Mathematical Tools for Engineering and Management

Lecture 13

25 Jan 2012

- Models, Data and Algorithms
- Linear Optimization
- Mathematical Background: Polyhedra, Simplex-Algorithm
- Sensitivity Analysis; (Mixed) Integer Programming
- MIP Modelling
- MIP Modelling: More Examples; Branch & Bound
- Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- TSP-Heuristics
- Network Flows
- Shortest Path Problem
- Complexity Theory
- Nonlinear Optimization
- Scheduling
- Lot Sizing & Intro to Multiobjective Optimization (Feb 01)
- Summary (Feb 08)
- Oral exam (Feb 15)

Printing machine

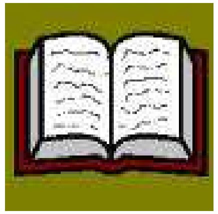
Printing machine

Jobs

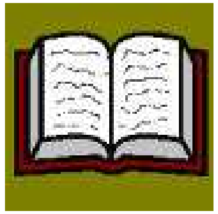
Example: printing machine

Printing machine

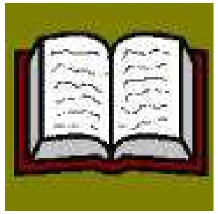
Jobs



Job 1: Book
200 pages, 500 copies
3h printing time



Job 2: Book
60 pages, 2500 copies
4h printing time

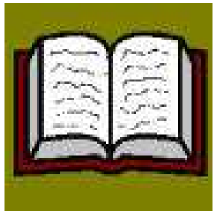


Job 3: Thesis
170 pages, 10 copies
1h printing time

Example: printing machine

Printing machine

Jobs



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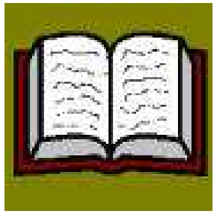

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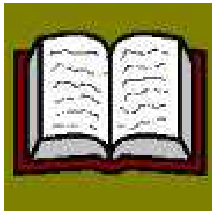

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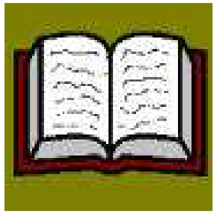
▷ Determine an optimal order for the jobs to be processed...

Printing machine

Jobs



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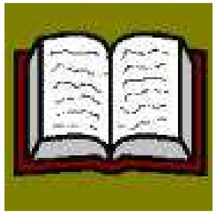

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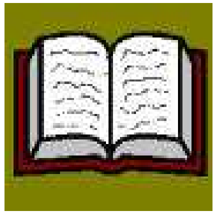

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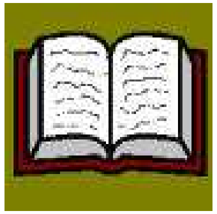
- Determine an optimal order for the jobs to be processed...

➡ ...if jobs have to be finished at a given time

Printing machine

Jobs



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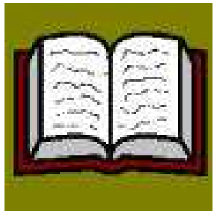

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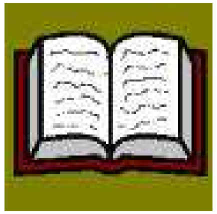

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▷ Determine an optimal order for the jobs to be processed...

➡ ...if jobs have to be finished at a given time

➡ ...if some jobs are more important than others

Printing machine

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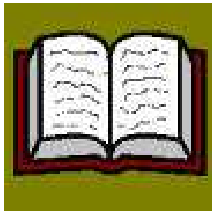

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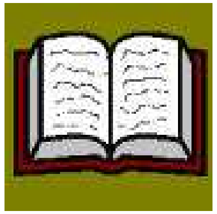

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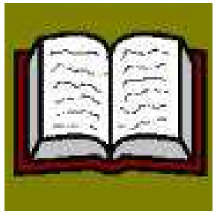

Job 3: Thesis
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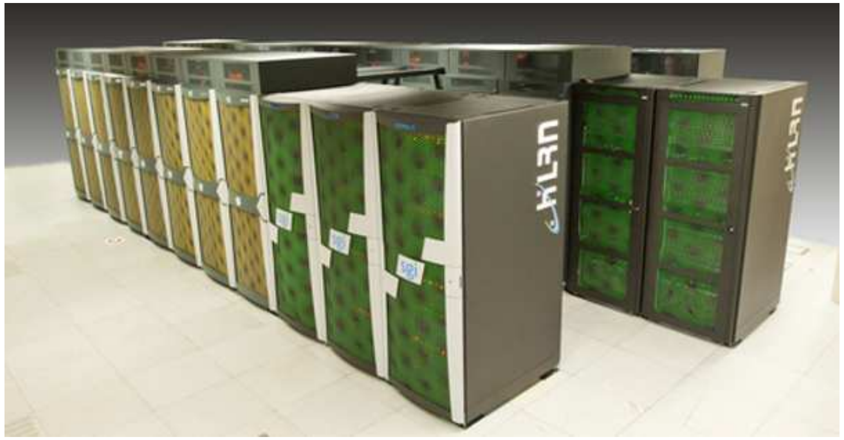
▷ Determine an optimal order for the jobs to be processed...

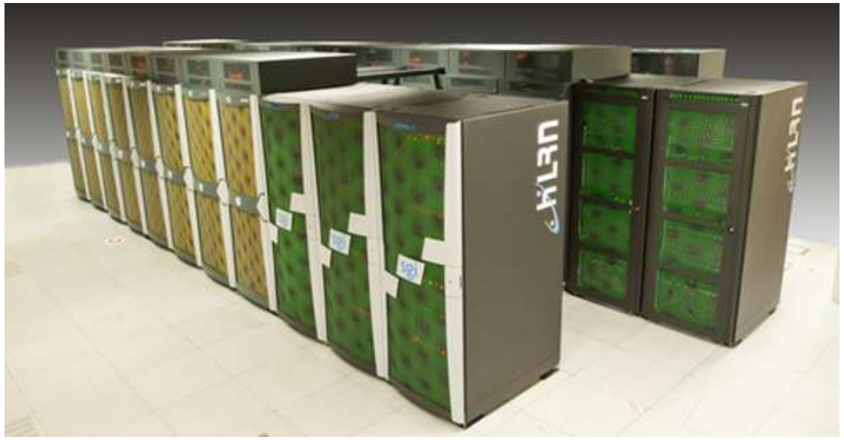
- ➡ ...if jobs have to be finished at a given time
- ➡ ...if some jobs are more important than others
- ➡ ...if there is more than one machine (identical or different machines)

- Supercomputing at ZIB



- Supercomputing at ZIB

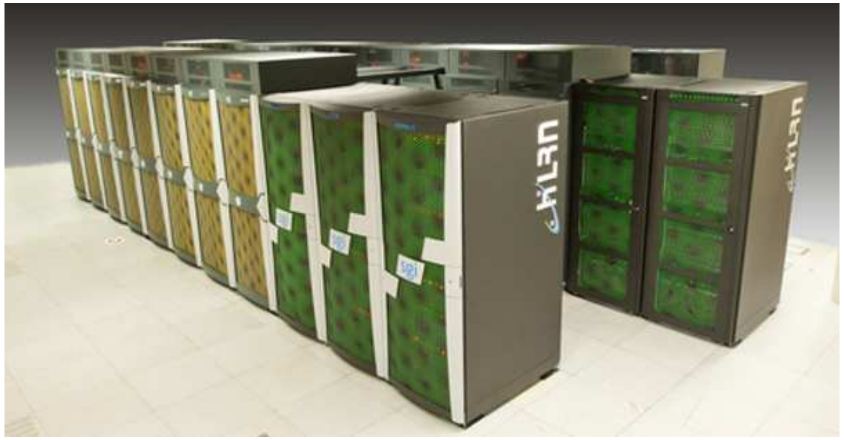
Norddeutscher Verbund für Hoch- und Höchstleistungsrechnen



- ~1500 compute nodes with ~13000 cores

- Supercomputing at ZIB

HLRN
Norddeutscher Verbund für Hoch- und Höchstleistungsrechnen



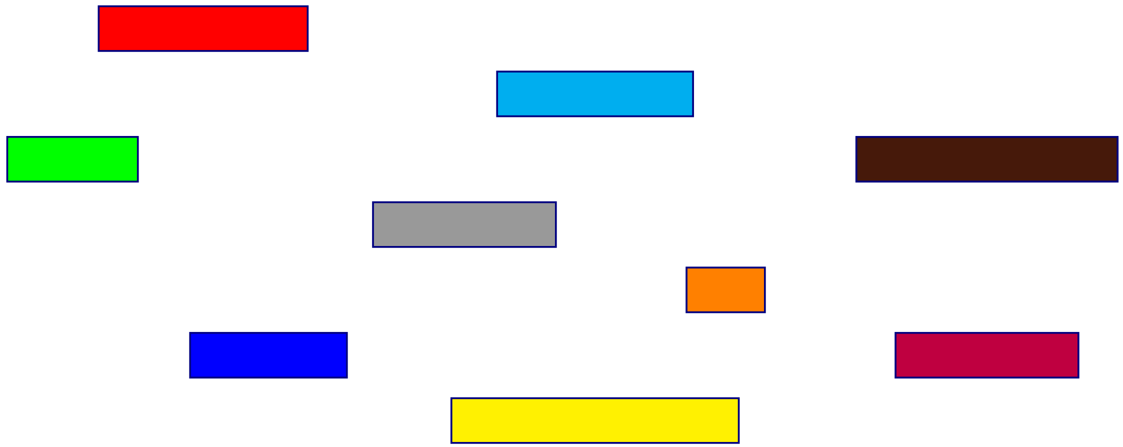
- ∼1500 compute nodes with ∼13000 cores
- Schedule computation jobs...

