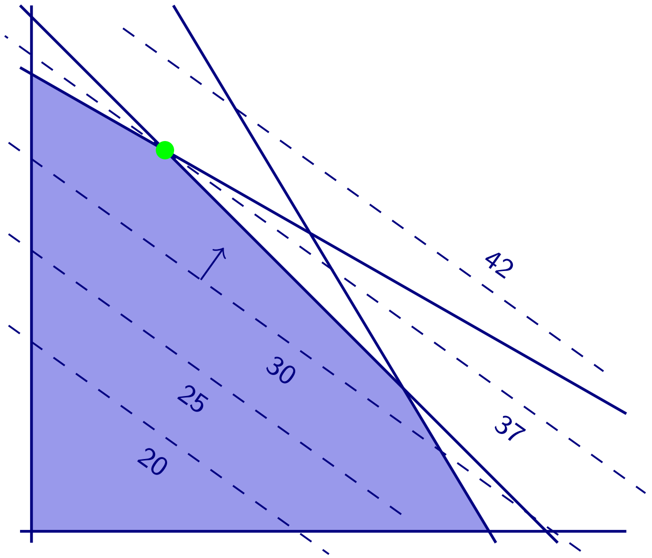

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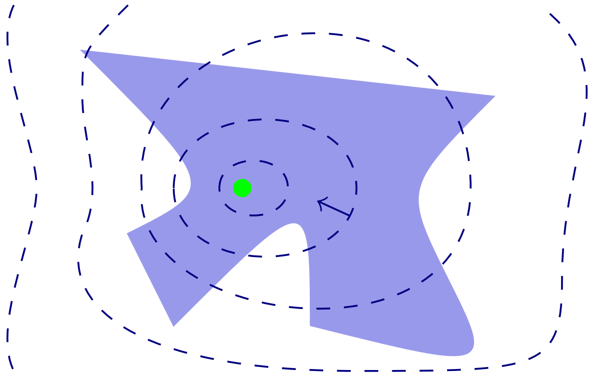

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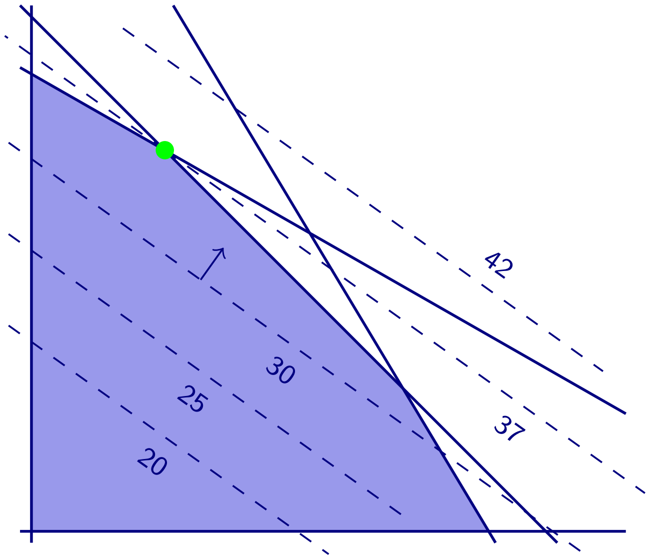

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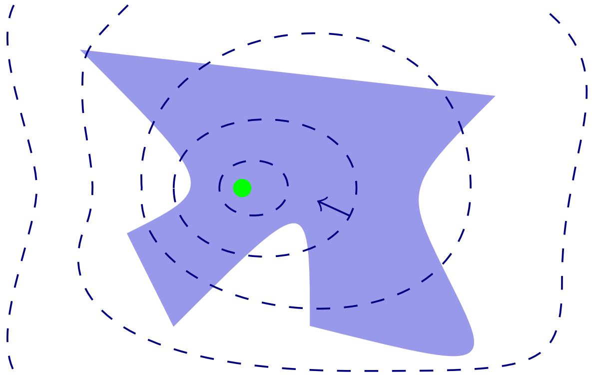

➡ Finding optimal solution can be difficult

Local and global optima

- Non-linear model:

$$\begin{aligned} \max \quad & \sqrt{(x-4)^2+(y-4)^2} \\ \text{s.t.} \quad x &\geq 2 \\ x &\leq 5 \\ -x+y &\leq 2 \\ x+y &\leq 10 \\ x-3y &\leq -4 \end{aligned}$$

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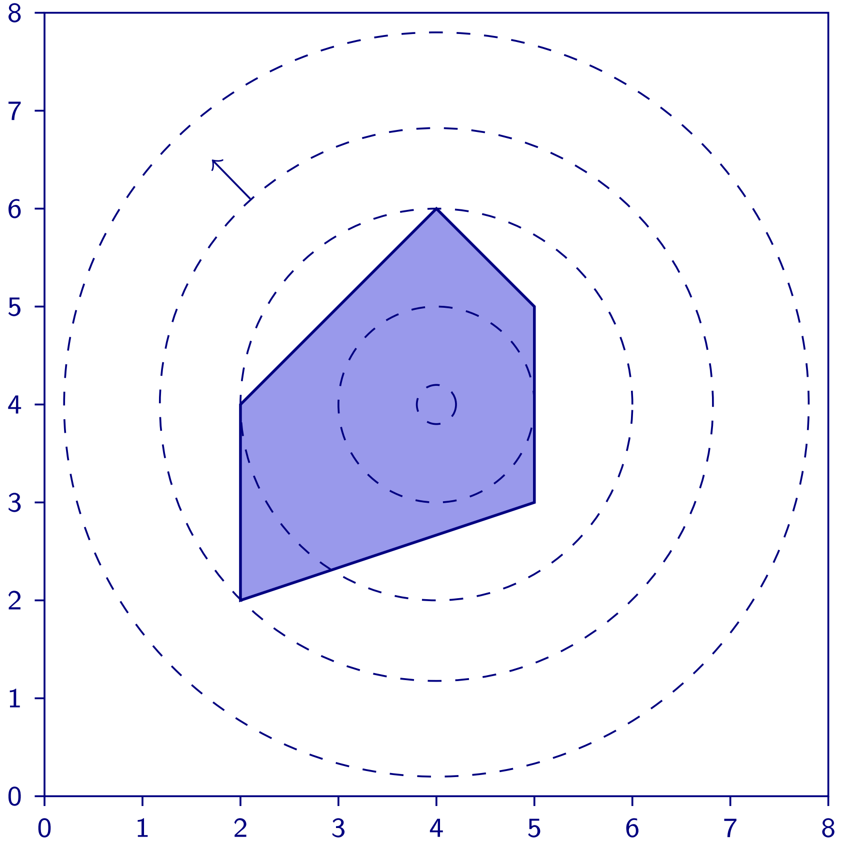

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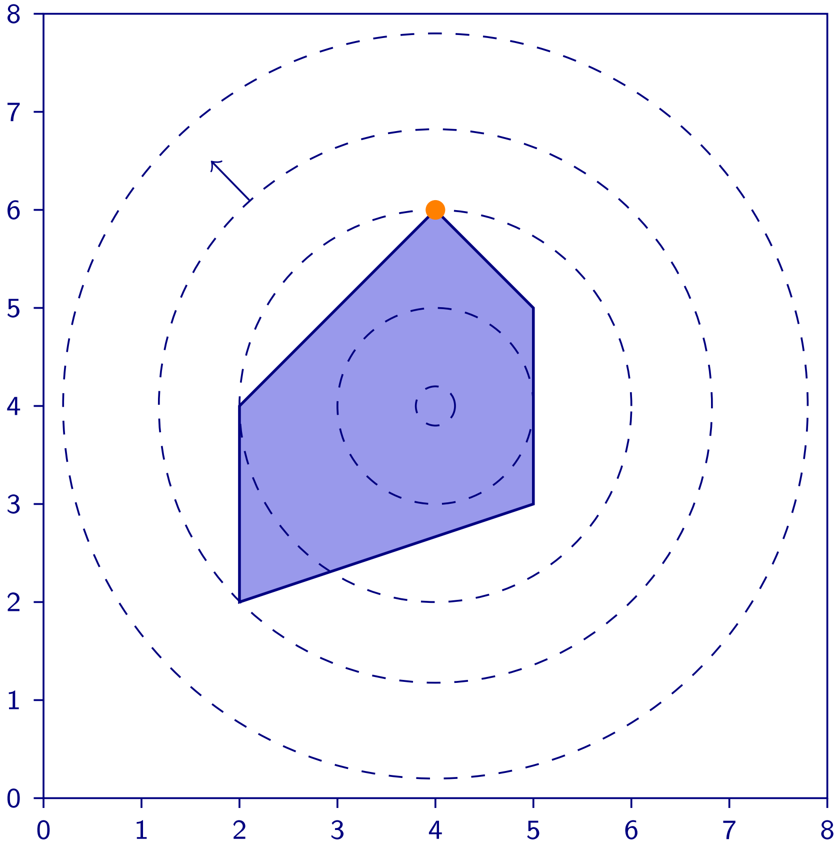

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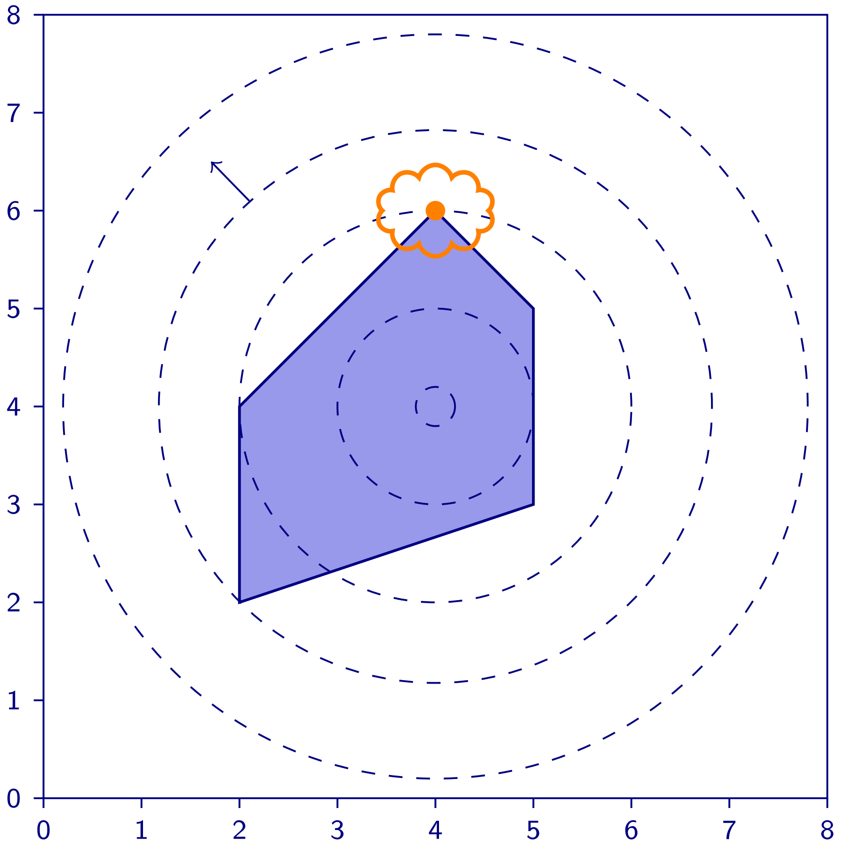

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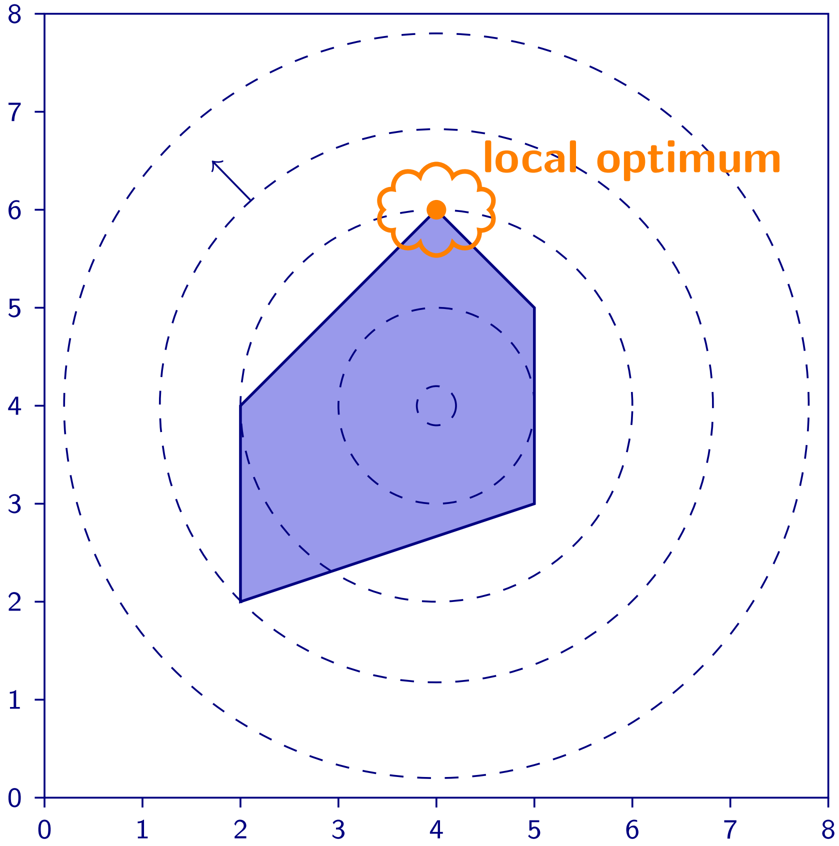

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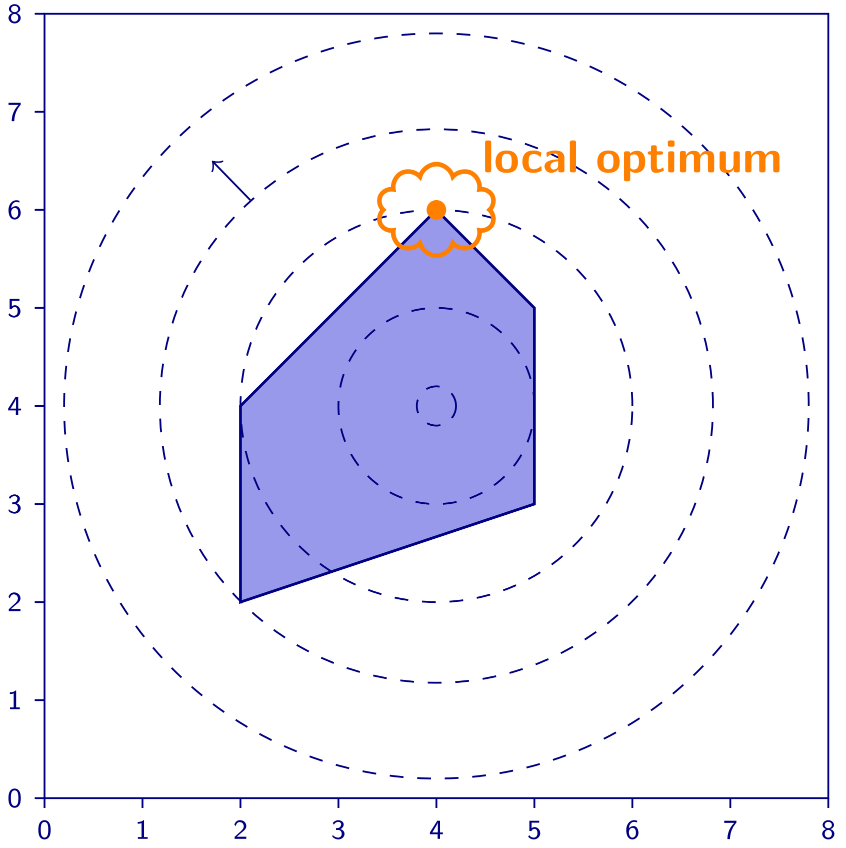

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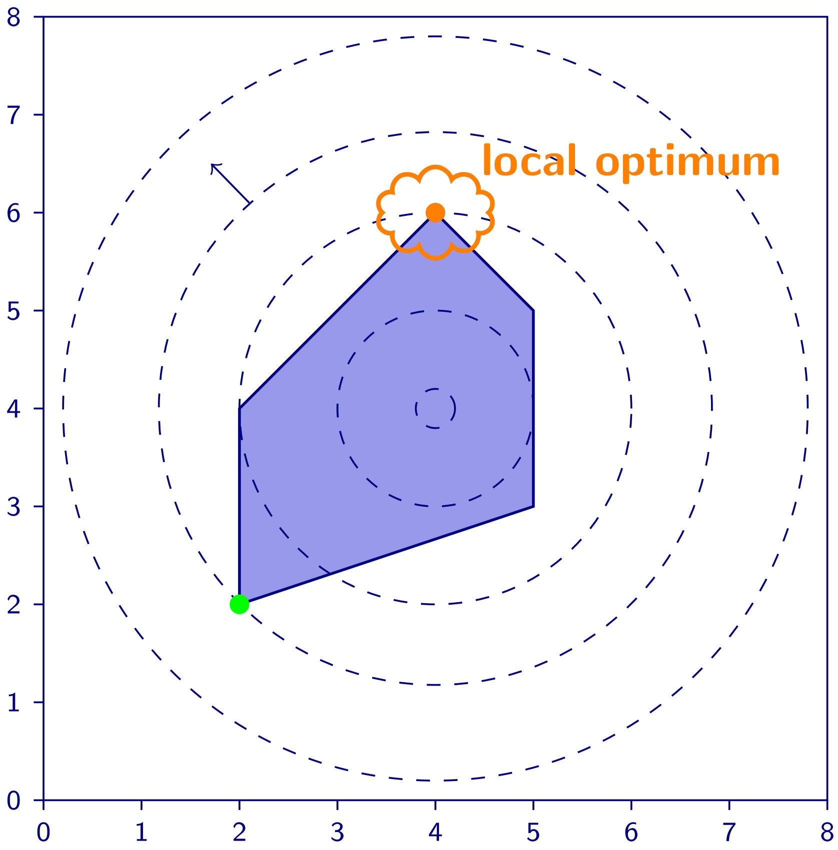

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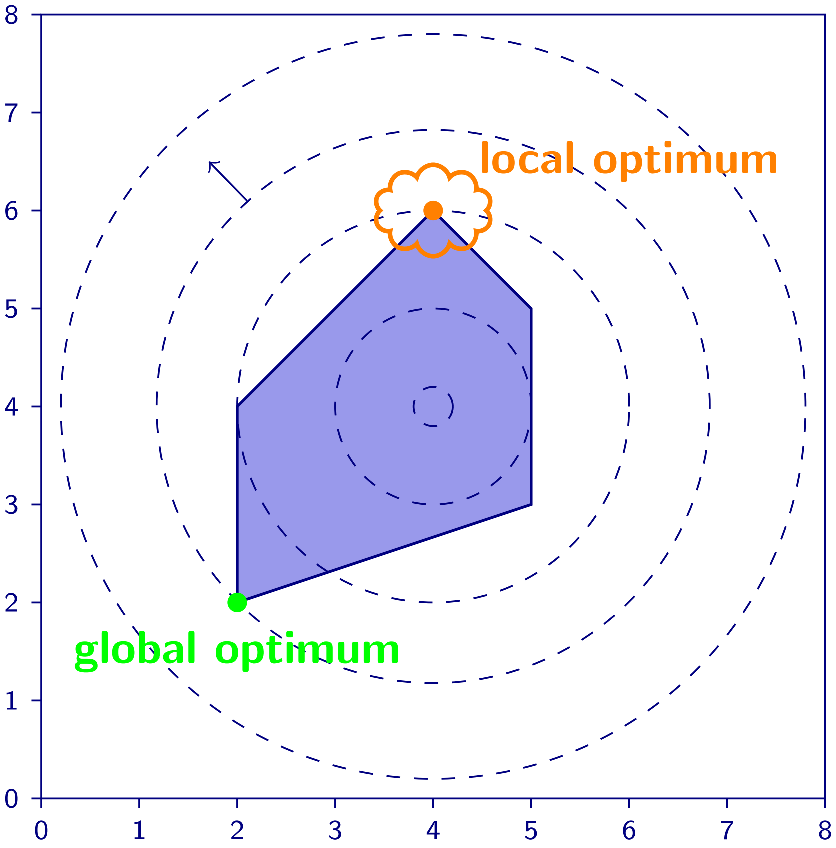

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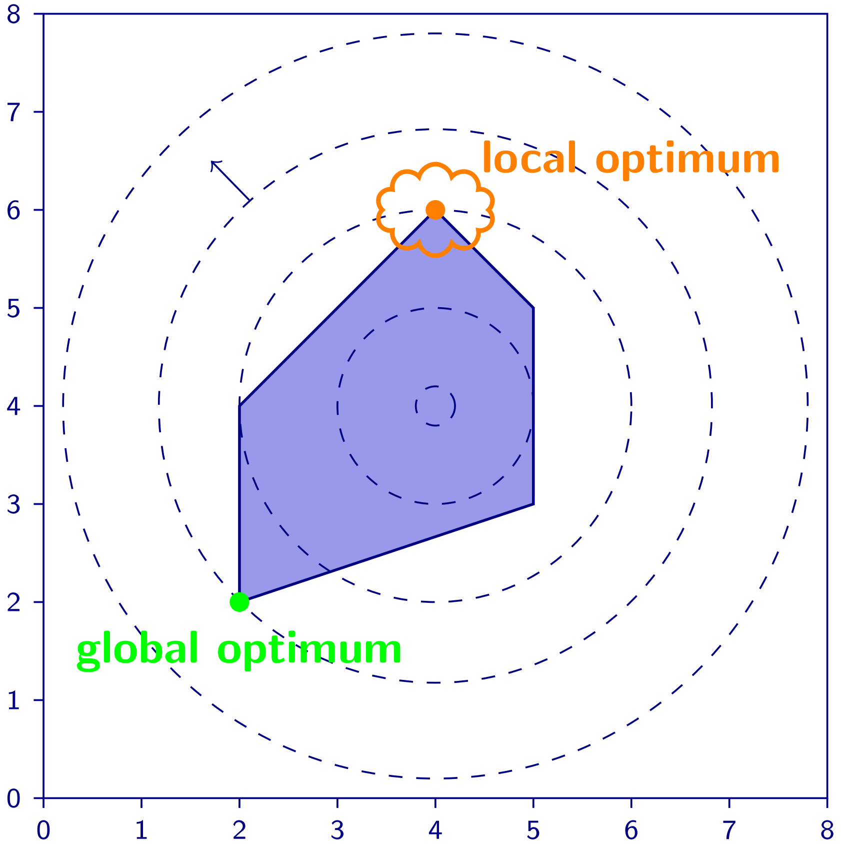

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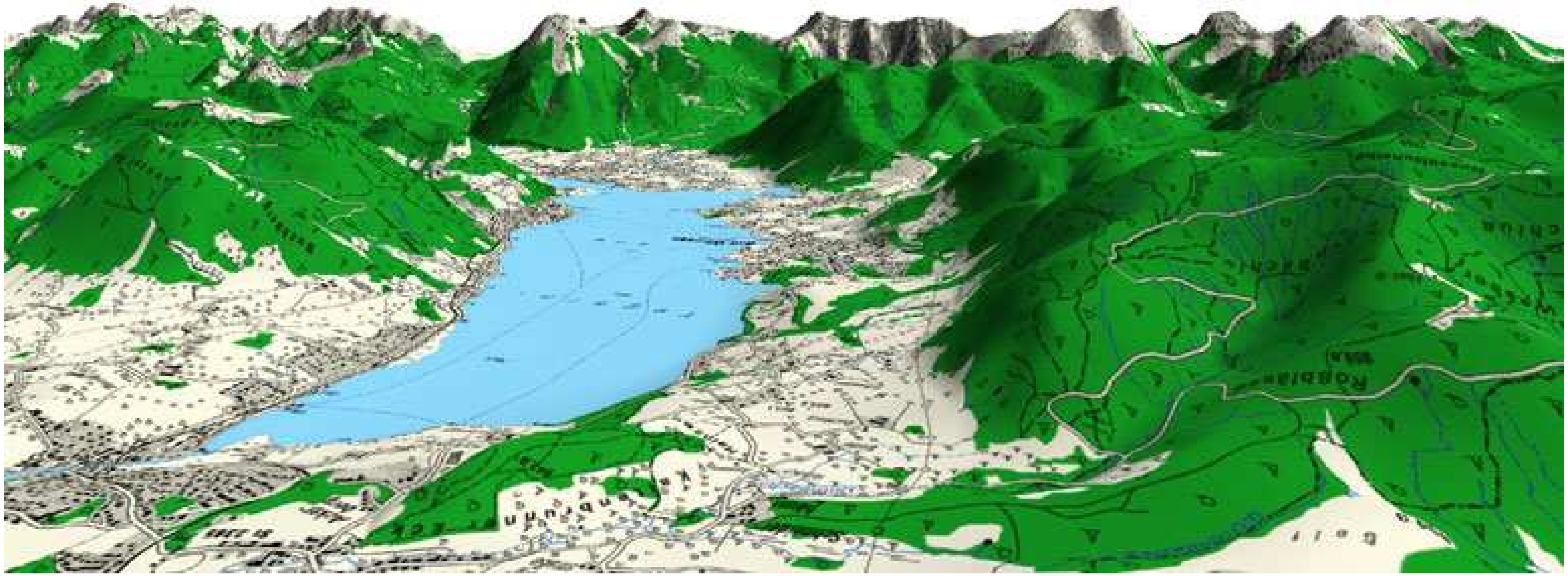
Mountain climbing

- Non-linear optimization is like mountain-climbing in the fog



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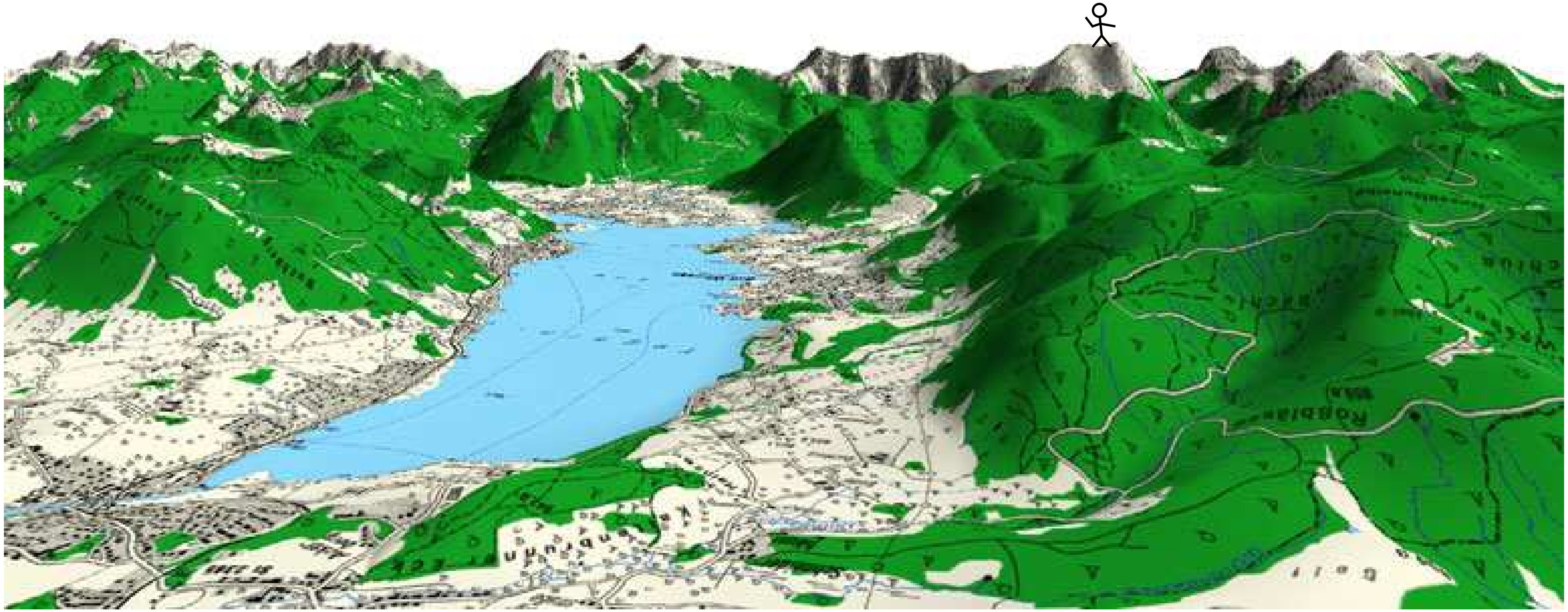

- How do you know that you're on the highest mountain if you can't see the other peaks?

Interior point methods and attempts to get out of local maxima

- Basic principle of interior point methods for finding a local maximum:

Interior point methods and attempts to get out of local maxima

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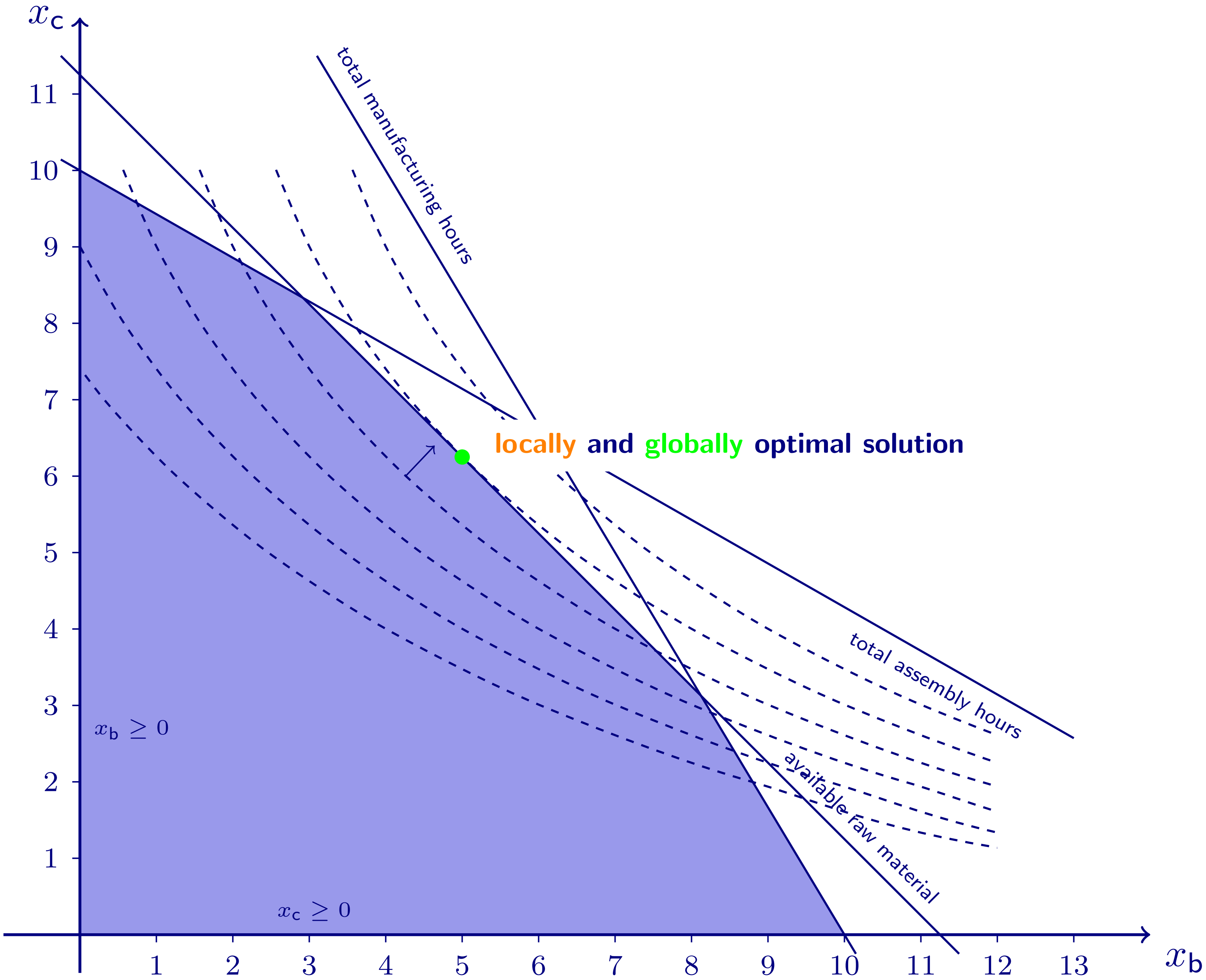
- ▷ Examples:
  - Simulated annealing, Tabu search, Genetic algorithms, Ant colony simulation
- ▷ Big disadvantage: no optimality information (as gaps in branch & bound)!
  - ➡ You have to rely on luck to get an optimal solution...

When are local optima automatically global?



locally and globally optimal solution

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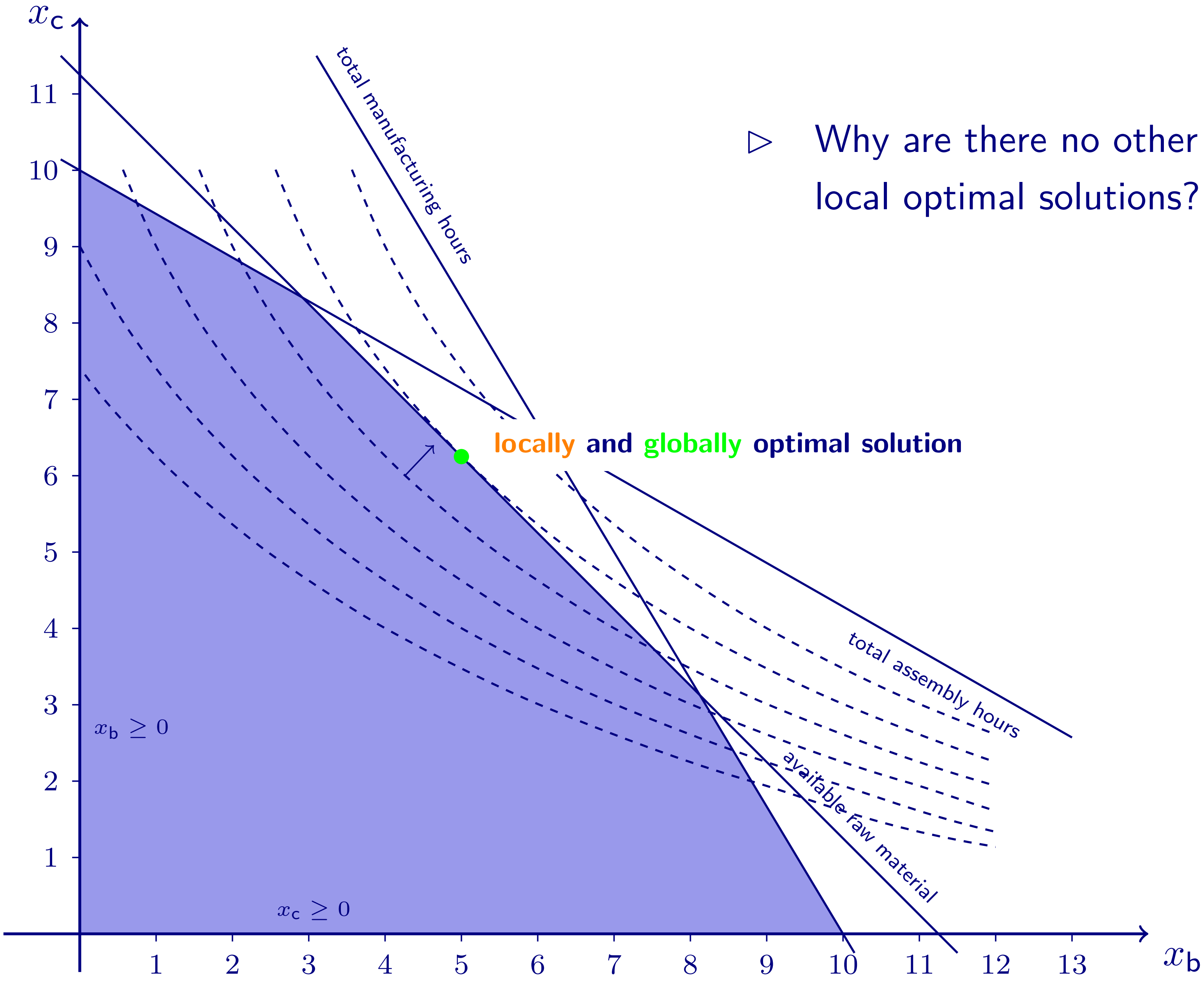
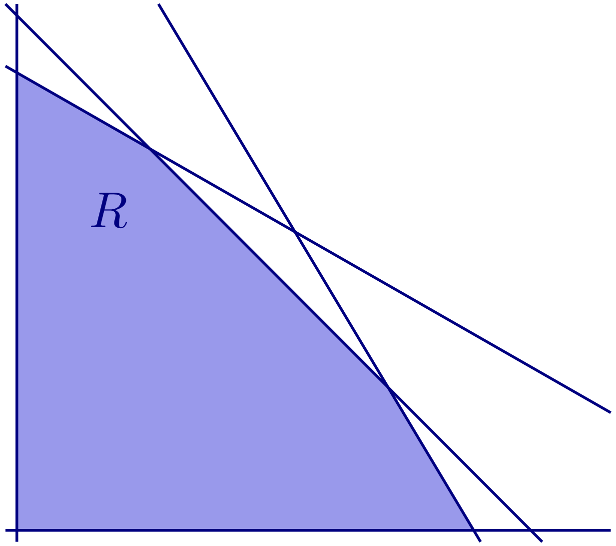


- ▷ Why are there no other local optimal solutions?