  ...consisting of thousands of parallel processes

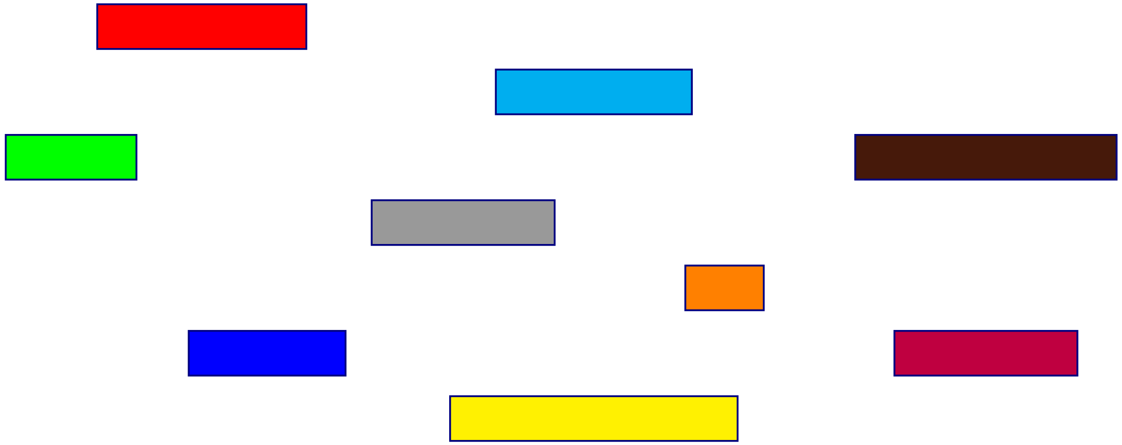
  ...according to their release times

General principle

- Jobs:

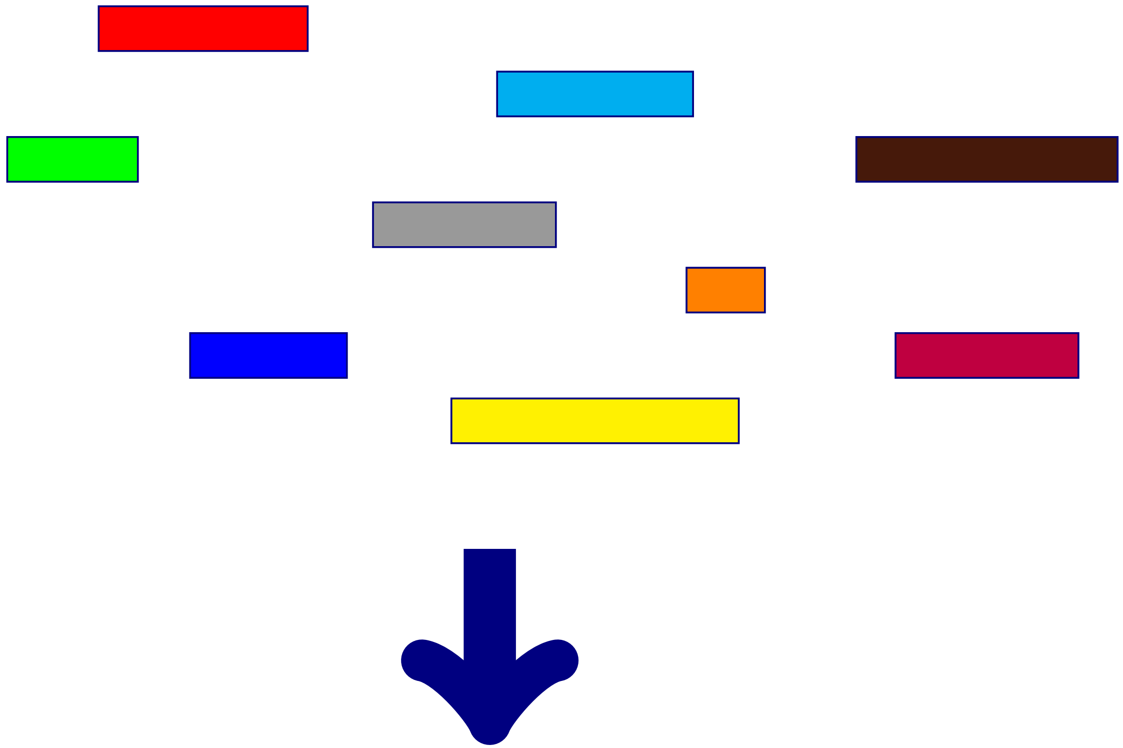


- Jobs:



- Schedule (Gantt chart):

- Jobs:



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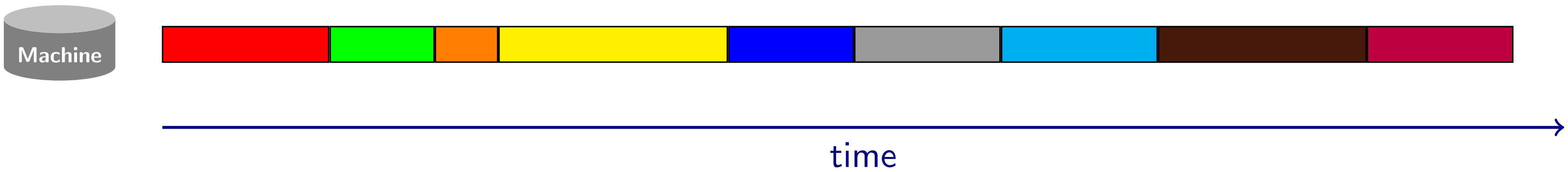
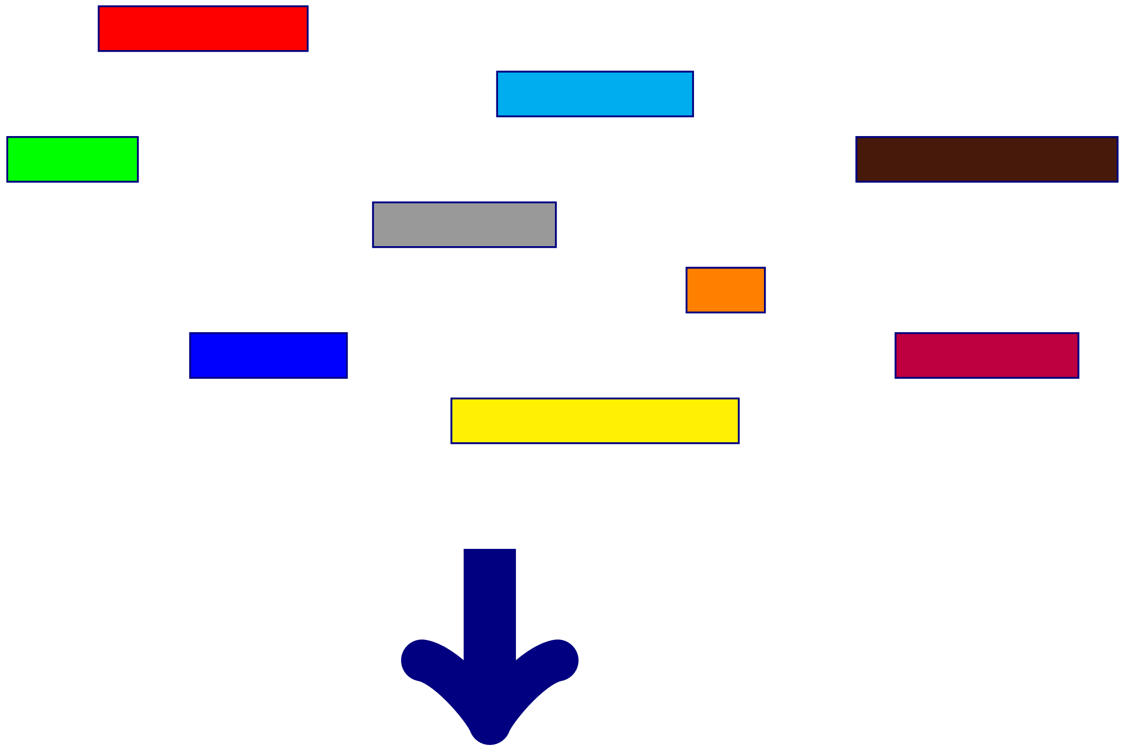
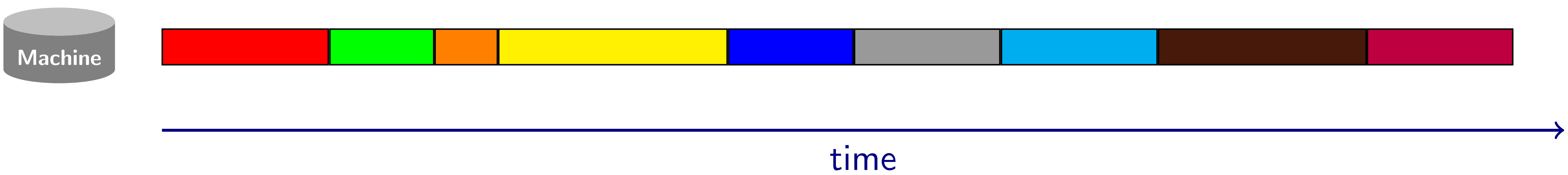