Convex regions

- Feasible region: $$R = \{(x_{\mathsf{b}}, x_{\mathsf{c}}) \mid \begin{aligned} 4x_{\mathsf{b}} + 4x_{\mathsf{c}} &\leq 45 \\ 5x_{\mathsf{b}} + 3x_{\mathsf{c}} &\leq 50 \\ 4x_{\mathsf{b}} + 7x_{\mathsf{c}} &\leq 70 \\ x_{\mathsf{b}}, x_{\mathsf{c}} &\geq 0\} \end{aligned}$$



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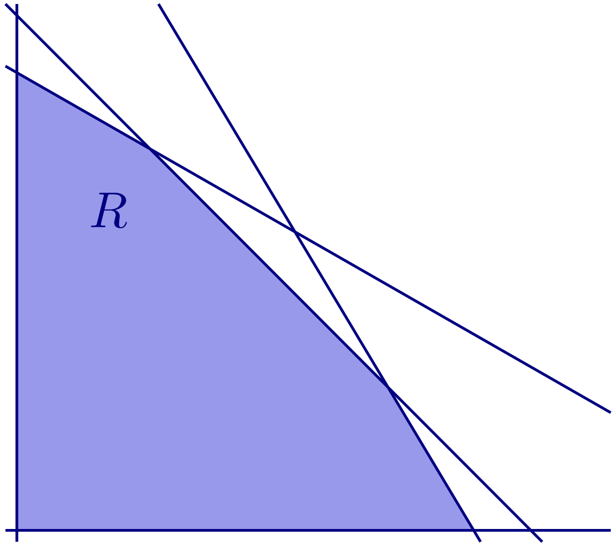

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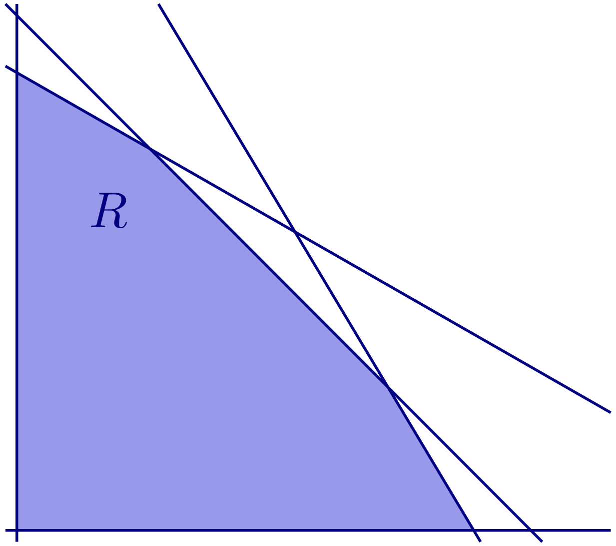
i.e. every straight line between two points in R also lies completely in R



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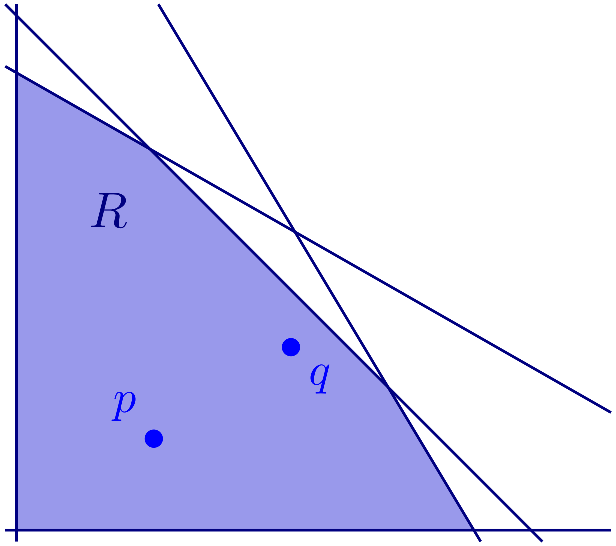

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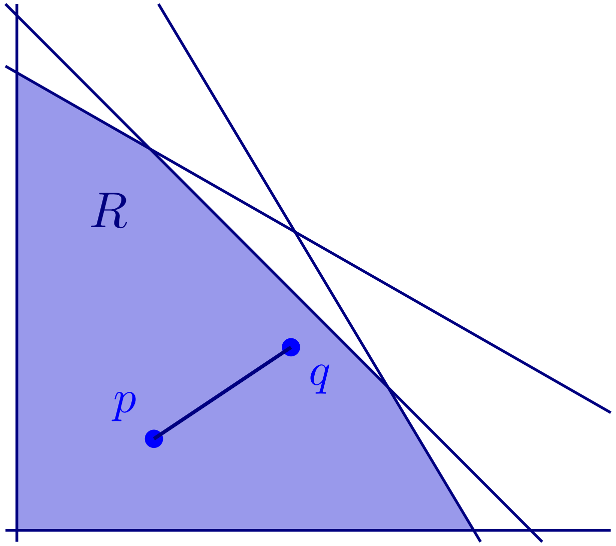

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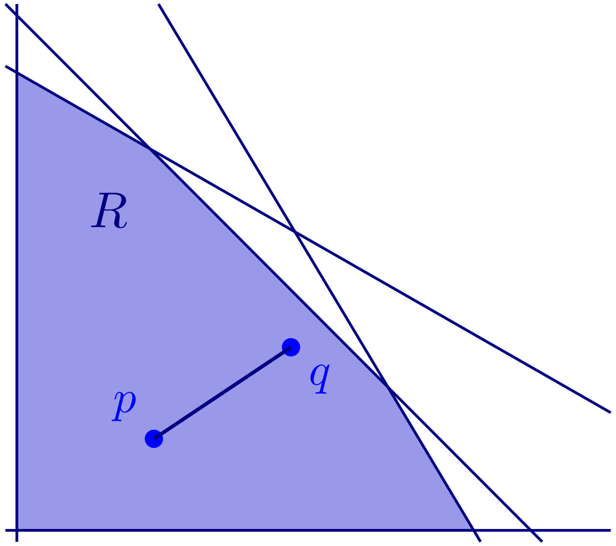

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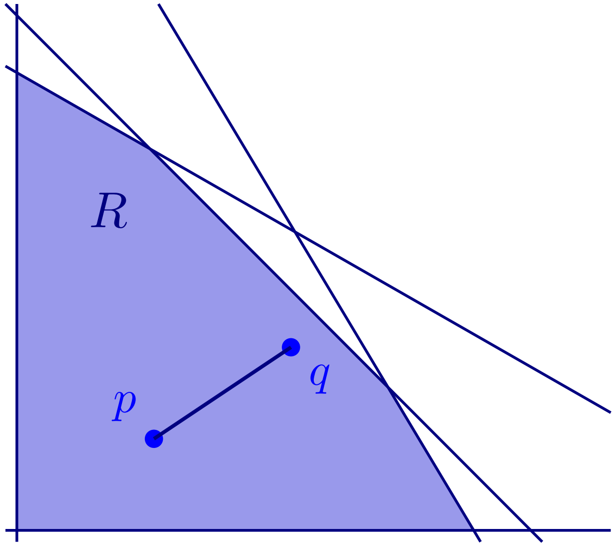

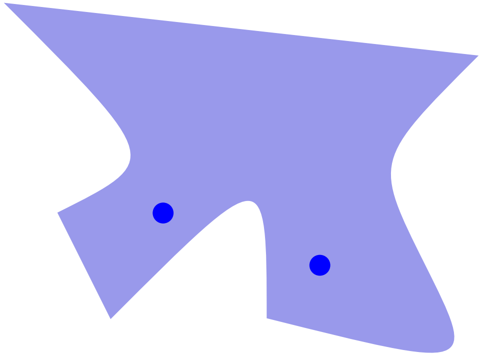

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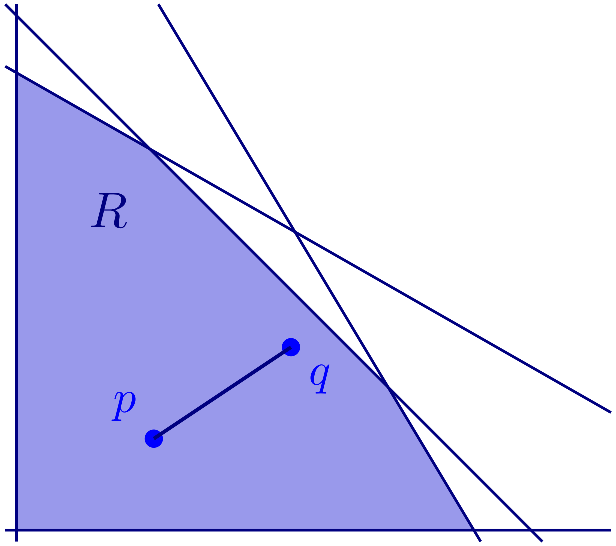

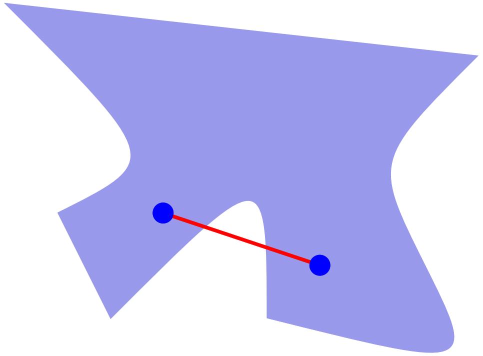

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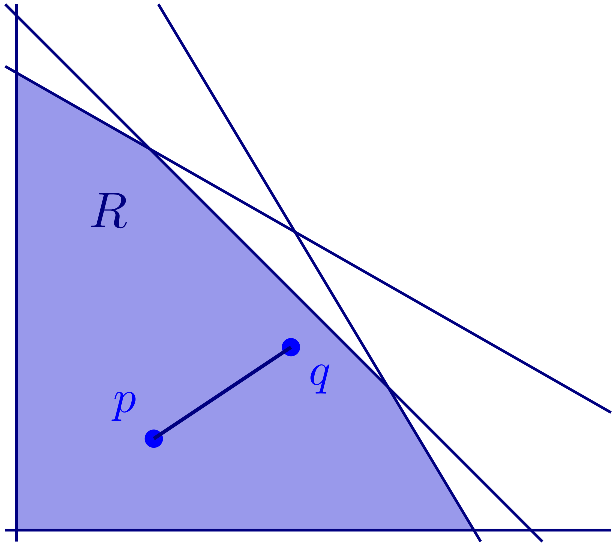
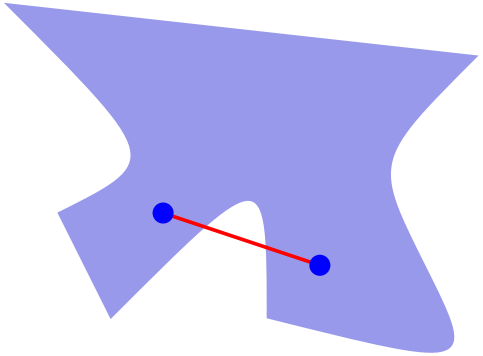
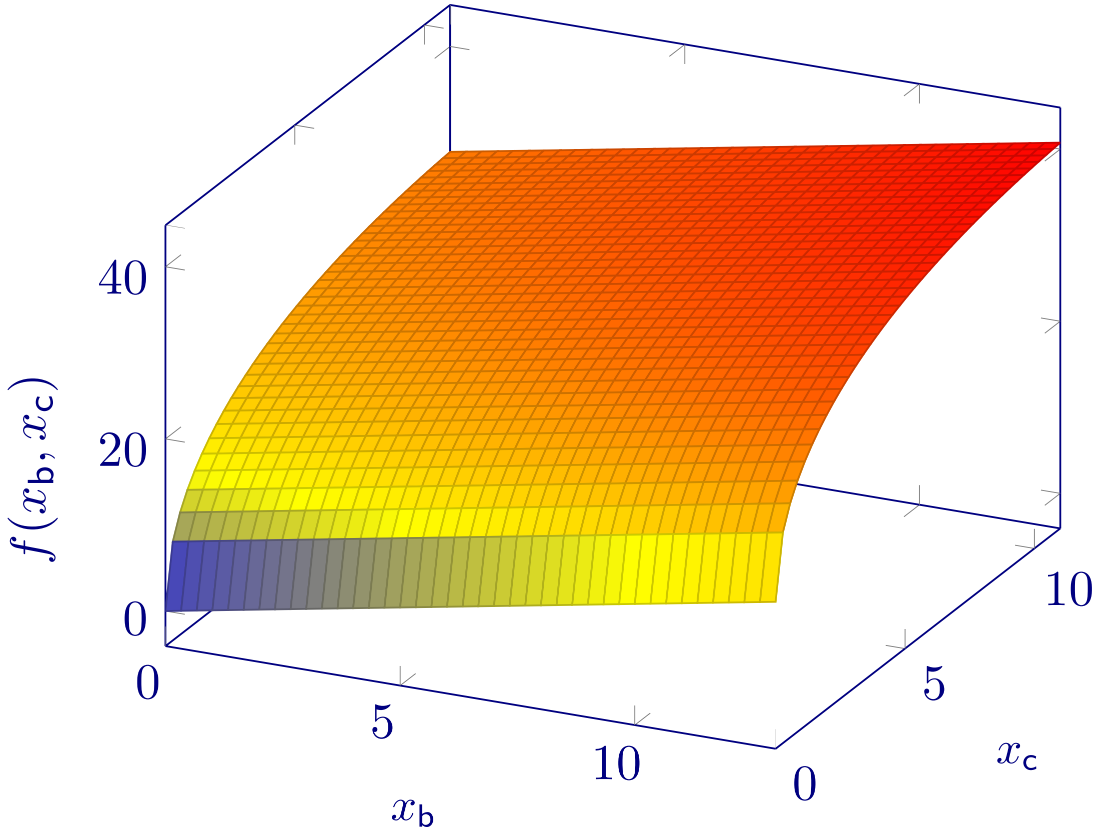




not convex!

Concave functions

- Objective function: $f(x_{\mathsf{b}}, x_{\mathsf{c}}) = x_{\mathsf{b}} + 15\sqrt{x_{\mathsf{c}}} - 2x_{\mathsf{c}}$



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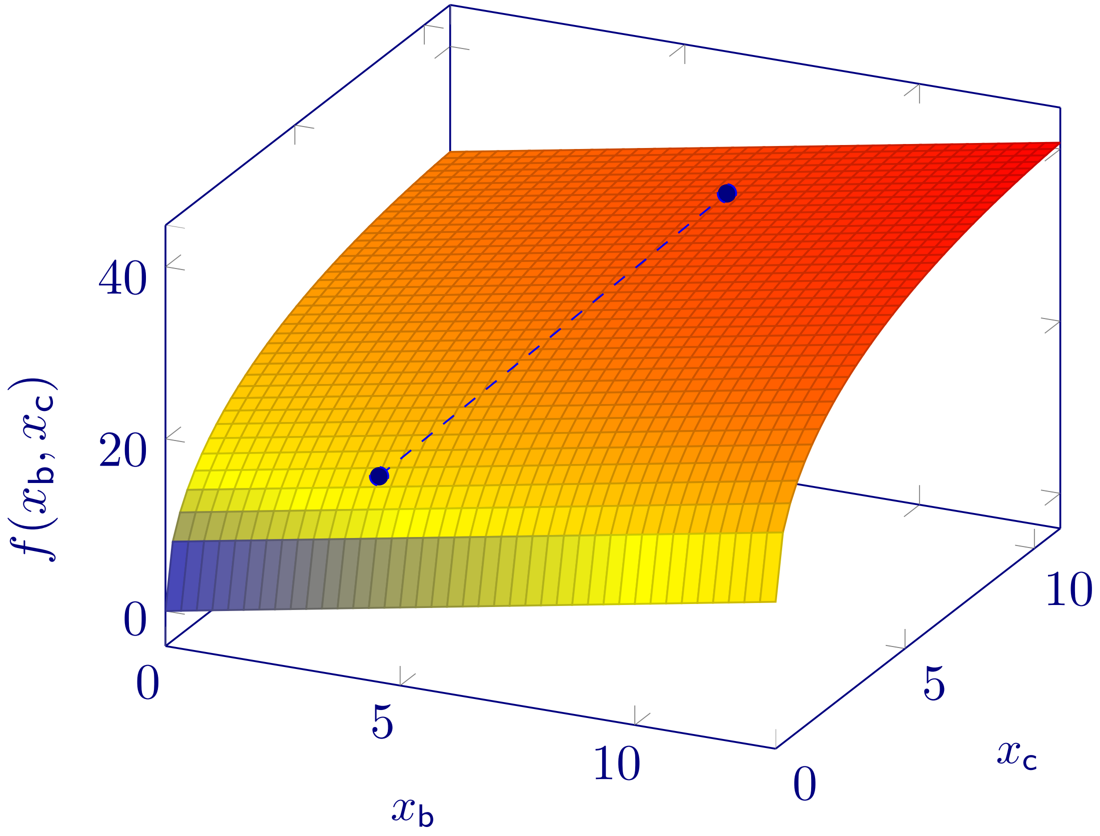

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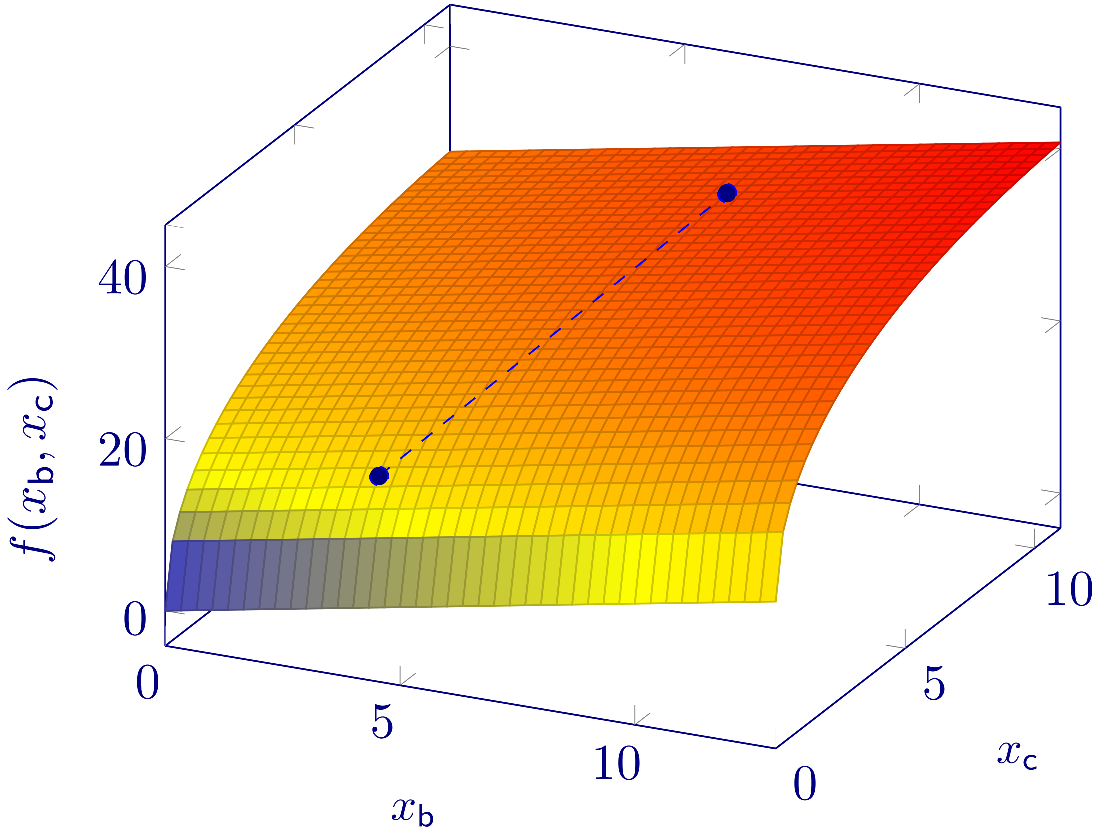