- ▷ Jobs:



- ▷ Schedule (Gantt chart): ➡ optimal with respect to an objective to specify!



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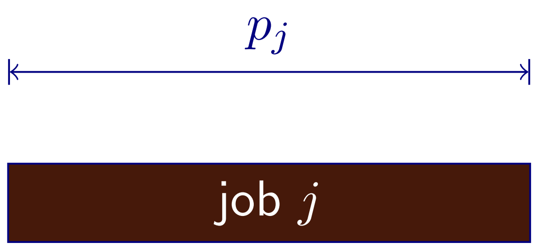

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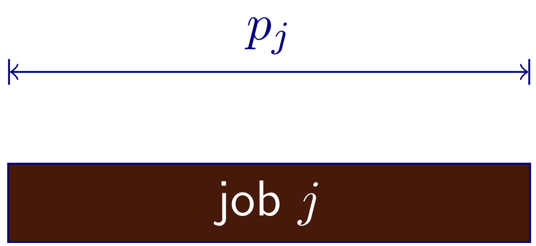
- A schedule has to provide: a **start time** s_j



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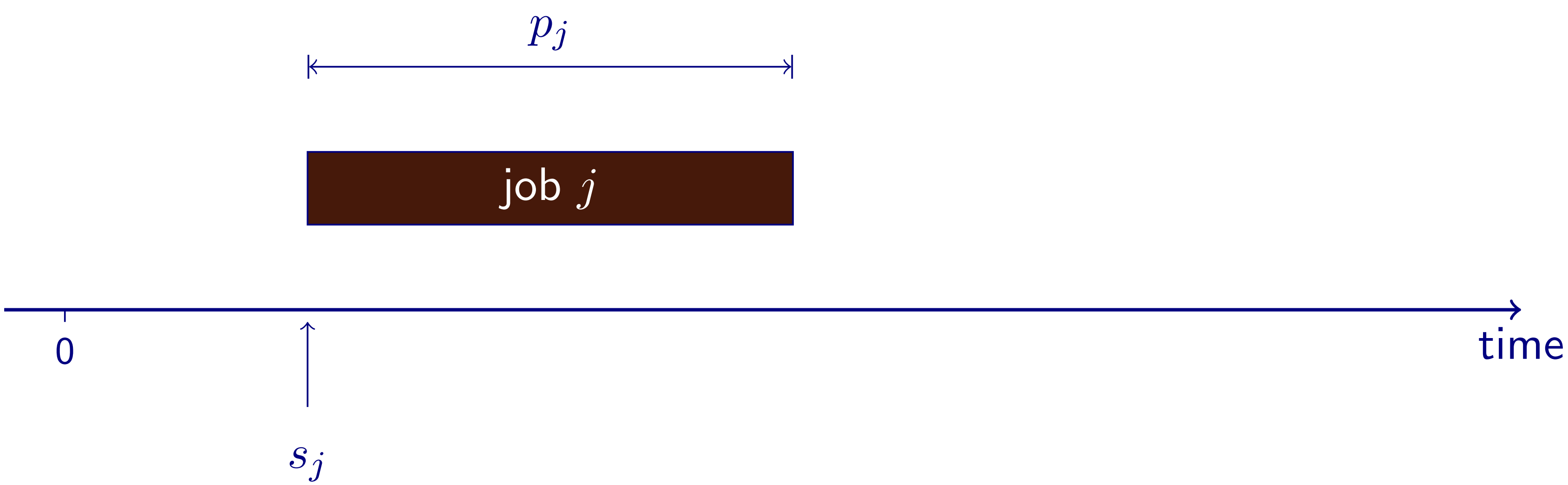

- Jobs usually have: a **processing time** p_j
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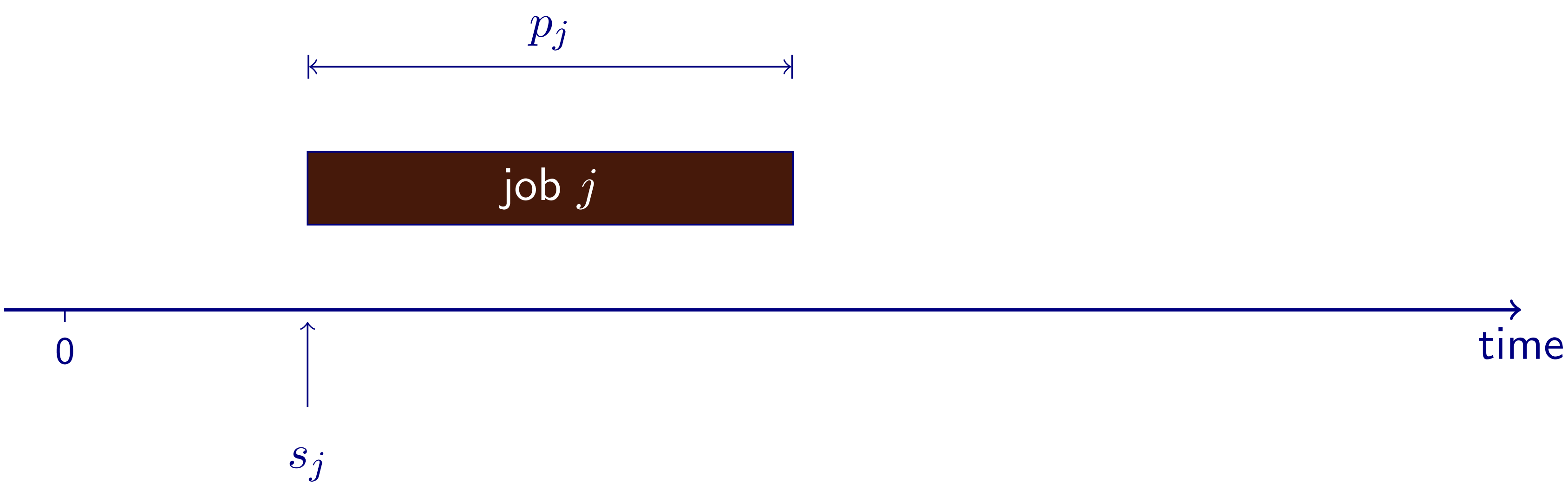

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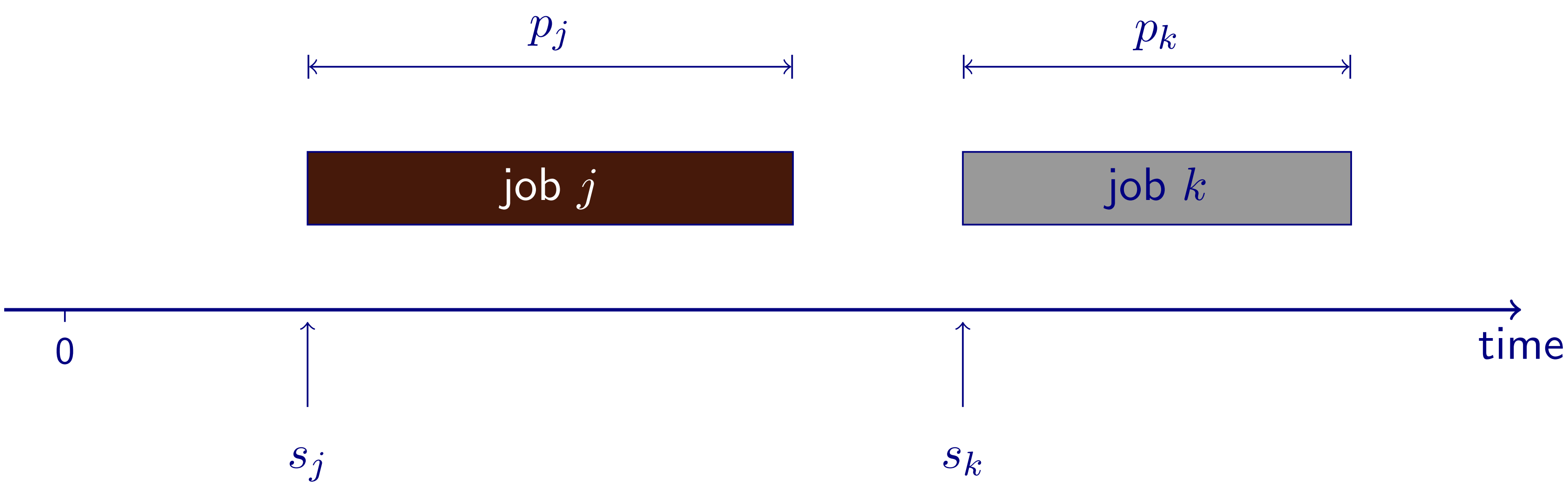

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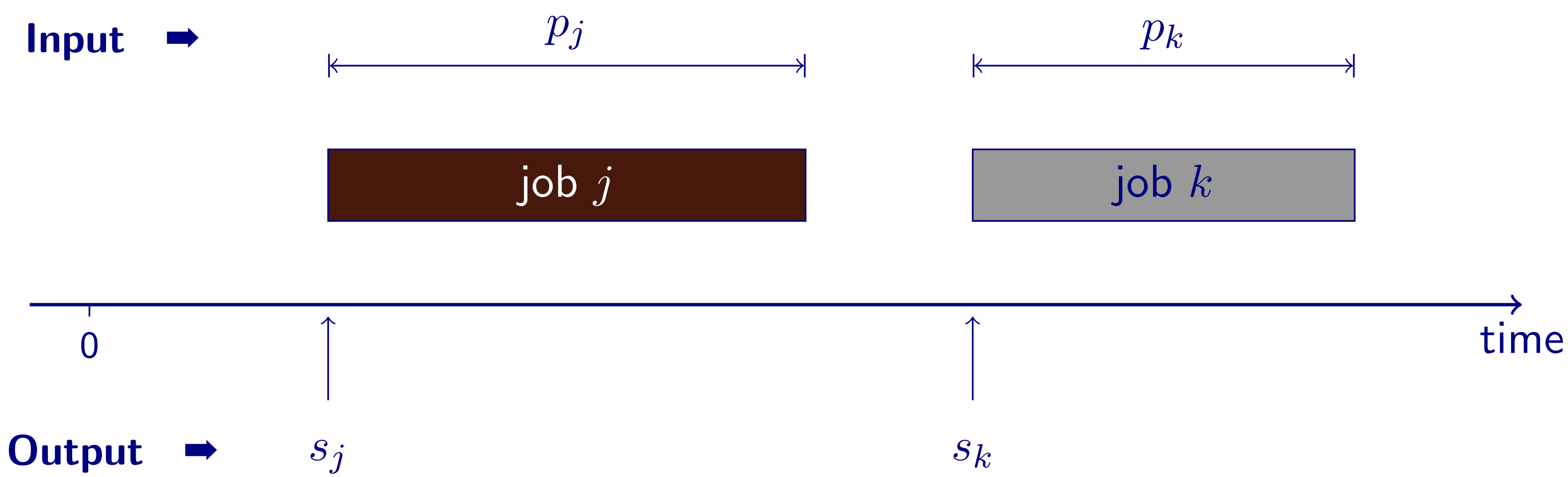

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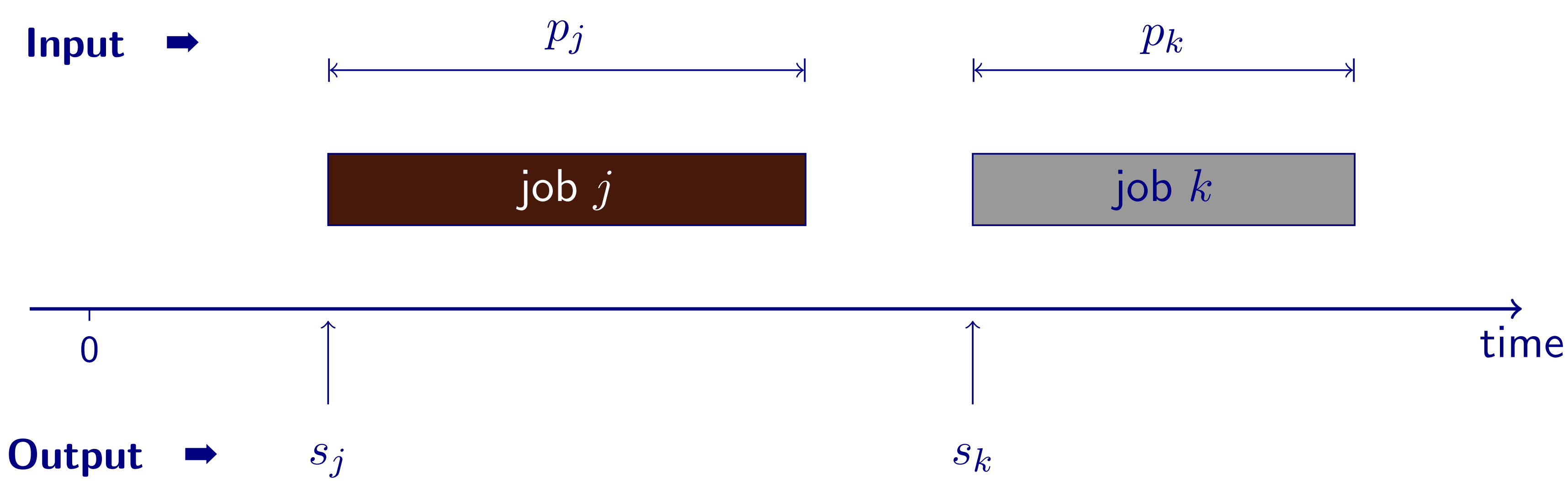
➡ Combinatorial optimization problem



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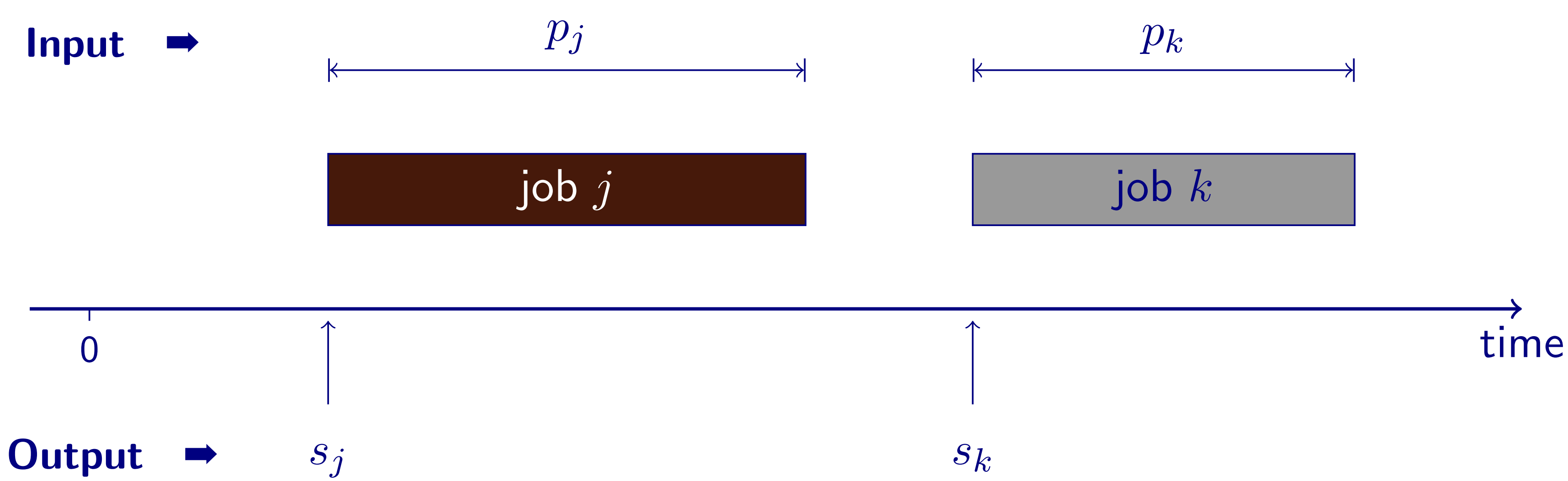
➡ Combinatorial optimization problem ➡ IP formulations for most scheduling problems