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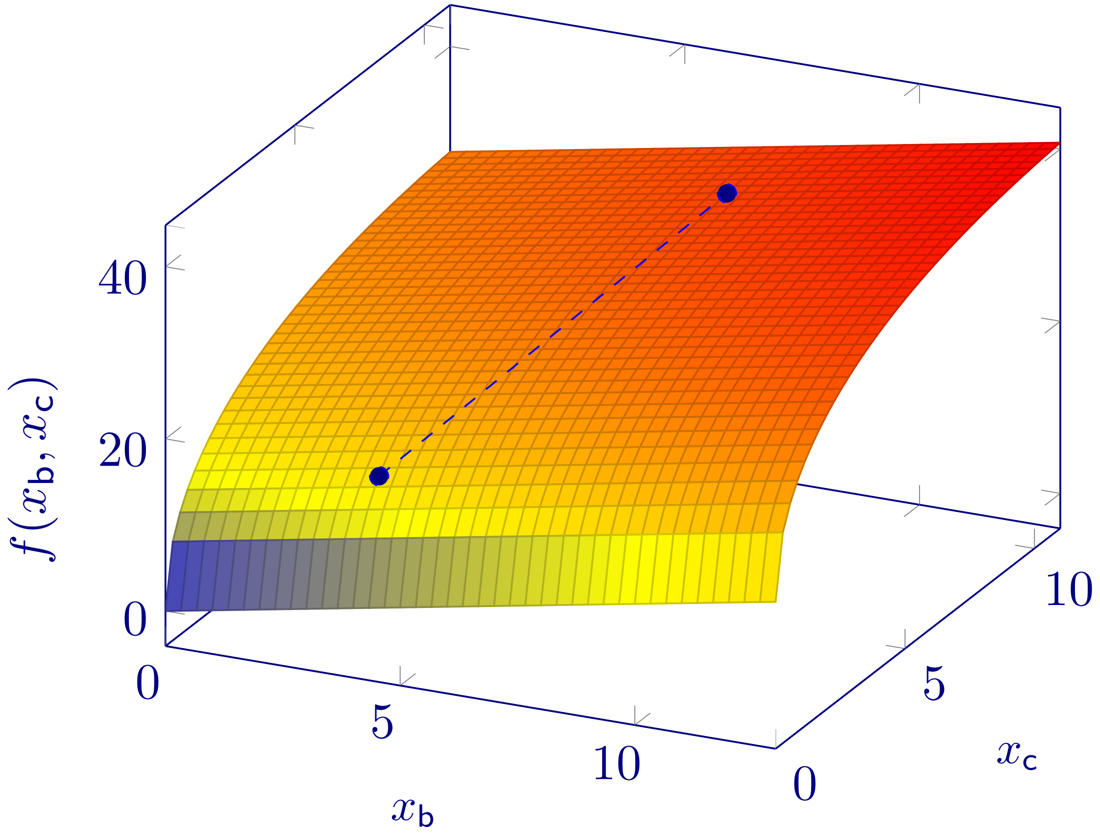

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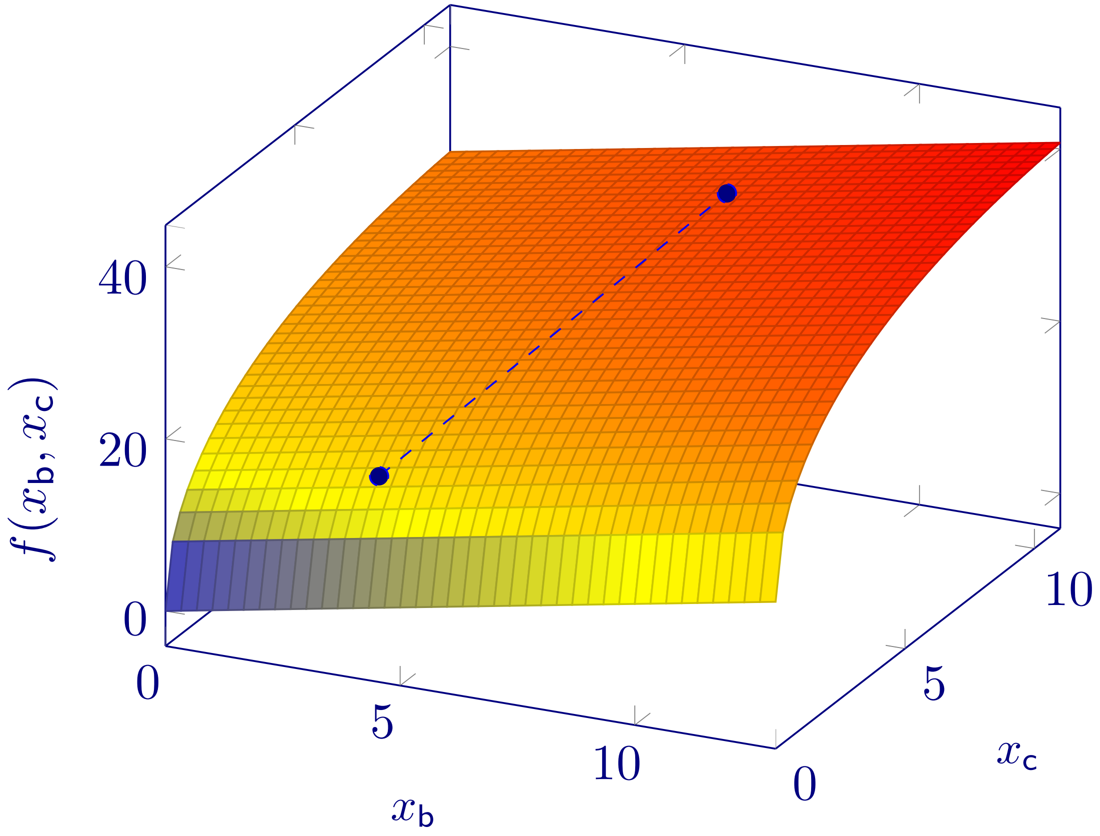

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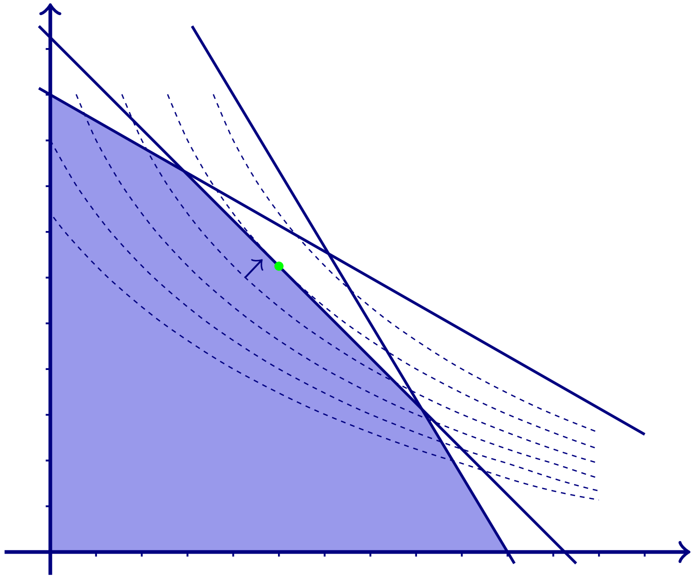

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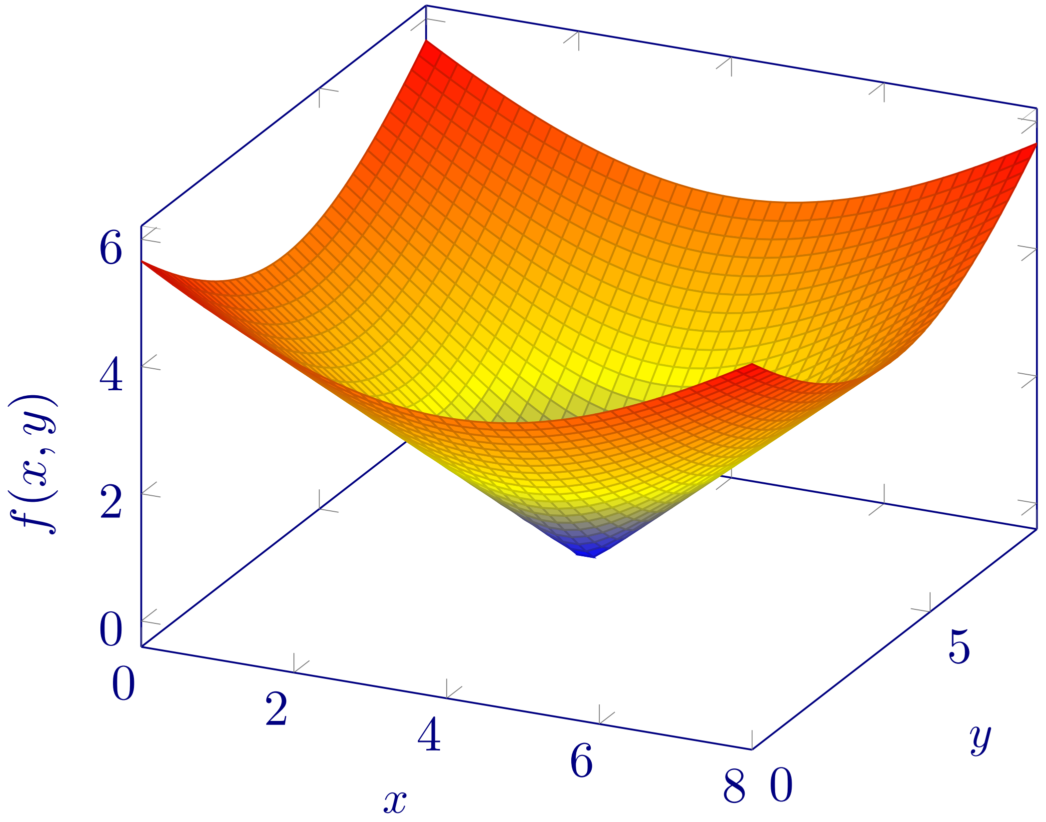

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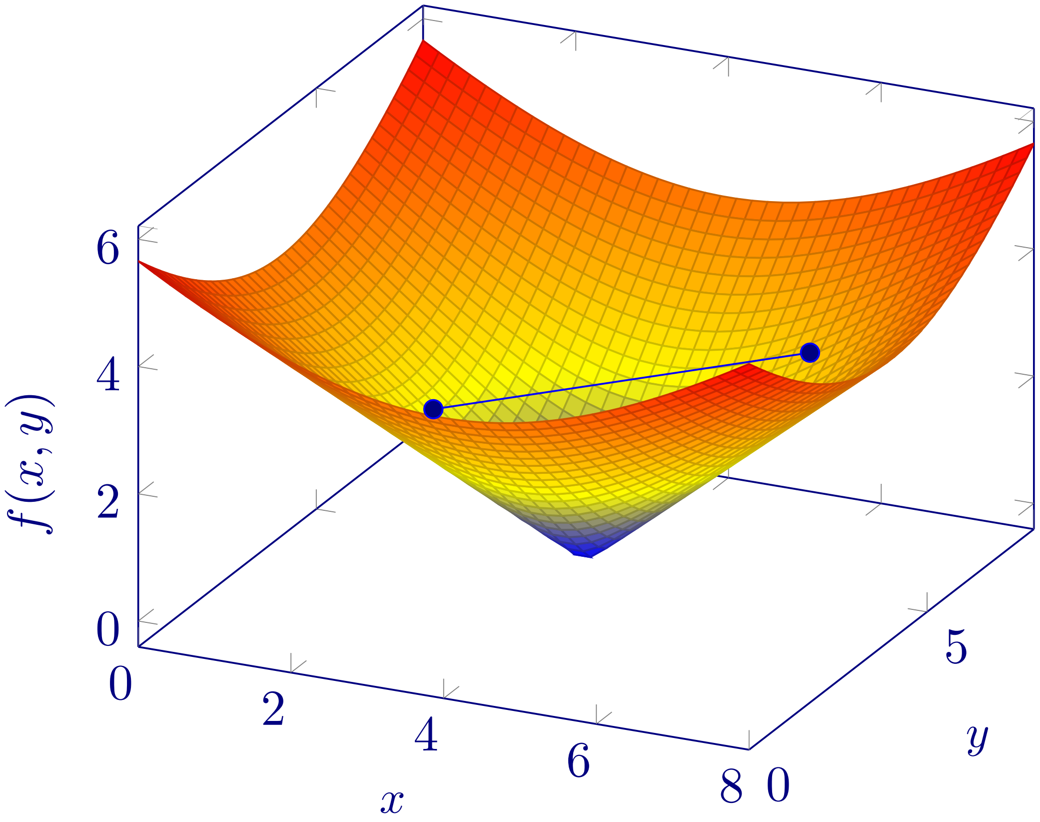

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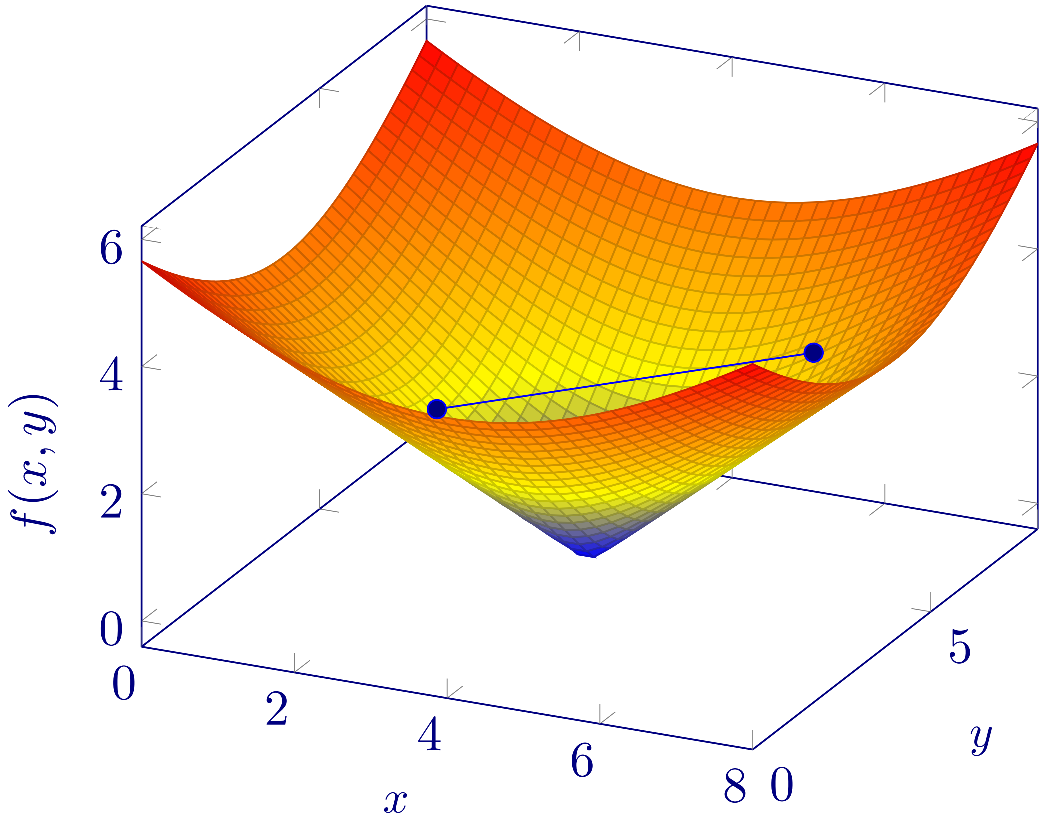

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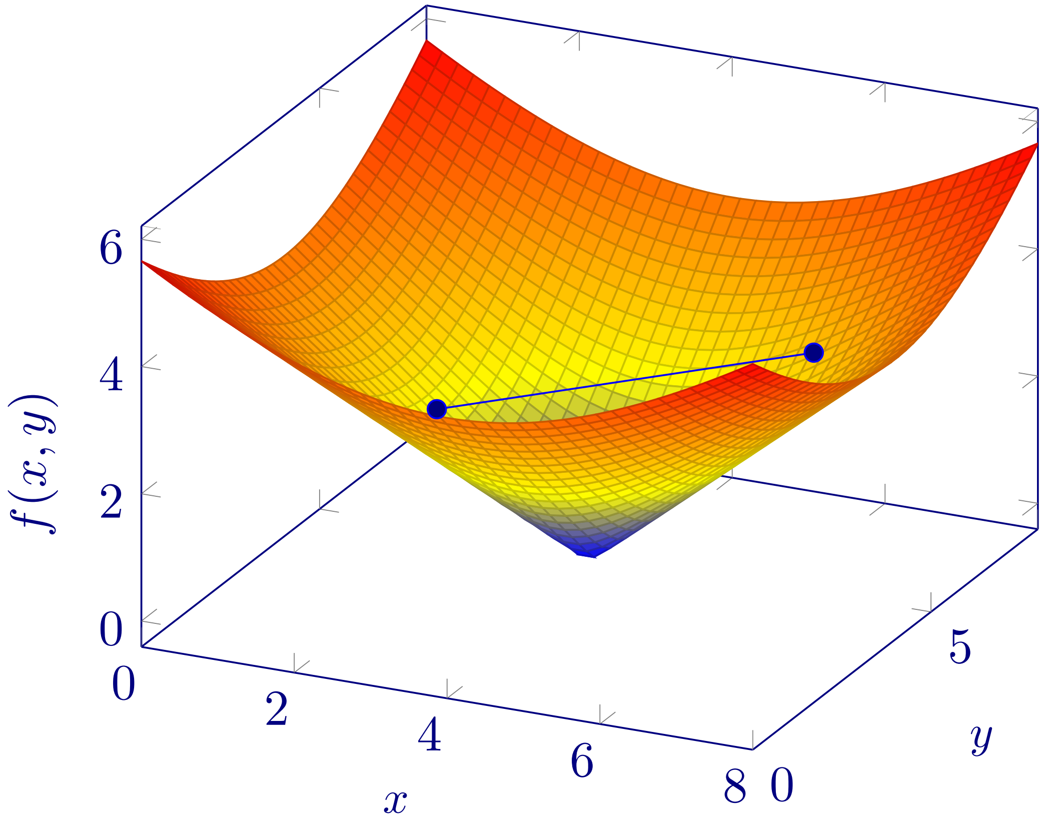