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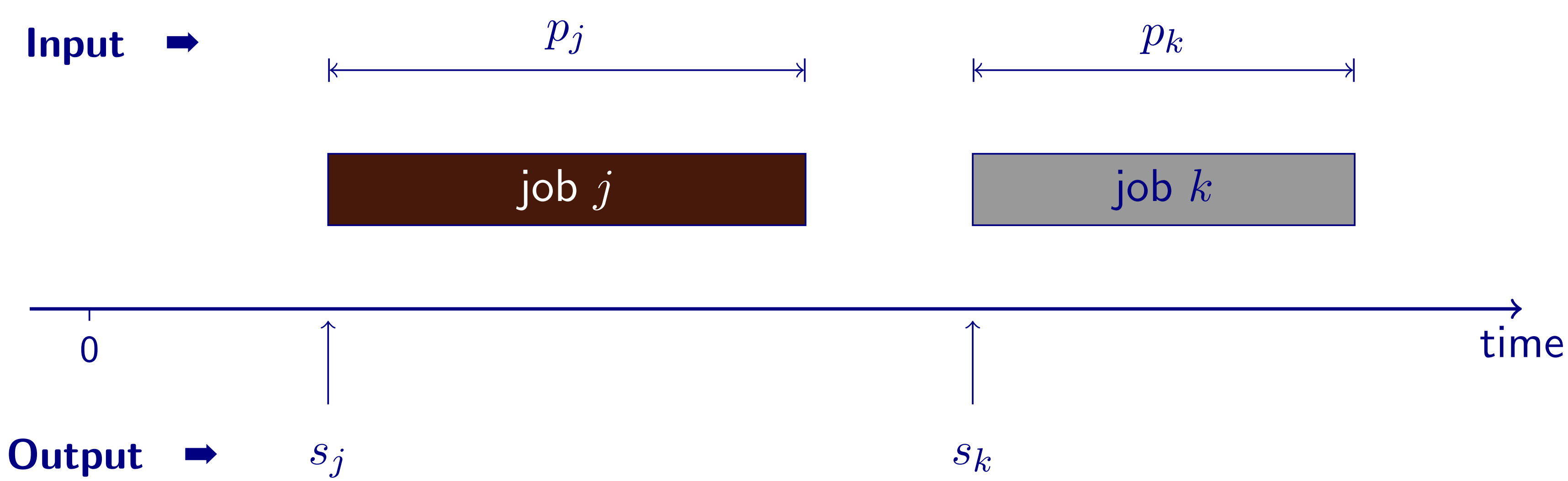
- ➡ **Completion time** $C_j := s_j + p_j$



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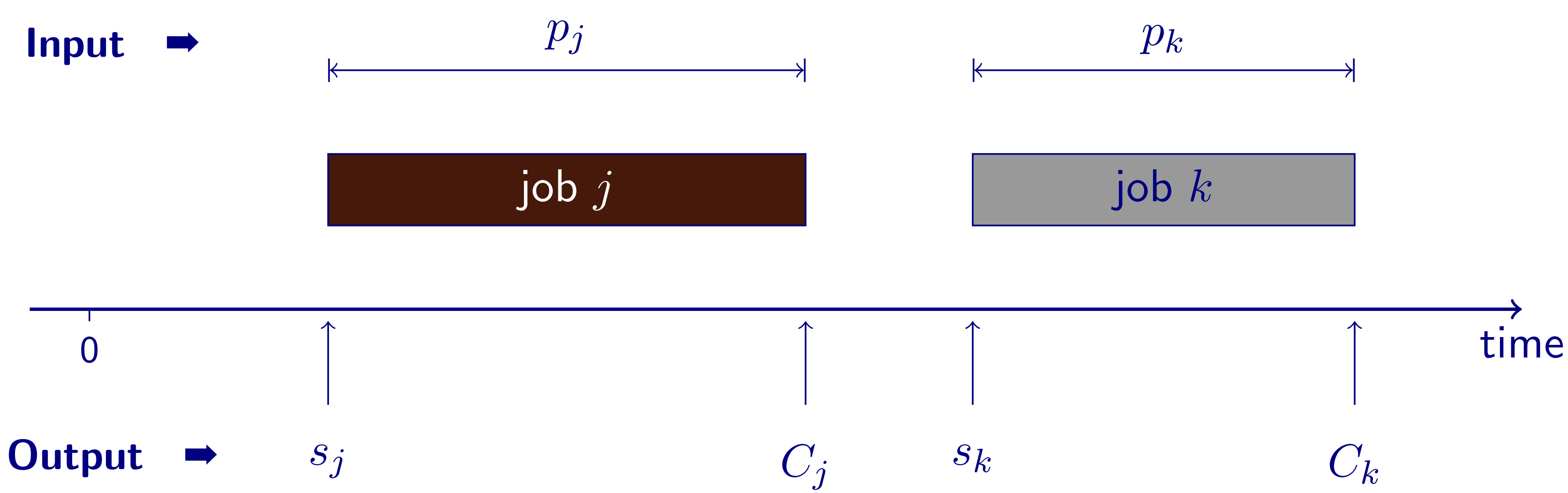


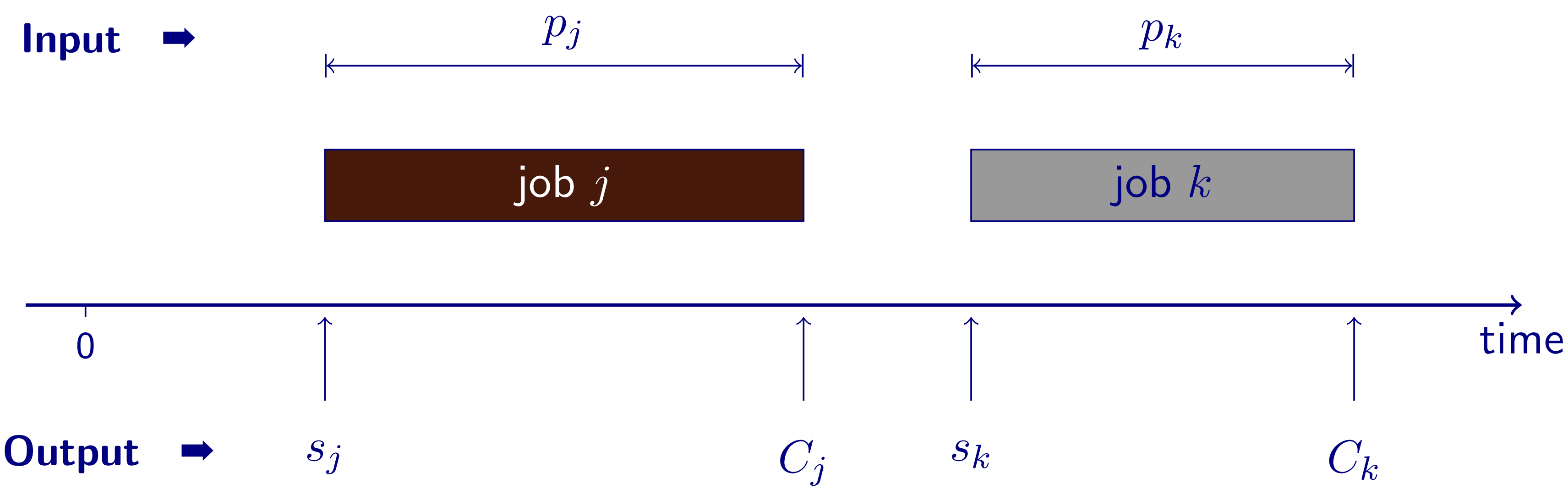
Notation

- Jobs usually have: a processing time p_j
- A schedule has to provide: a start time s_j, such that different jobs do not overlap

➡ Combinatorial optimization problem ➡ IP formulations for most scheduling problems

➡ Completion time $C_j := s_j + p_j$

➡ Average completion time for n jobs: $\frac{1}{n}\sum_{j=1}^{n} C_j$



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Minimize sum of completion times

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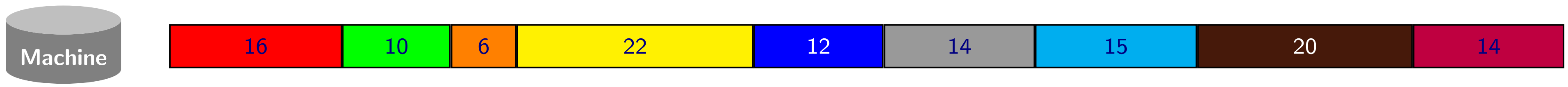