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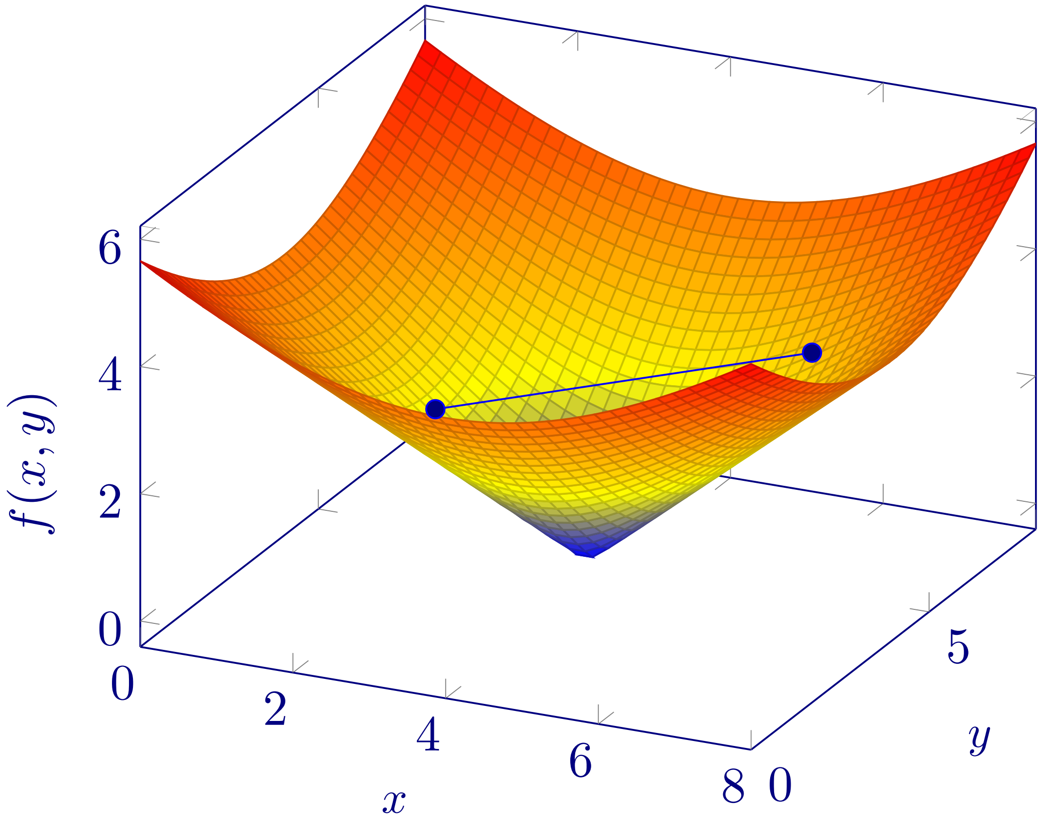

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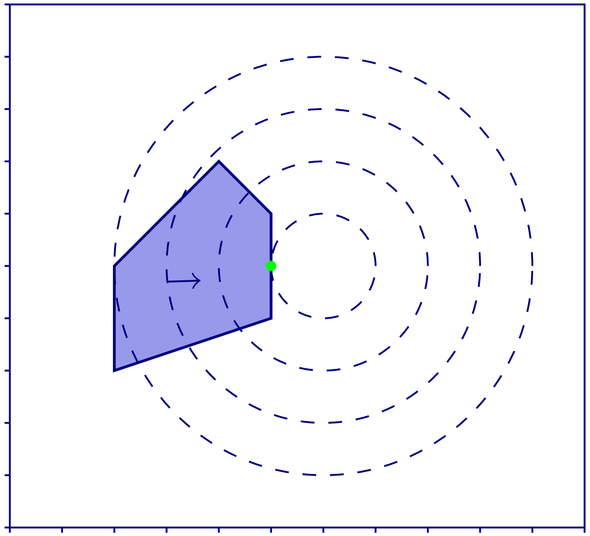
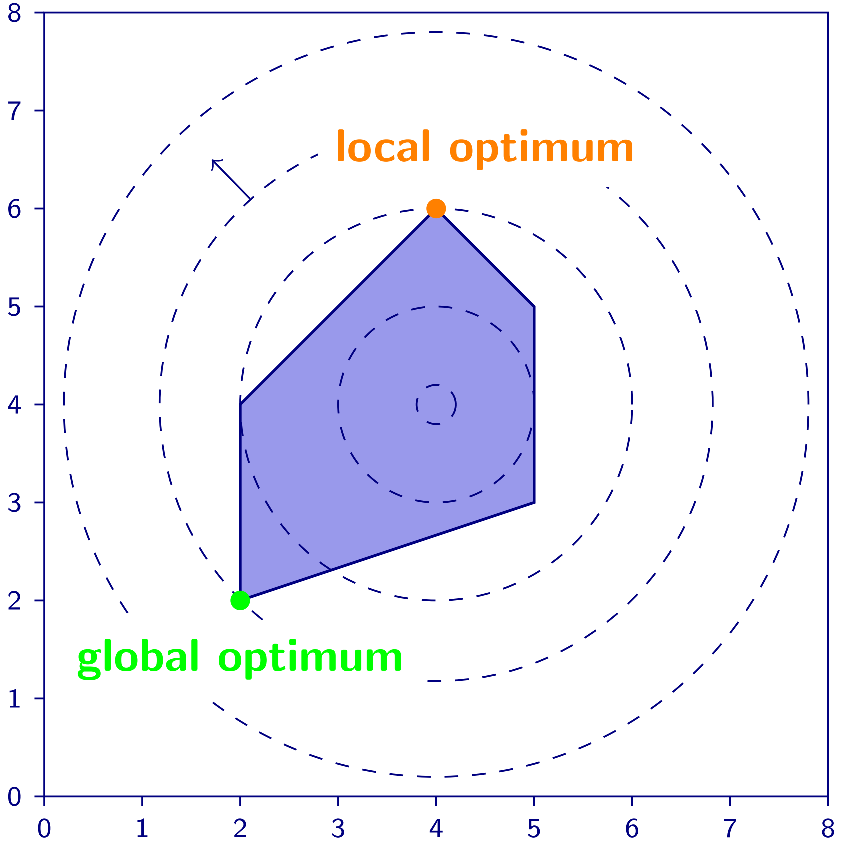


Convex maximization doesn't work

$$\begin{aligned} \max \quad & \sqrt{(x-4)^2+(y-4)^2} \\ \text{s.t.} \quad x &\geq 2 \\ x &\leq 5 \\ -x+y &\leq 2 \\ x+y &\leq 10 \\ x-3y &\leq -4 \end{aligned}$$



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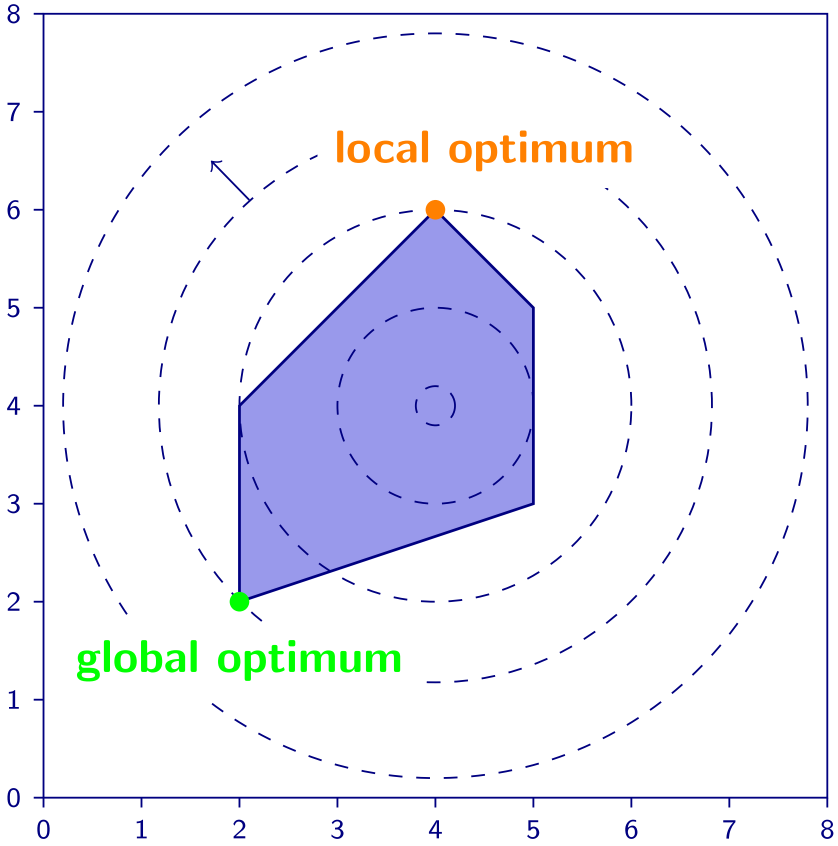

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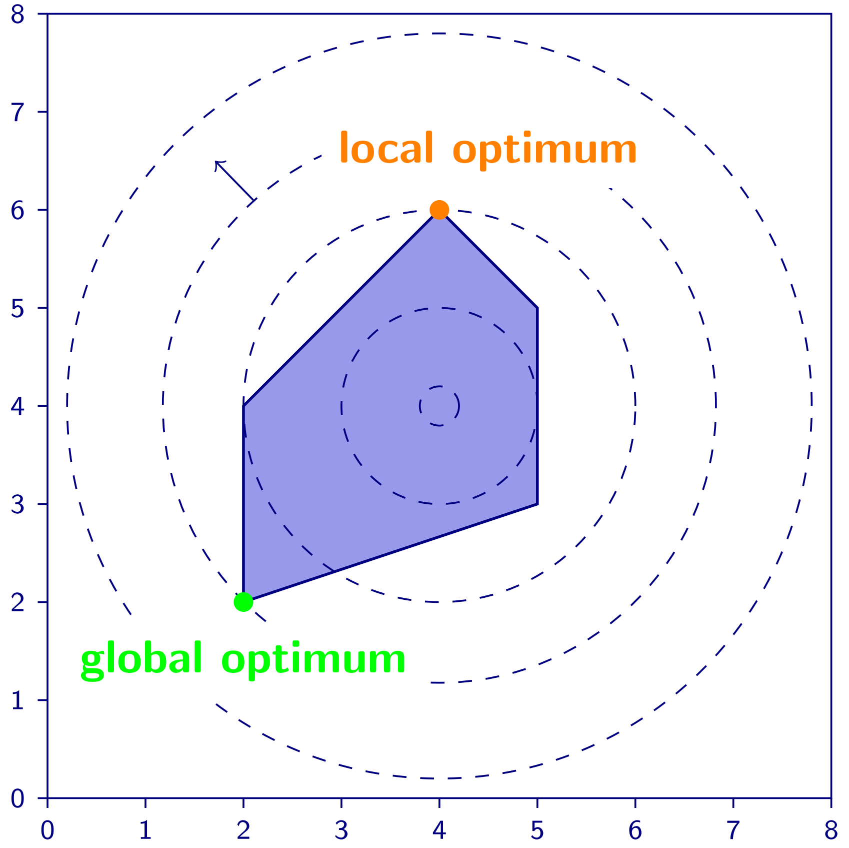

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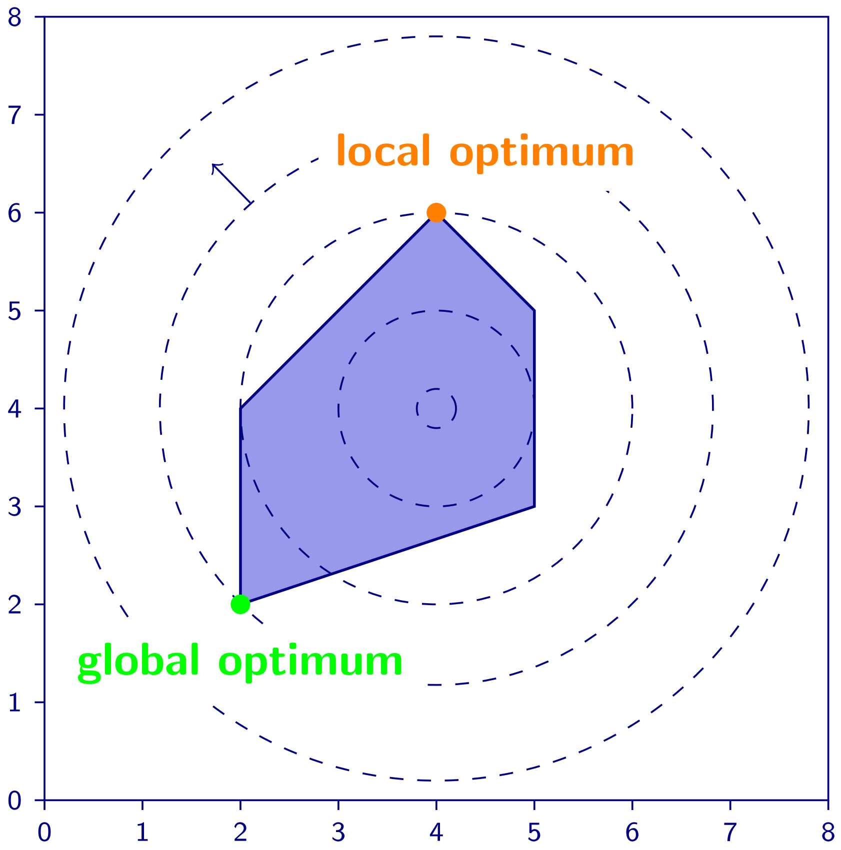

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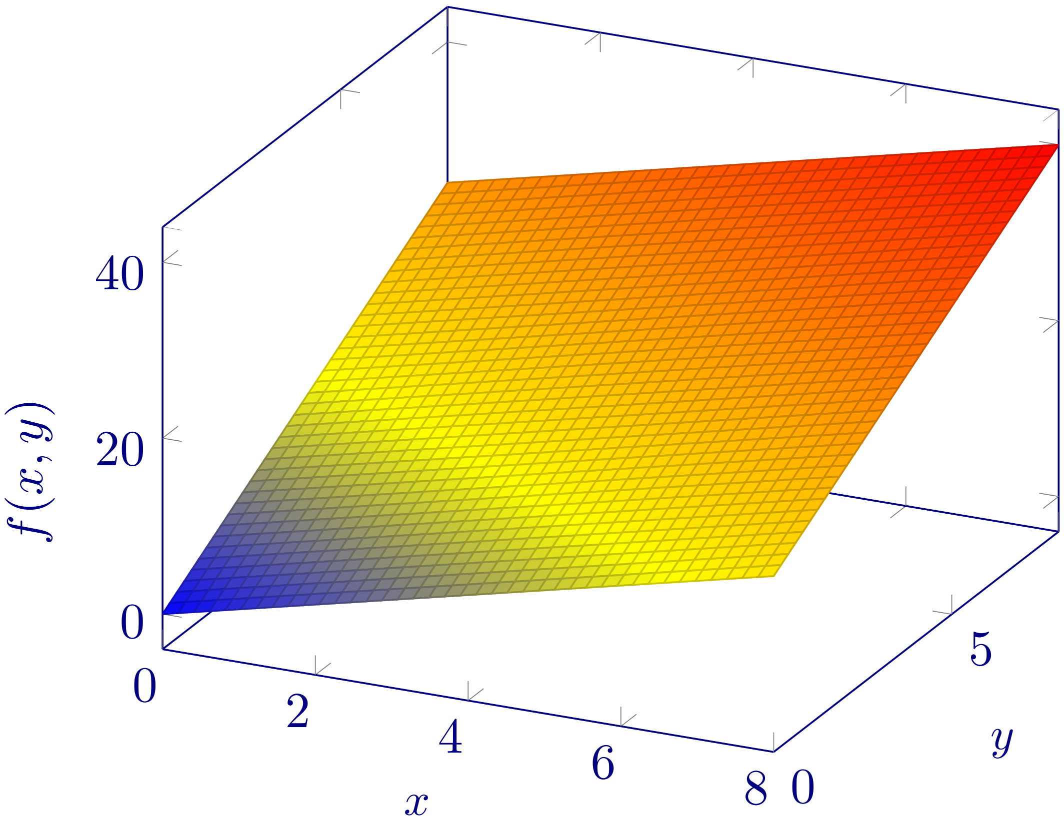
Linear optimization always works

- Linear objective function: $f(x, y) = 2x + 3y$



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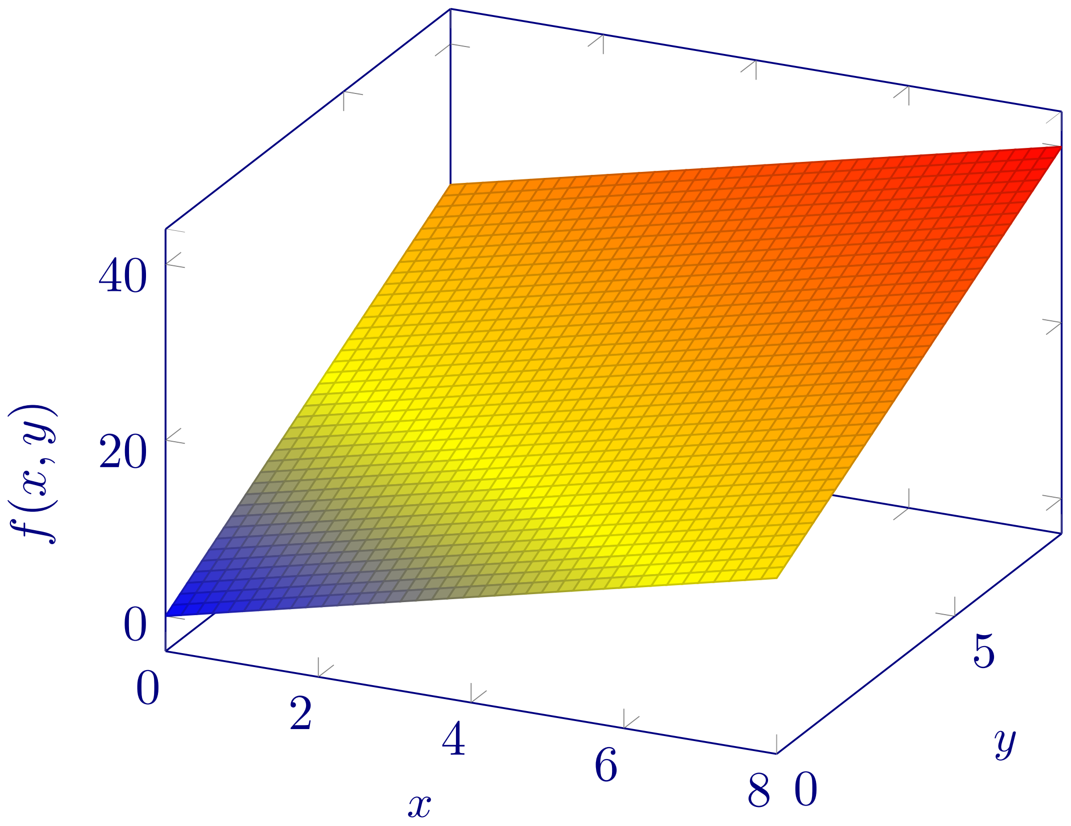

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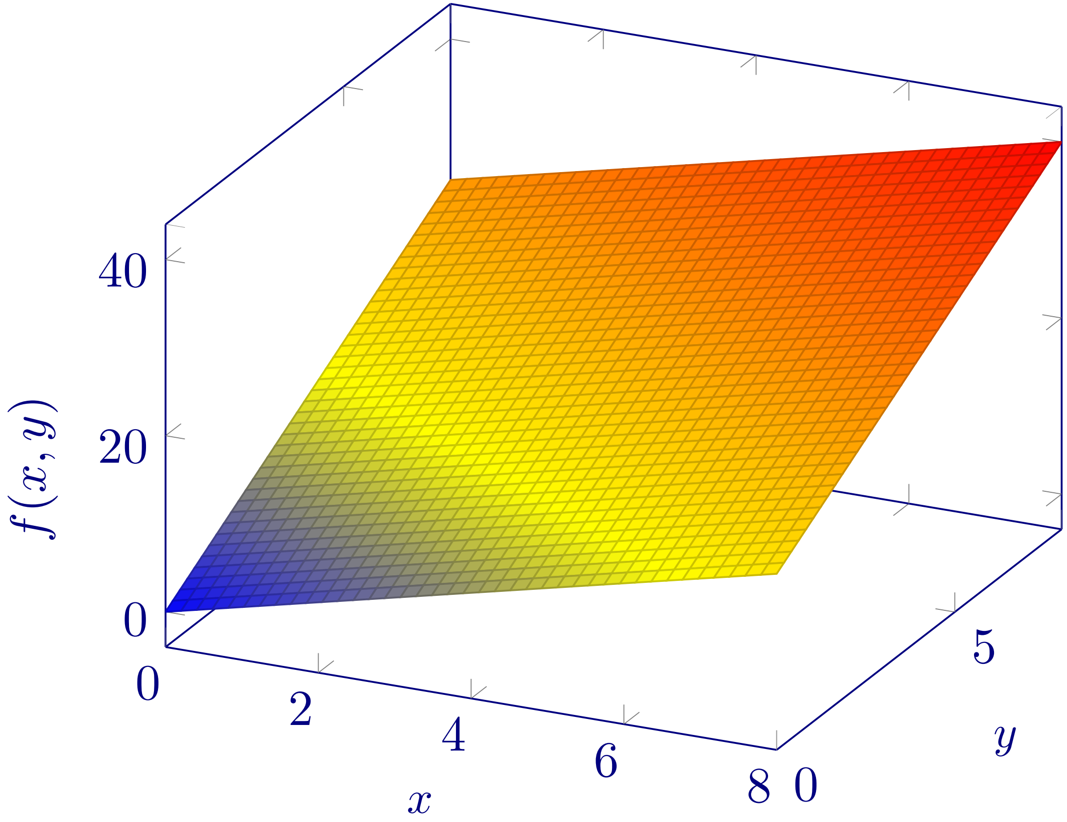

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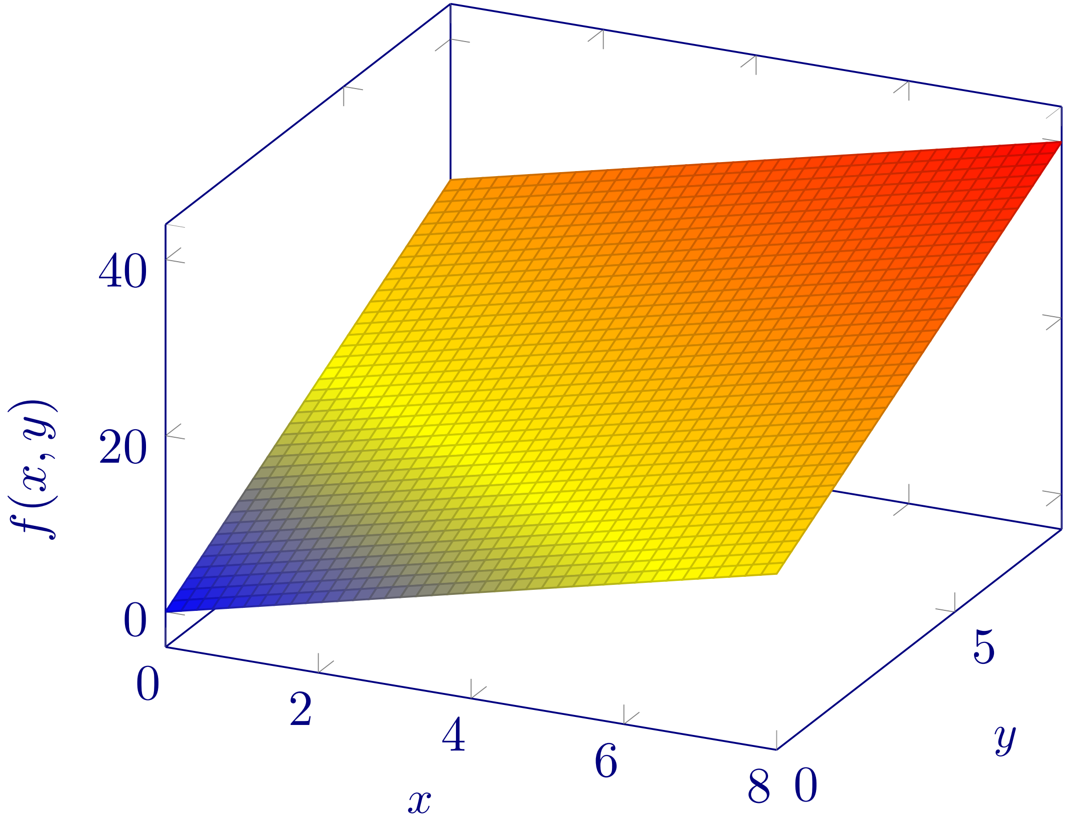

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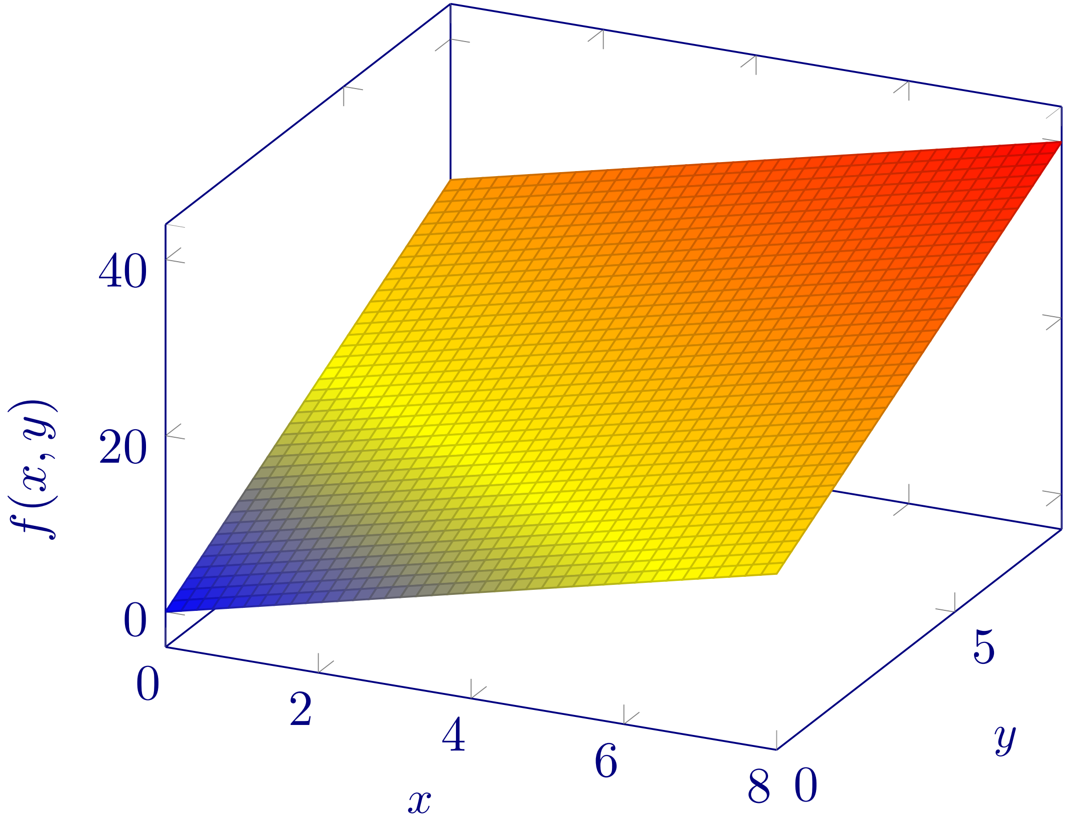

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Transforming special non-linear models into linear ones

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$$\begin{aligned} \text{minimize} \quad & \max_{k=1,\dots,\ell} f_k(x_1,\dots,x_n) \qquad (f_k \text{ are all linear}) \\ \text{subject to} \quad & \sum_{i=1}^{n} a_{ji} x_i \le b_j \qquad \forall j \end{aligned}$$

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➡ Rewrite as:

$$\begin{aligned} \text{minimize} \quad & z \\ \text{subject to} \quad & f_k(x_1,\ldots,x_n) \;\leq\; z \qquad (1 \leq k \leq \ell) \\ & \sum_{i=1}^{n} a_{ji}x_i \;\leq\; b_j \qquad \forall j \end{aligned}$$

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- ▷ Linear constraints and objective function to minimize is convex of the form

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Transforming special non-linear models into linear ones

- ▷ Some special cases of non-linear models can be transformed directly into linear models
- ▷ Linear constraints and objective function to minimize is piecewise linear and convex

➡ Non-linear: minimize $\max_{k=1,\dots,\ell} f_k(x_1,\dots,x_n)$ (f_k are all linear)

subject to $\sum_{i=1}^{n} a_{ji}x_i \leq b_j \quad \forall j$

➡ Rewrite as: minimize z

subject to $f_k(x_1,\dots,x_n) - z \leq 0 \quad (1 \leq k \leq \ell)$

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- ▷ Linear constraints and objective function to minimize is convex of the form

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➡ Can be similarly rewritten into linear constraints

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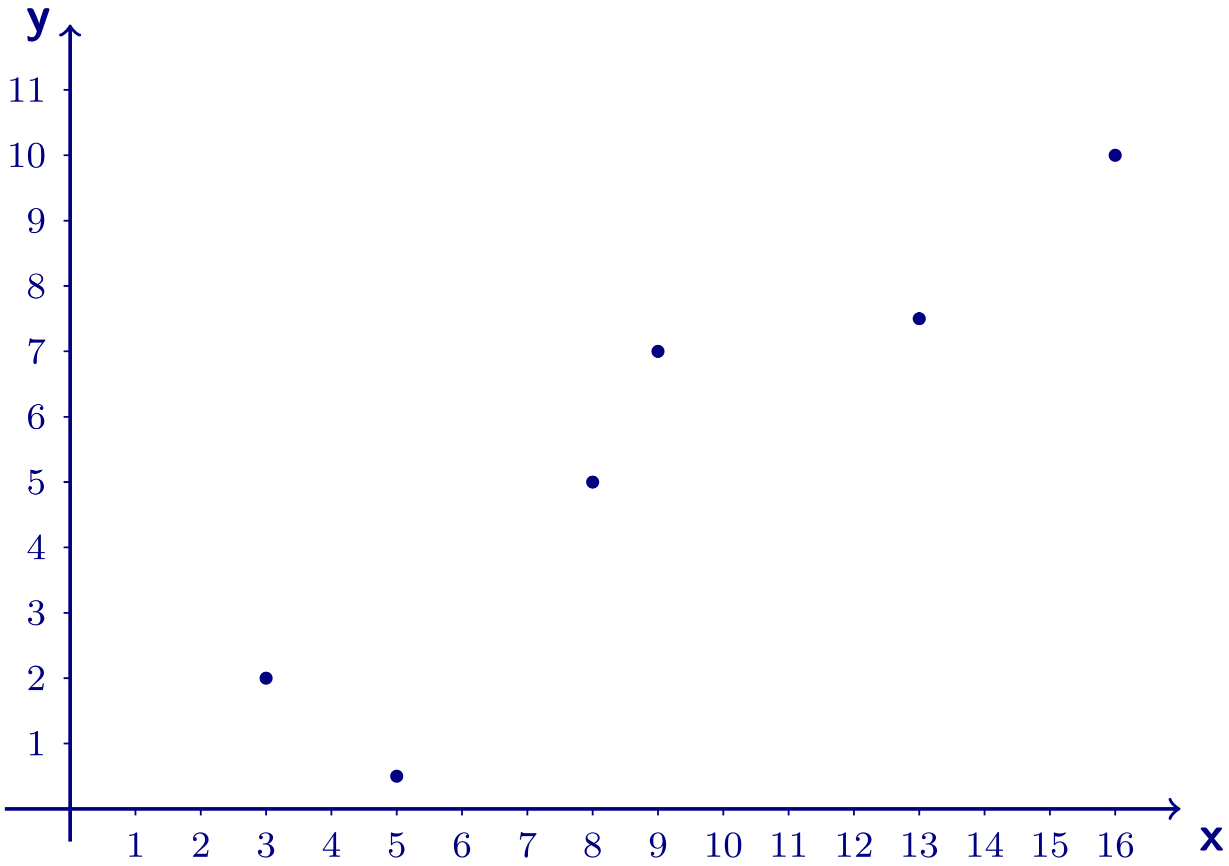

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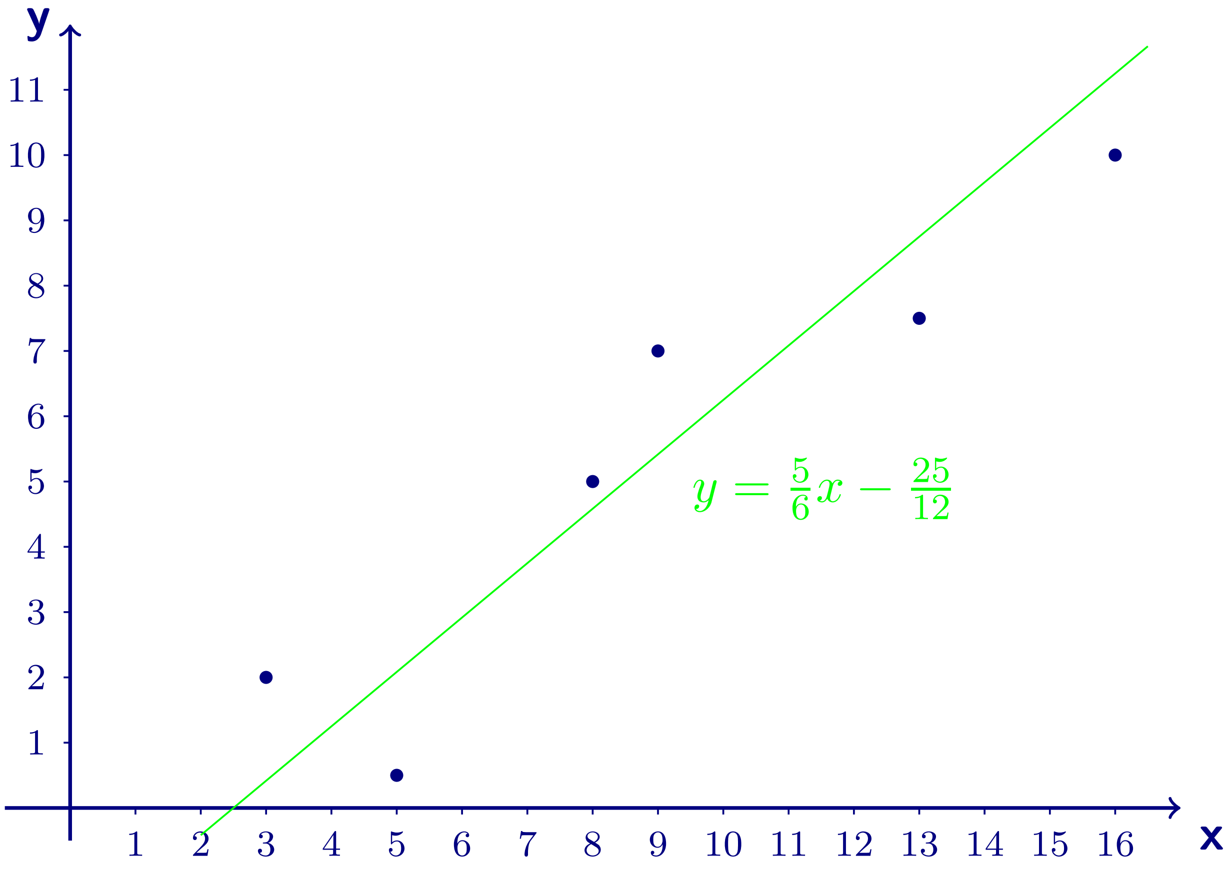

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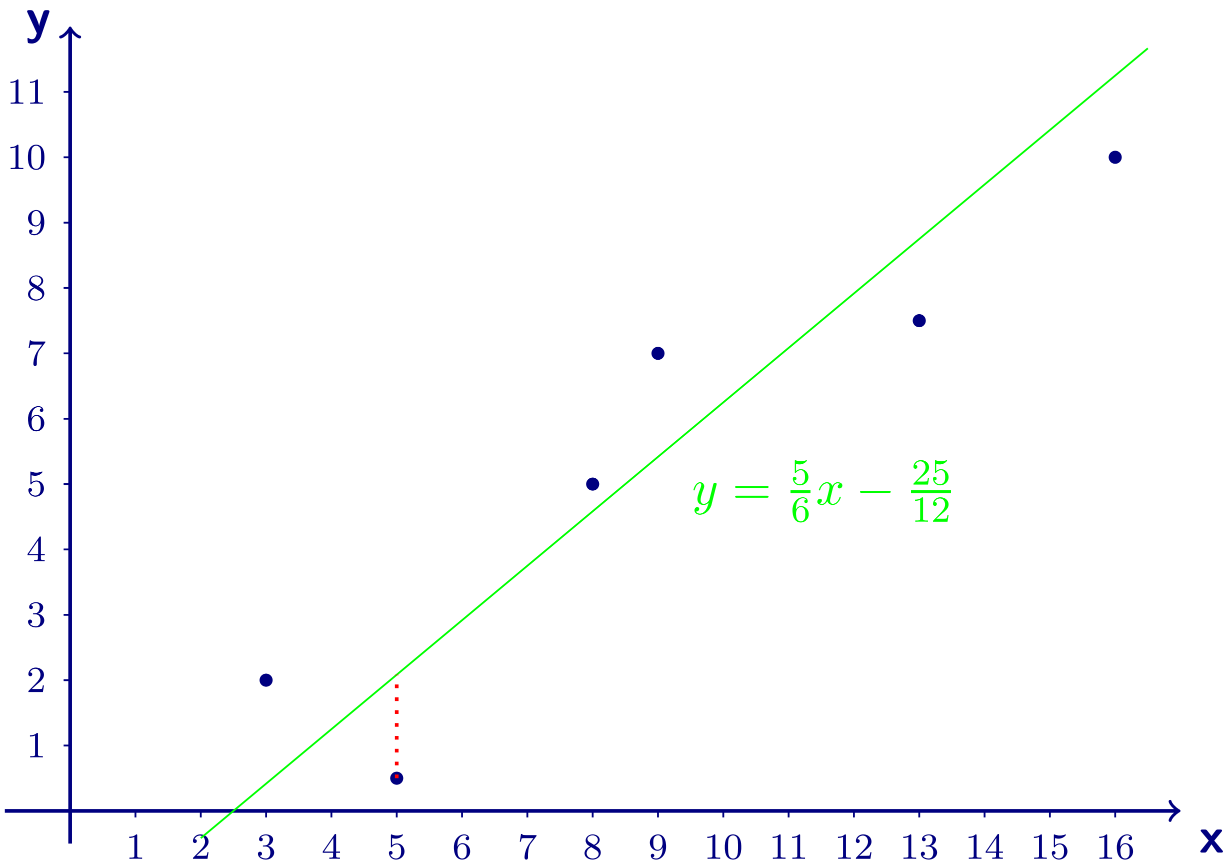