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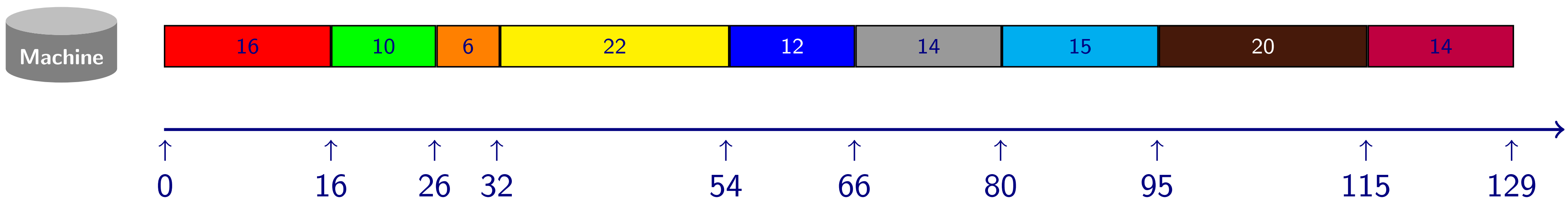

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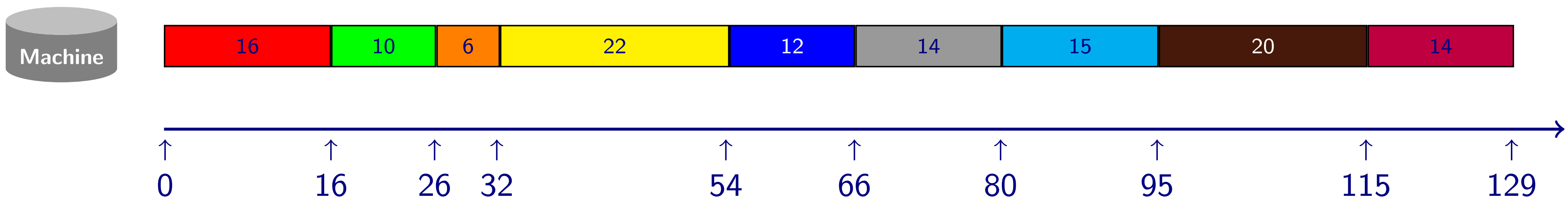

$$\Rightarrow \sum_{j=1}^{n} C_j = 16 + 26 + 32 + 54 + 66 + 80 + 95 + 115 + 129 = 613$$

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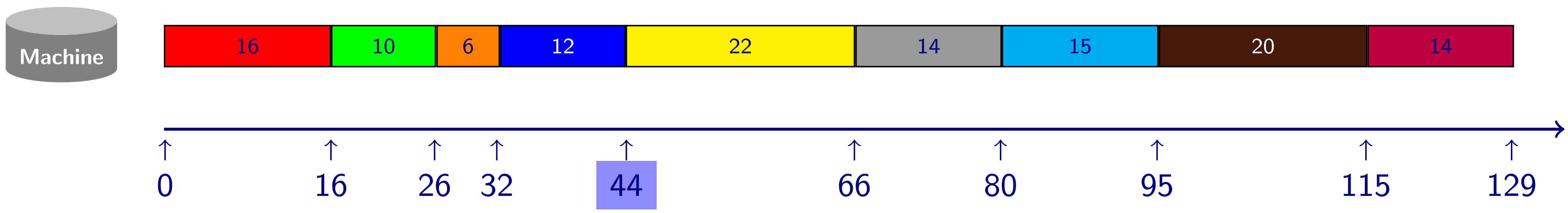

$$\Rightarrow \sum_{j=1}^{n} C_j = 16 + 26 + 32 + 44 + 66 + 80 + 95 + 115 + 129 = 603$$

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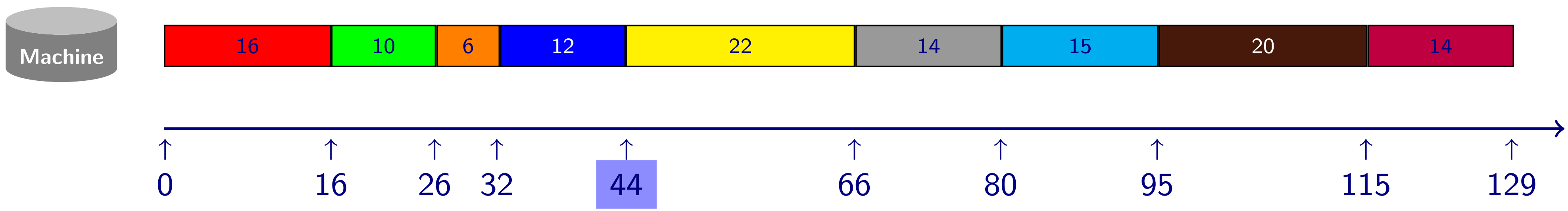

➡ $$\sum_{j=1}^{n} C_j = 16 + 26 + 32 + \boxed{44} + 66 + 80 + 95 + 115 + 129 = \boxed{603}$$

➡ Idea: Schedule jobs in order of non-decreasing processing time!

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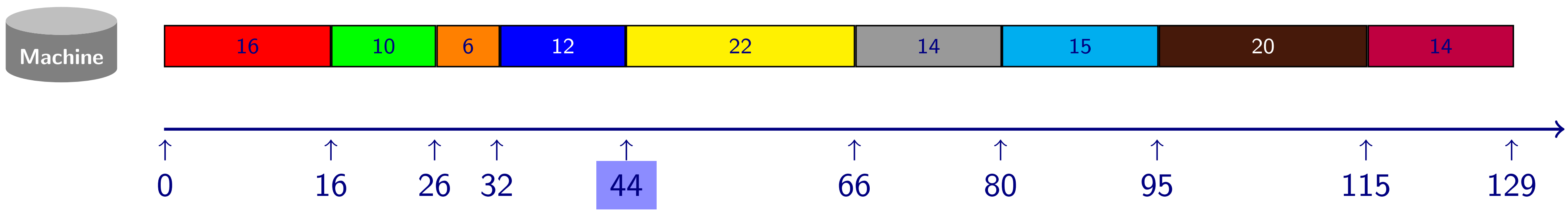

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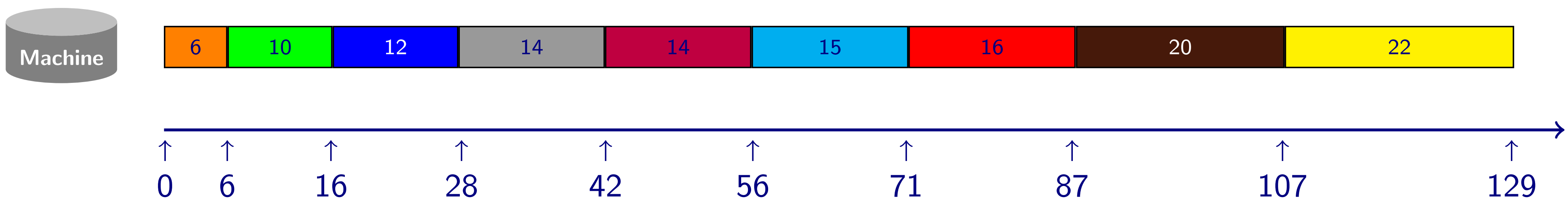

$$\Rightarrow \sum_{j=1}^{n} C_j = 6 + 16 + 28 + 42 + 56 + 71 + 87 + 107 + 129 = 542$$

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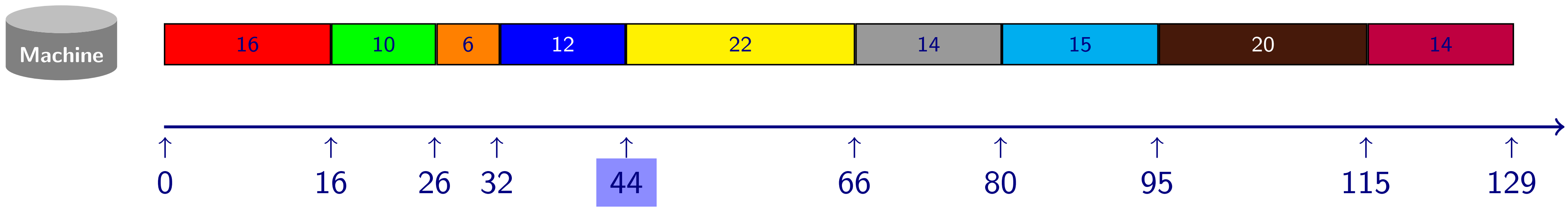
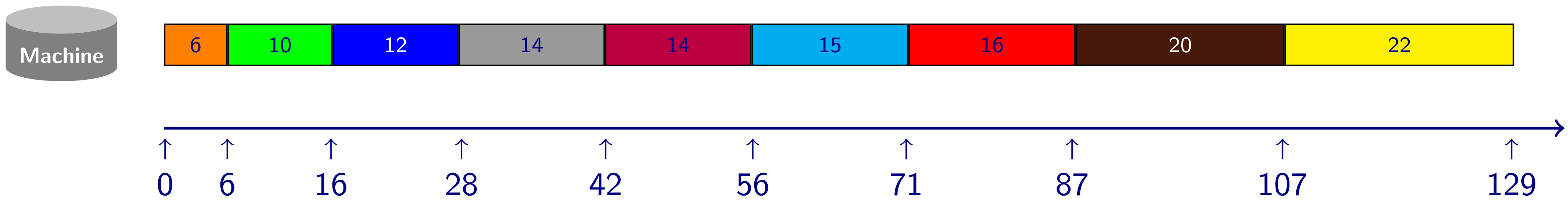