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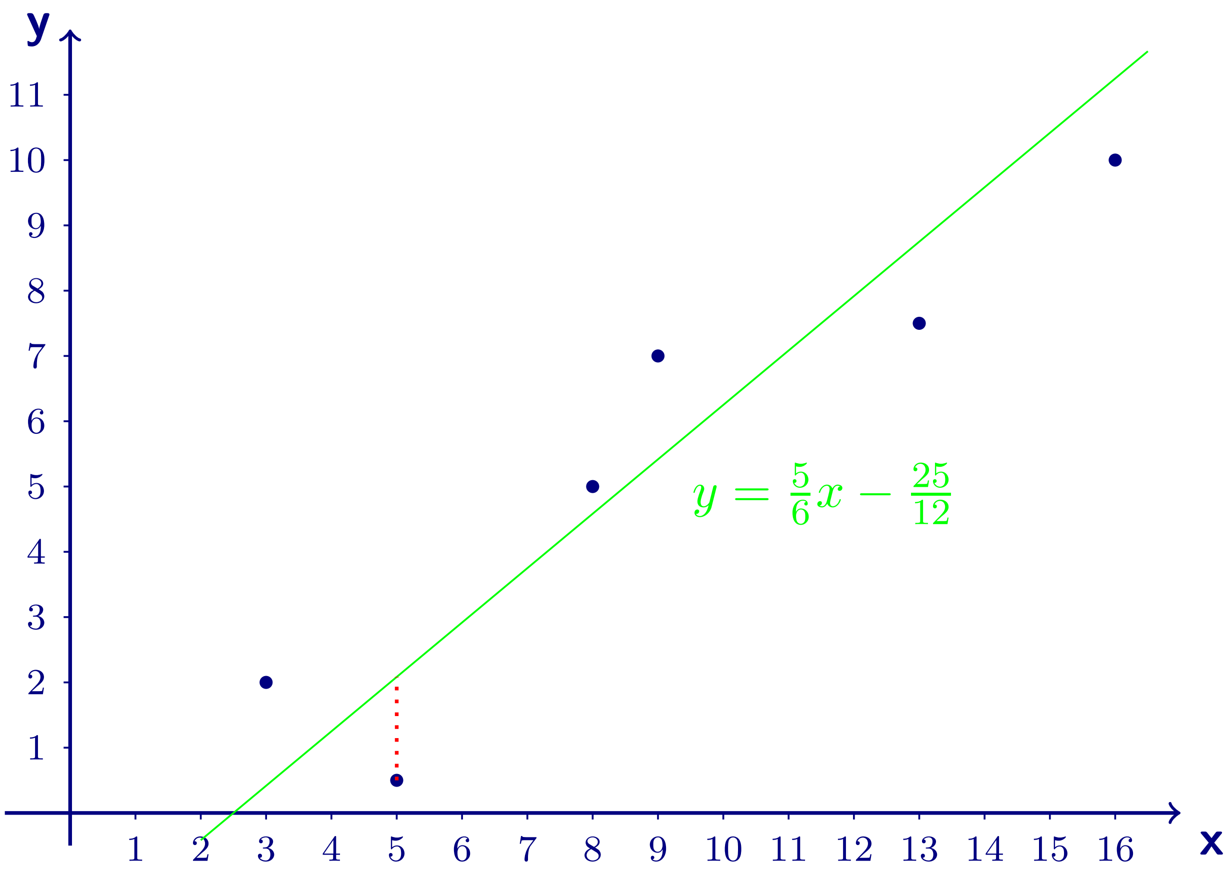

➡ Minimize the largest occuring vertical distance between the wanted line and the data points!

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- Minimize the largest occuring vertical distance between the wanted line $y = a \cdot x + b$ and the given data points (x_i, y_i)

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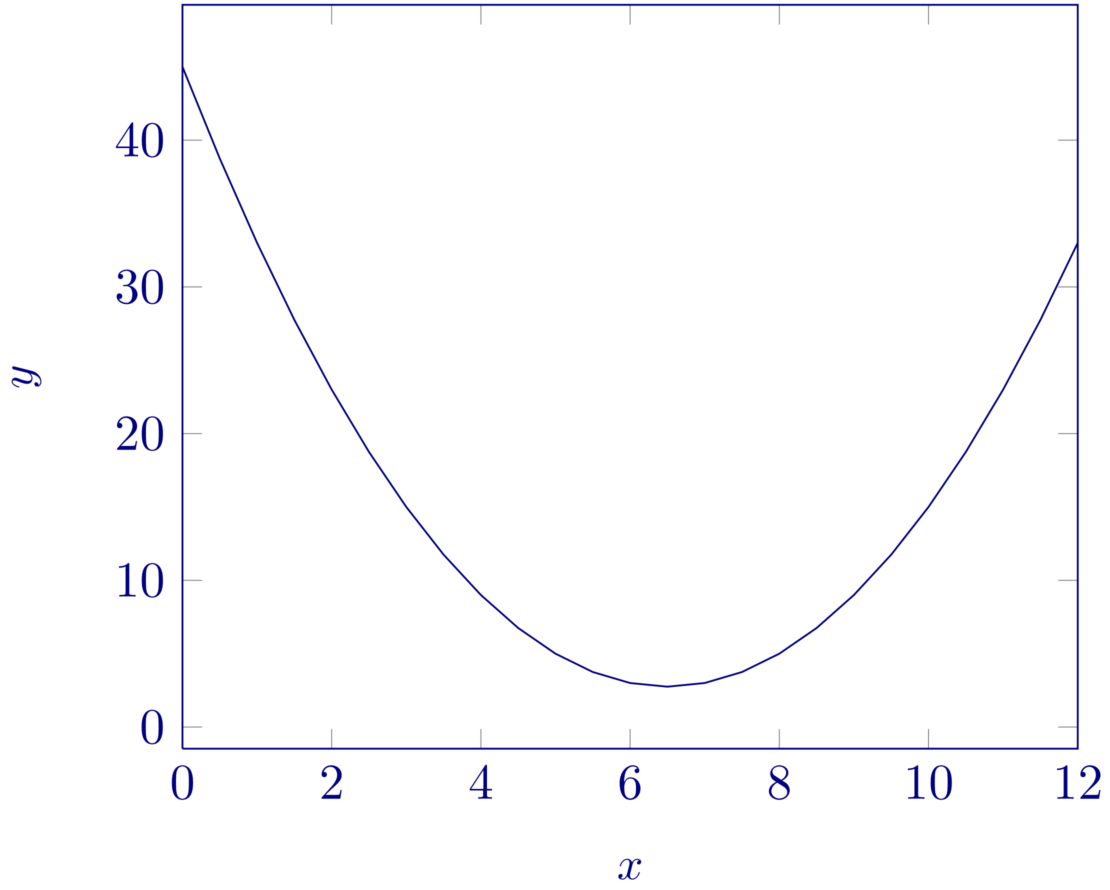
- ▷ Variants: minimize sum of distances, square of distances, fit a higher-order curve

Approximation of non-linear functions

- Approximate the function $y = x^2 - 13x + 45$ by piecewise linear functions



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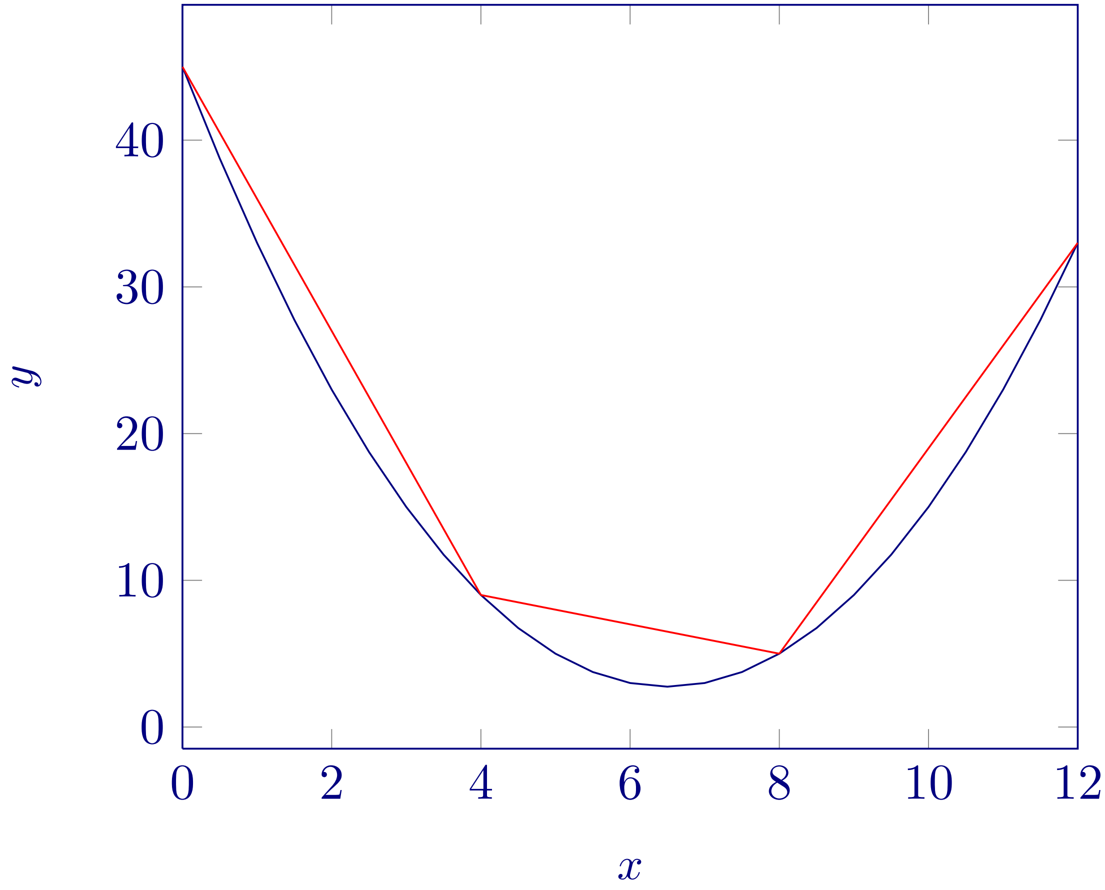

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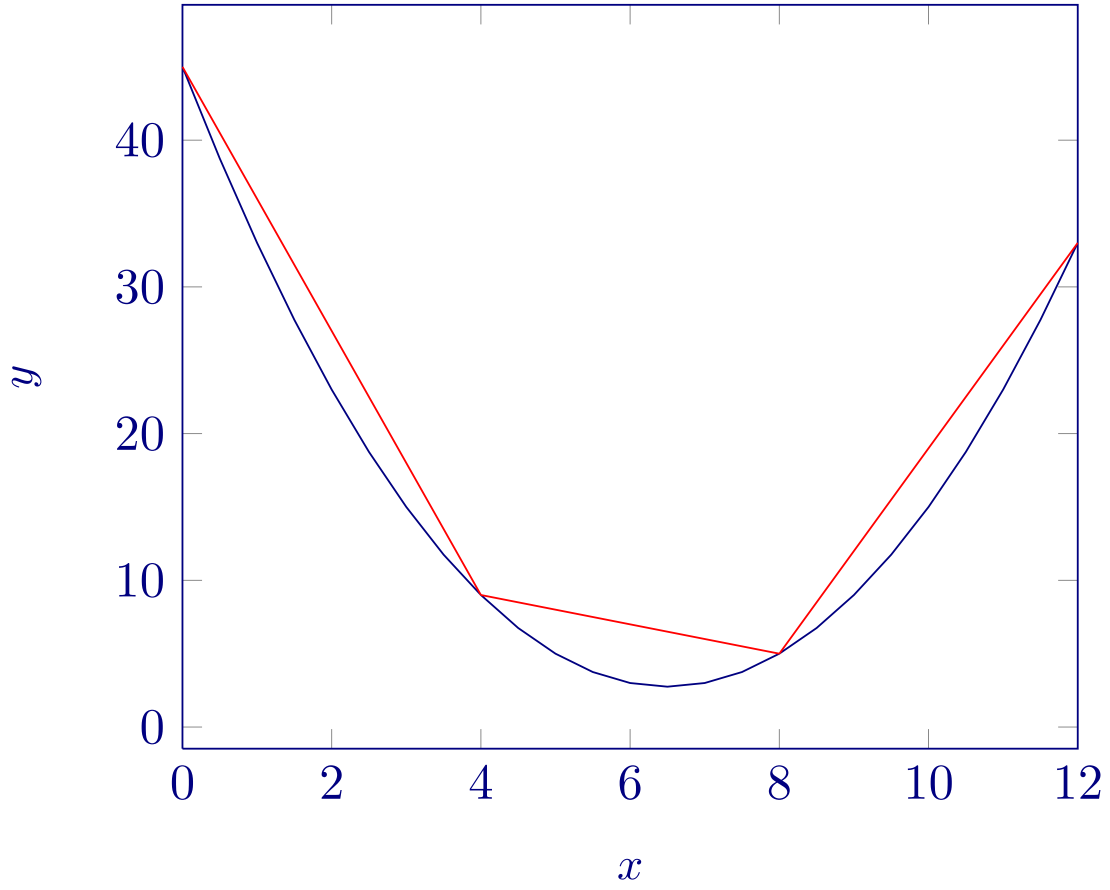

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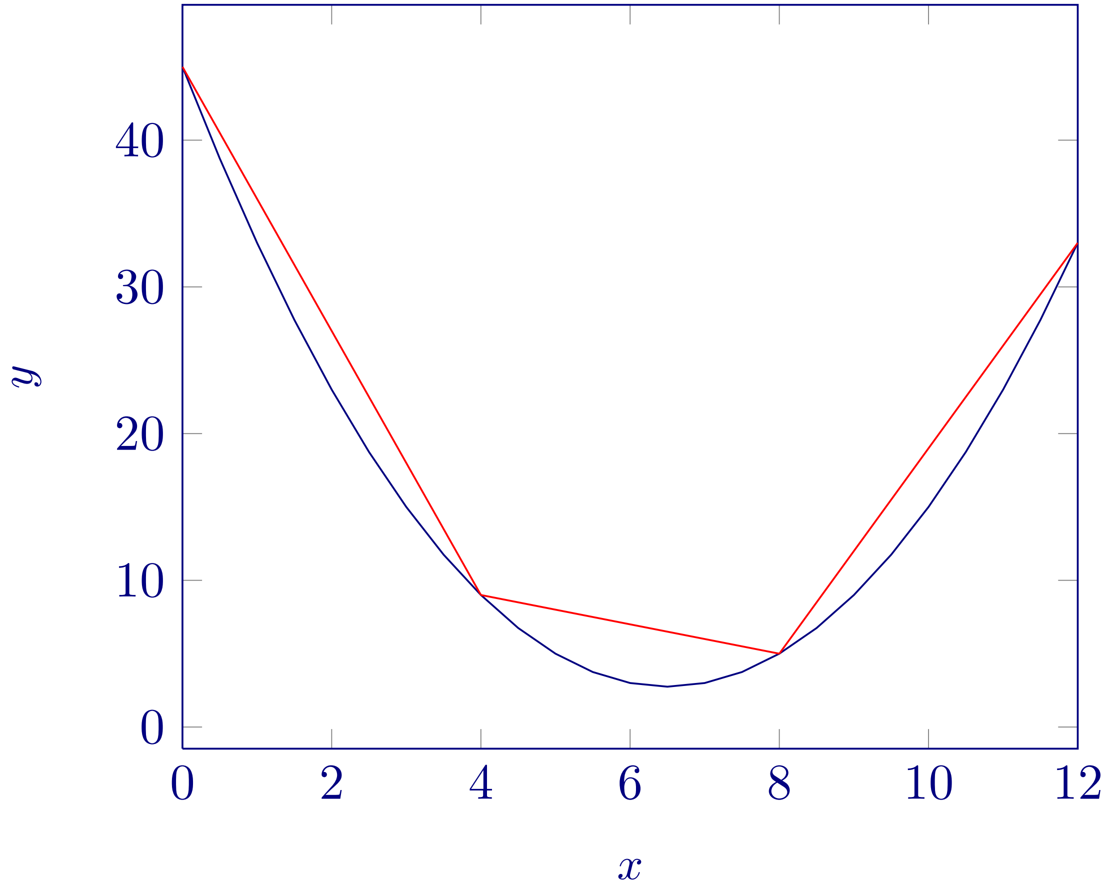

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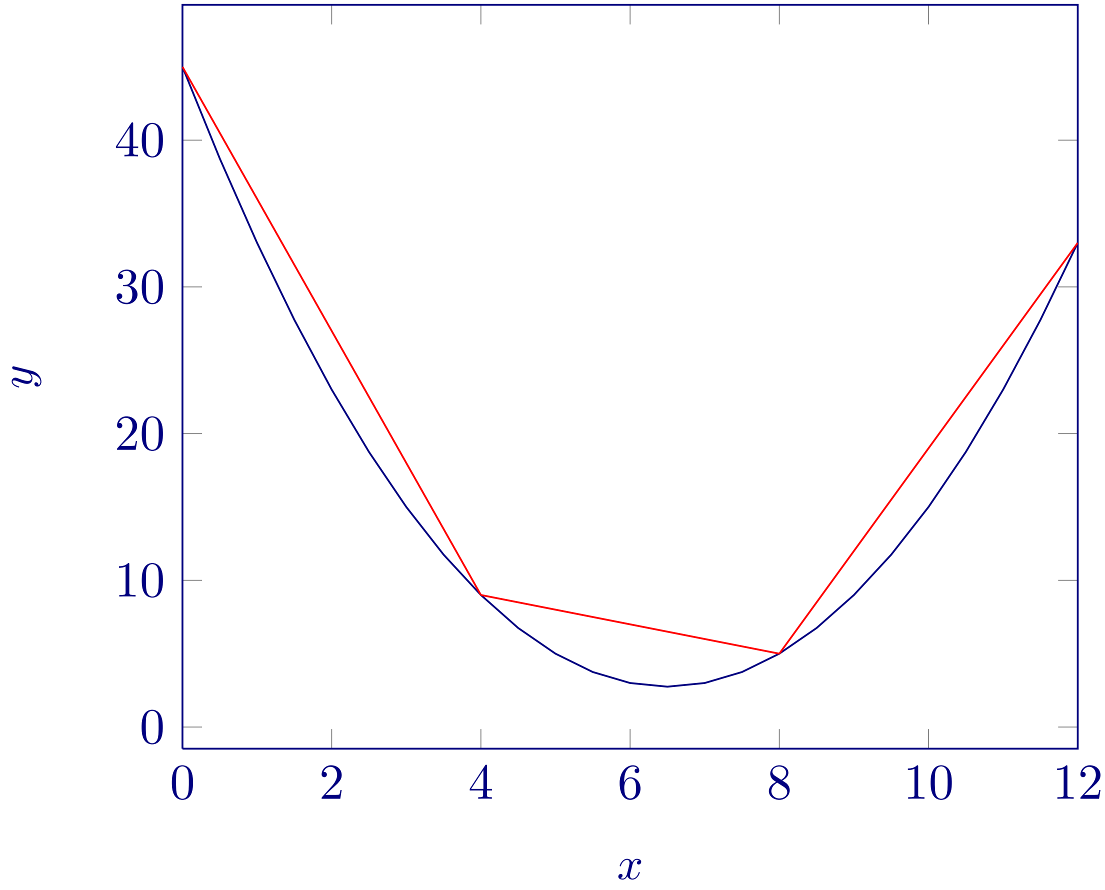