➡ $$\sum_{j=1}^{n} C_j = 16 + 26 + 32 + 44 + 66 + 80 + 95 + 115 + 129 = 603$$

➡ Idea: Schedule jobs in order of non-decreasing processing time! ➡ **provably optimal!**



➡ $$\sum_{j=1}^{n} C_j = 6 + 16 + 28 + 42 + 56 + 71 + 87 + 107 + 129 = 542$$

- Latest completion time ➡ minimize makespan $\max_{j=1,\ldots,n} C_j$

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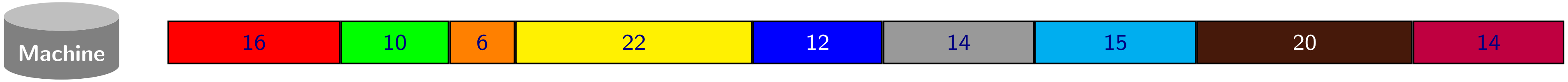

Minimize makespan

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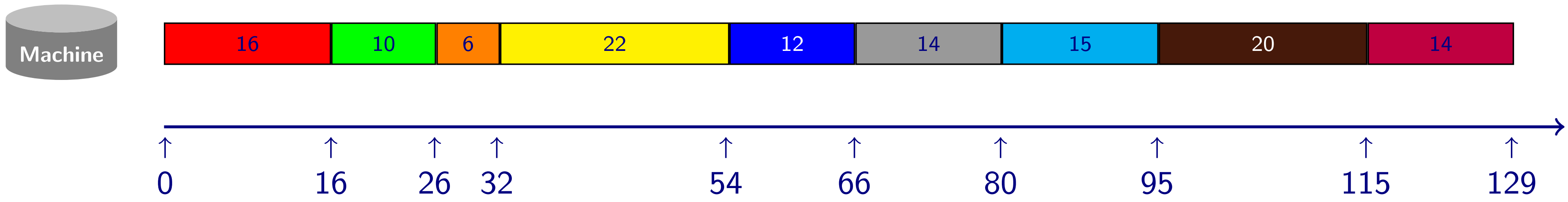

- ▷ Latest completion time ➡ minimize **makespan** $\max_{j=1,\ldots,n} C_j$
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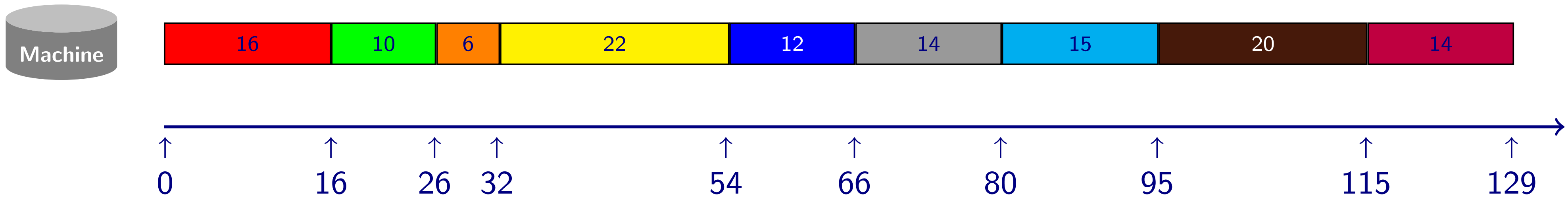

➡ makespan: 129

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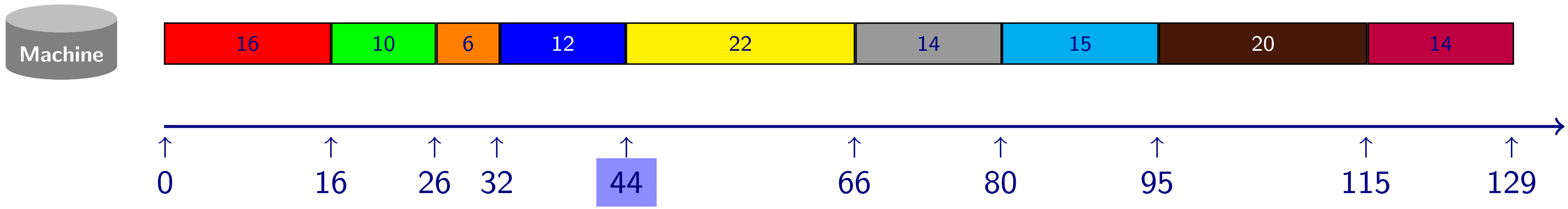

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Minimize makespan

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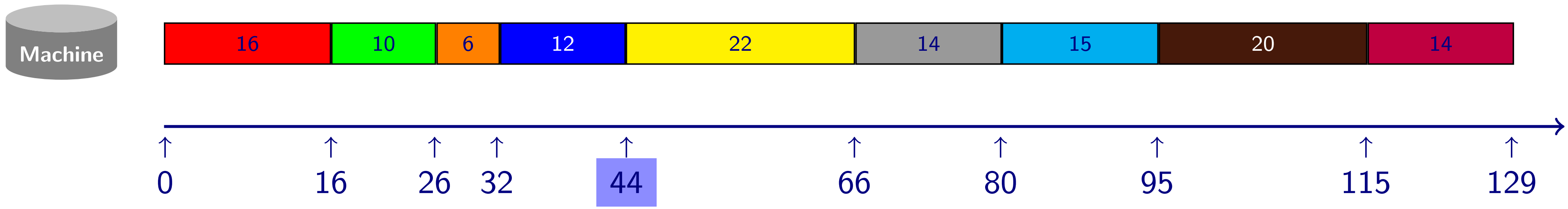

➡ makespan: 129

➡ Any schedule (without idle times) gives the same makespan!

- Latest completion time ➡ minimize makespan $\max_{j=1,\ldots,n} C_j$
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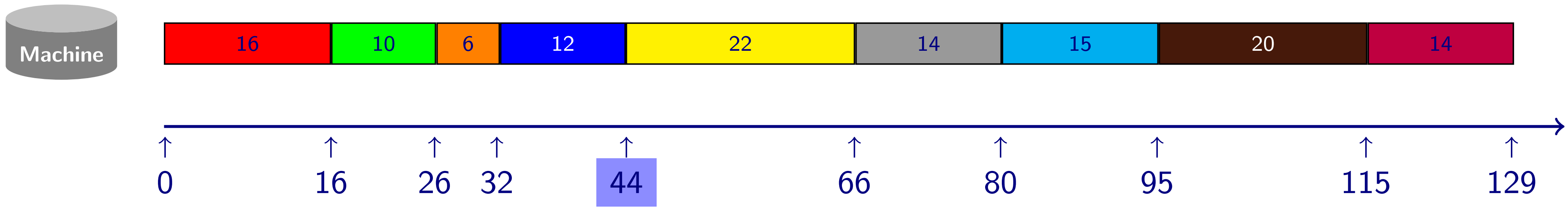


➡ makespan: 129

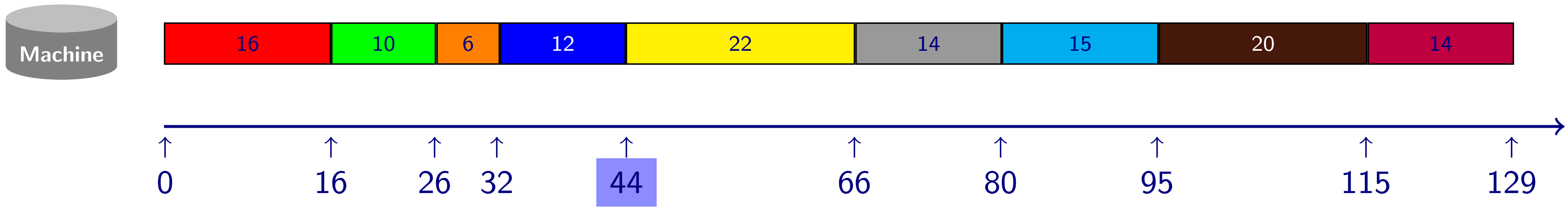
➡ Any schedule (without idle times) gives the same makespan!