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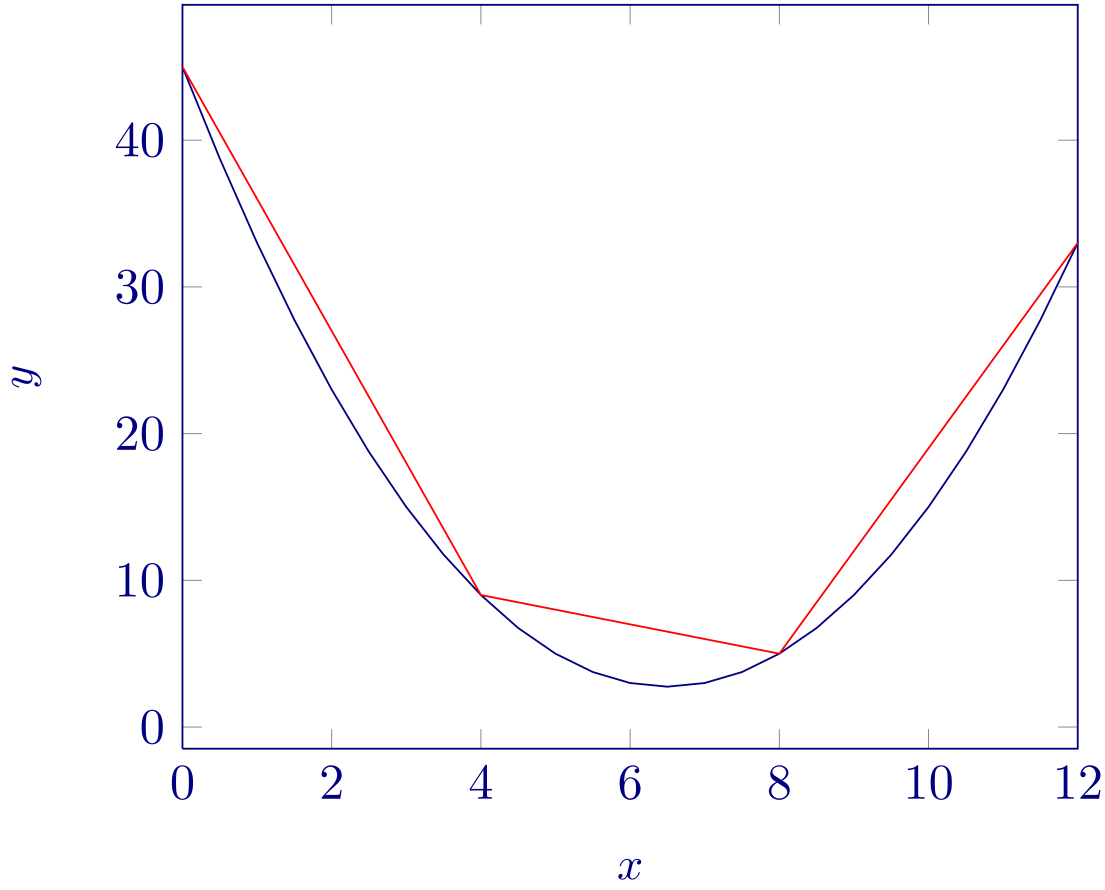

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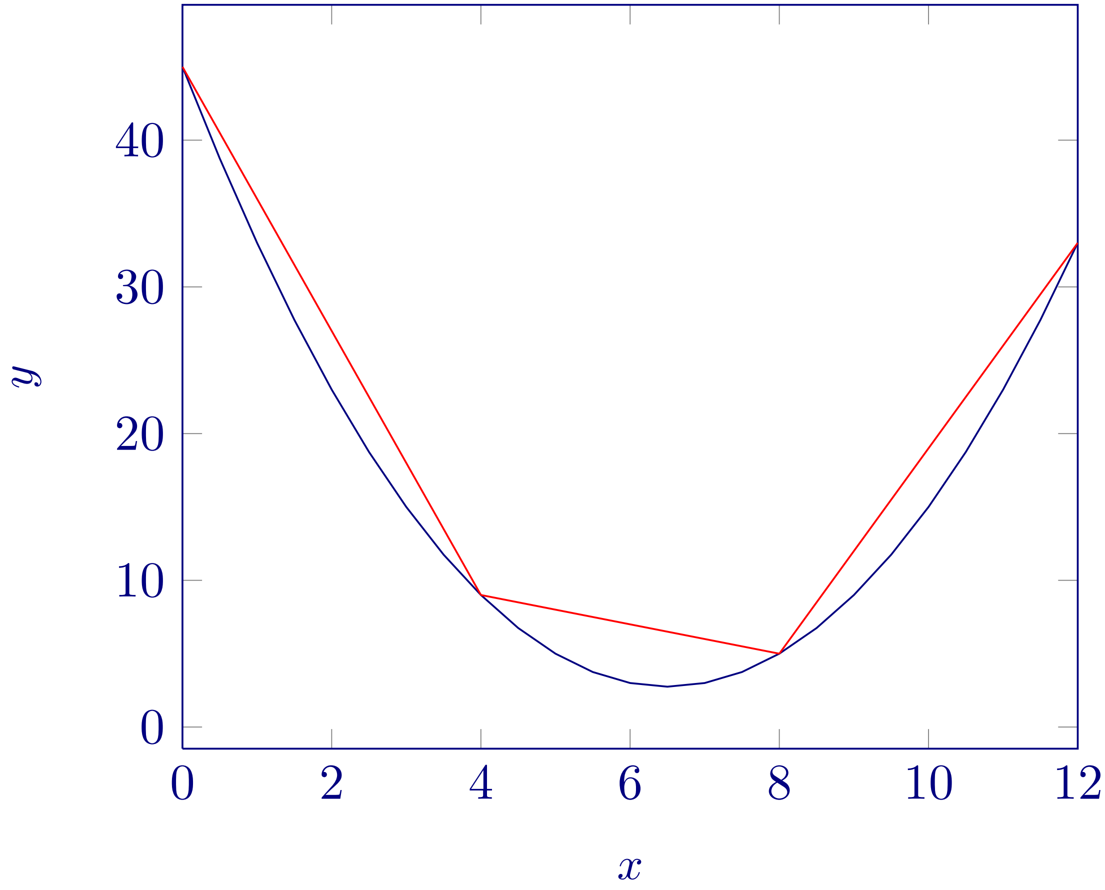

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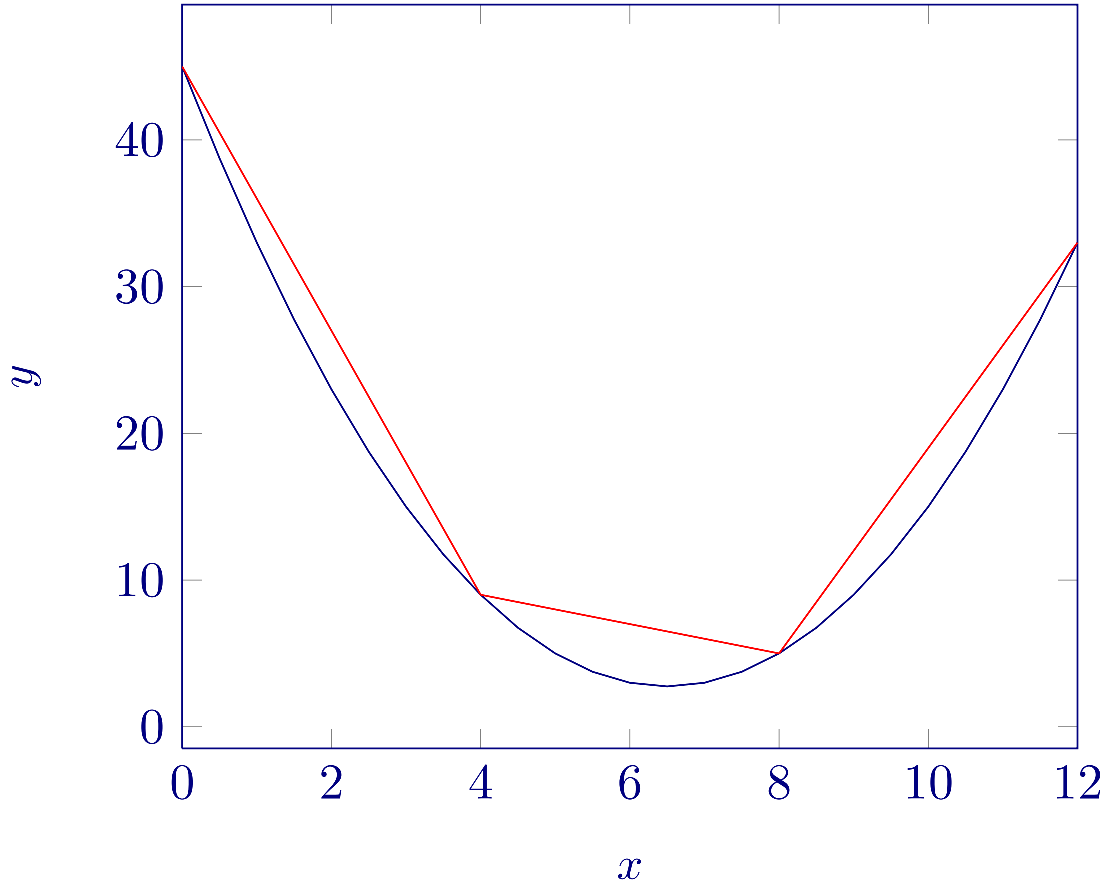

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- Last constraint can be expressed in integer variables

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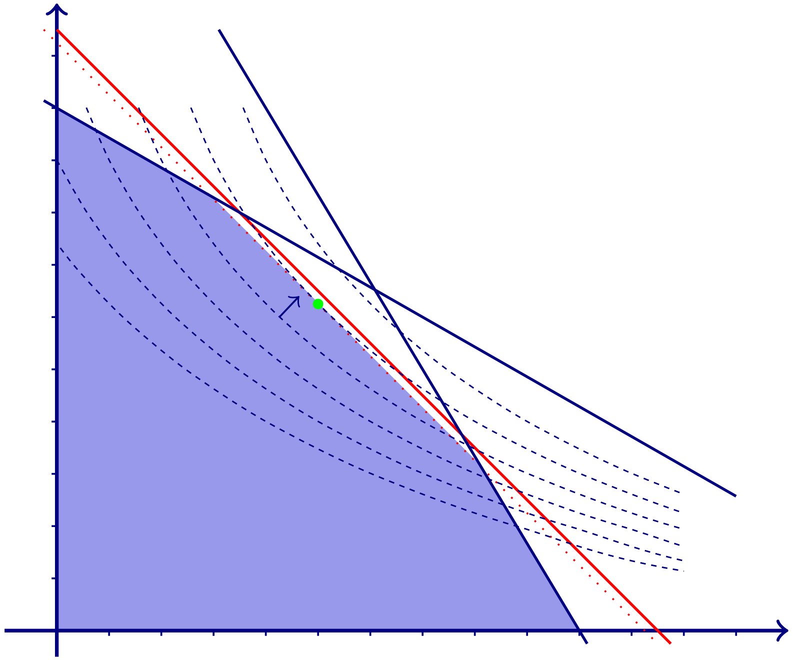

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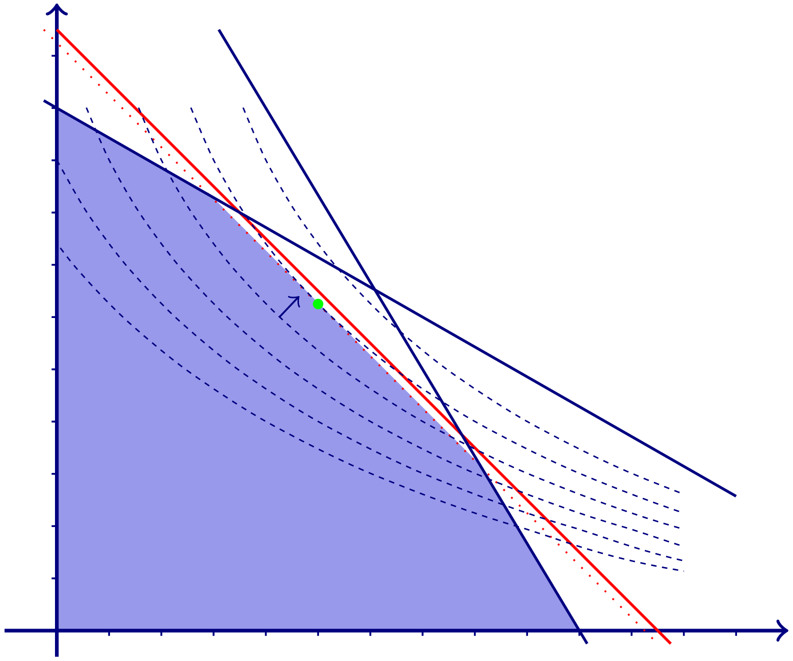

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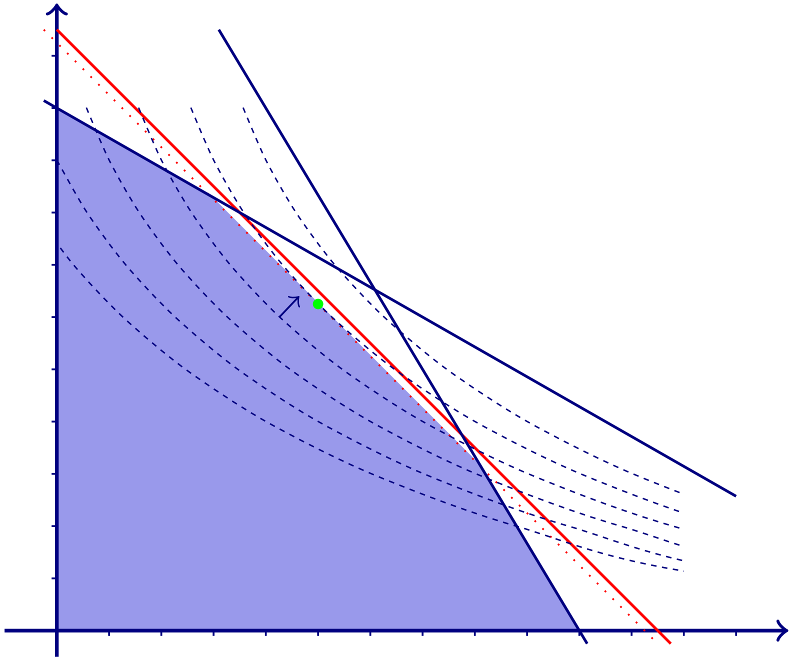

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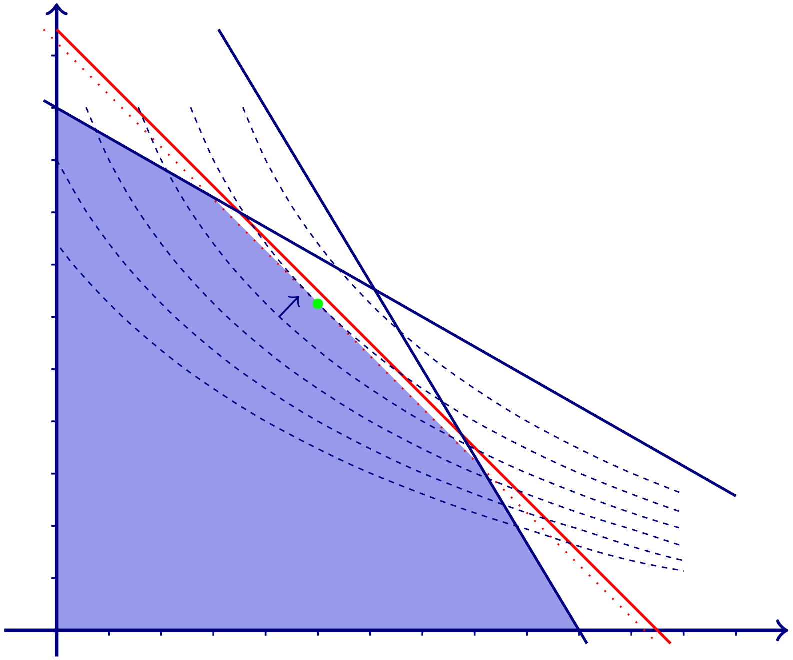

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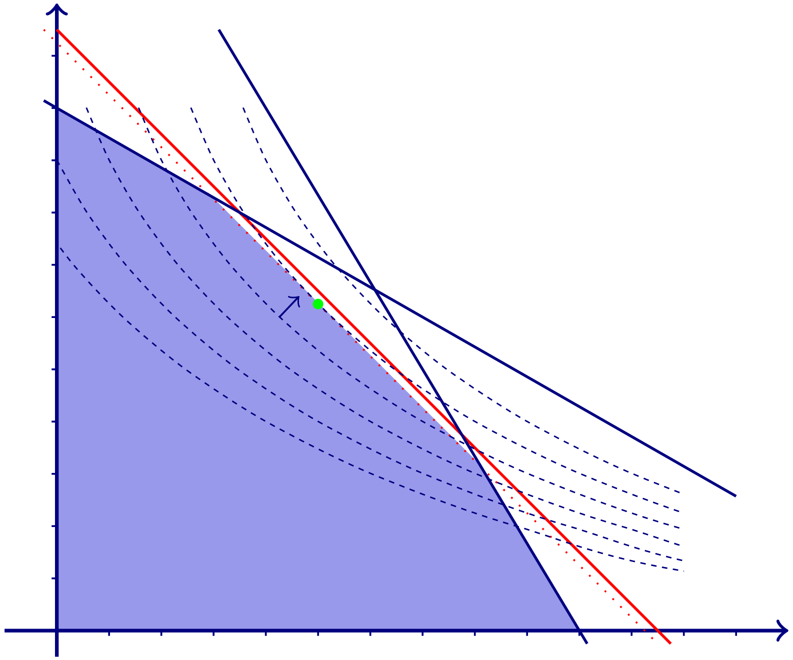

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  - Shadow prices of binding constraints may be 0 if the problem is degenerate

Overview

- ▷ Models, Data and Algorithms
- ▷ Linear Optimization
- ▷ Mathematical Background: Polyhedra, Simplex-Algorithm
- ▷ Sensitivity Analysis; (Mixed) Integer Programming
- ▷ MIP Modelling
- ▷ MIP Modelling: More Examples; Branch & Bound
- ▷ Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- ▷ TSP-Heuristics
- ▷ Network Flows
- ▷ Shortest Path Problem
- ▷ Complexity Theory
- ▷ Nonlinear Optimization
- ▷ Scheduling (Jan 25)
- ▷ Lot Sizing (Feb 01)
- ▷ Summary (Feb 08)
- ▷ Oral exam (Feb 15)