Obvious, since the makespan

$$\max_{j=1,\ldots,n} C_j = \sum_{j=1}^{n} p_j$$

Minimize makespan

- Latest completion time ➡ minimize makespan $\max_{j=1,\dots,n} C_j$
- Example schedule:



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➡ Any schedule (without idle times) gives the same makespan!

Obvious, since the makespan

$$\max_{j=1,\dots,n} C_j = \sum_{j=1}^{n} p_j$$

depends only on the input (processing times), not on the schedule itself

- Jobs can have: a release date r_j

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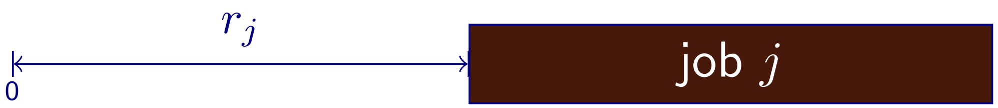

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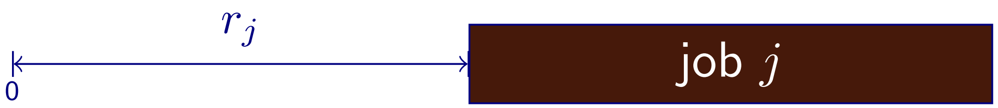

➡ Start time of job j cannot be before its release date

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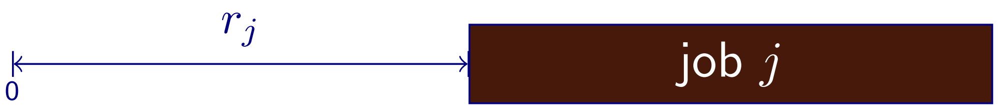

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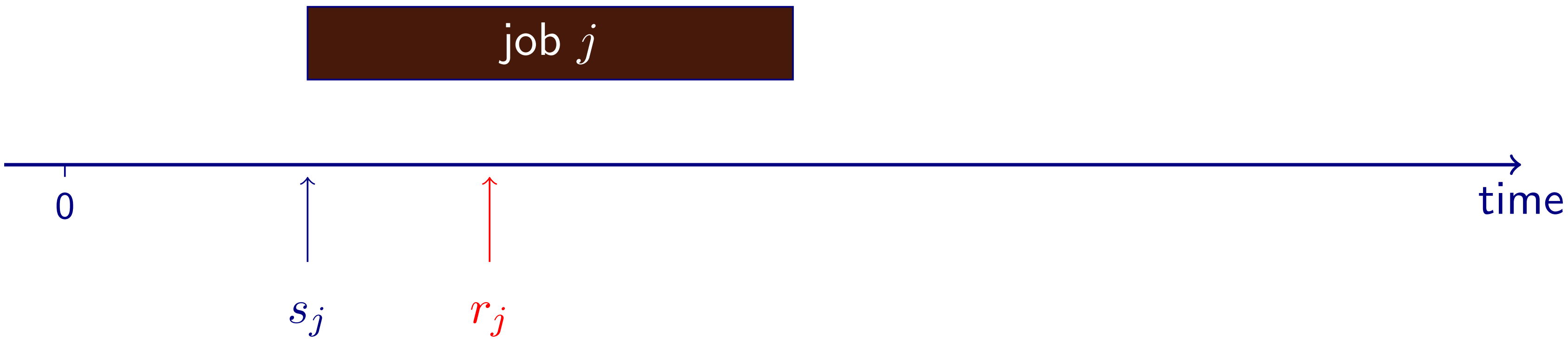

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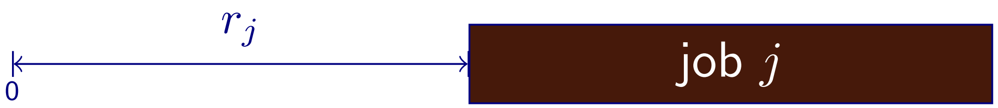

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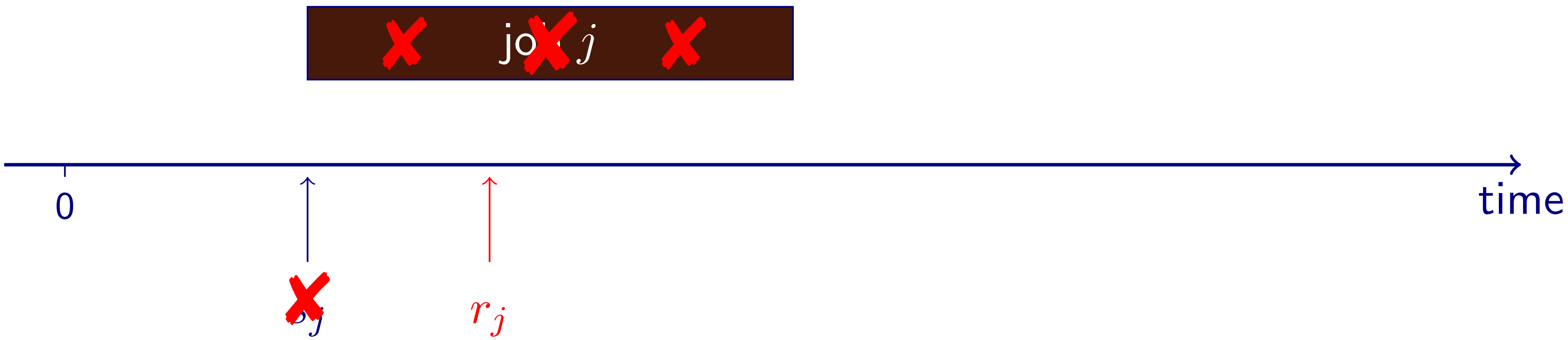

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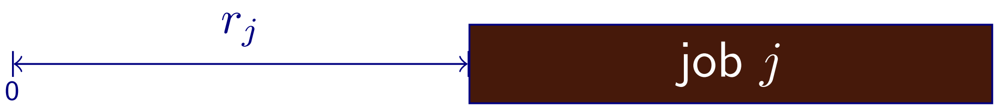

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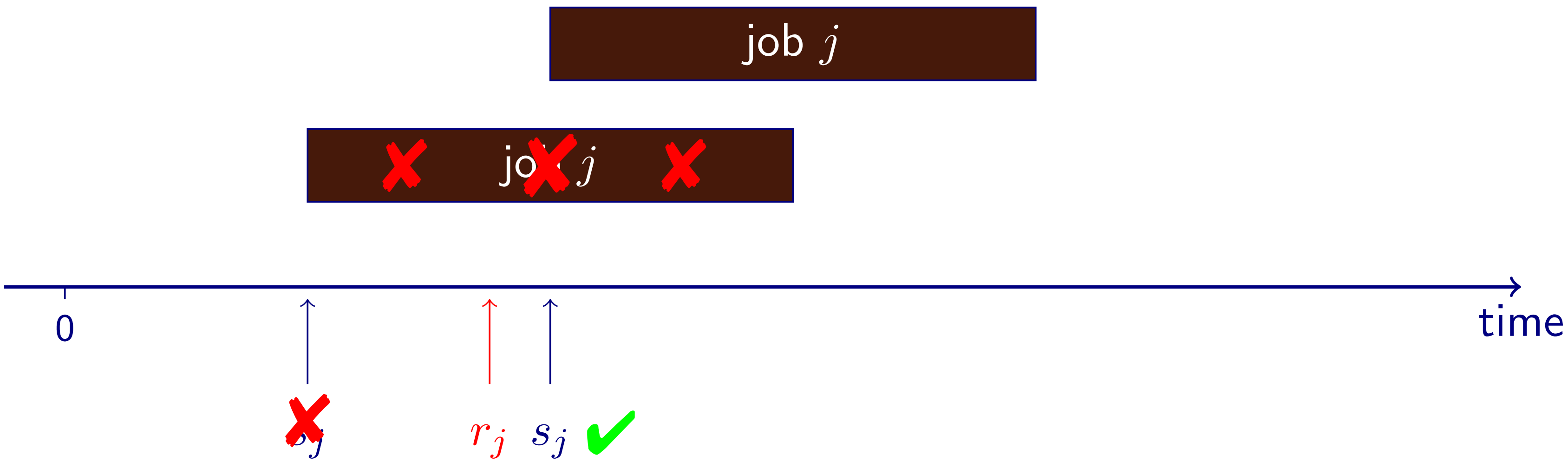

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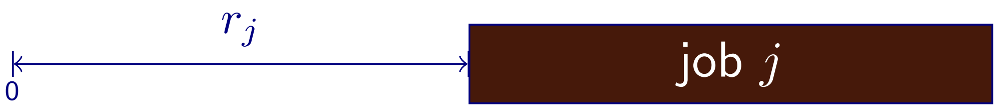

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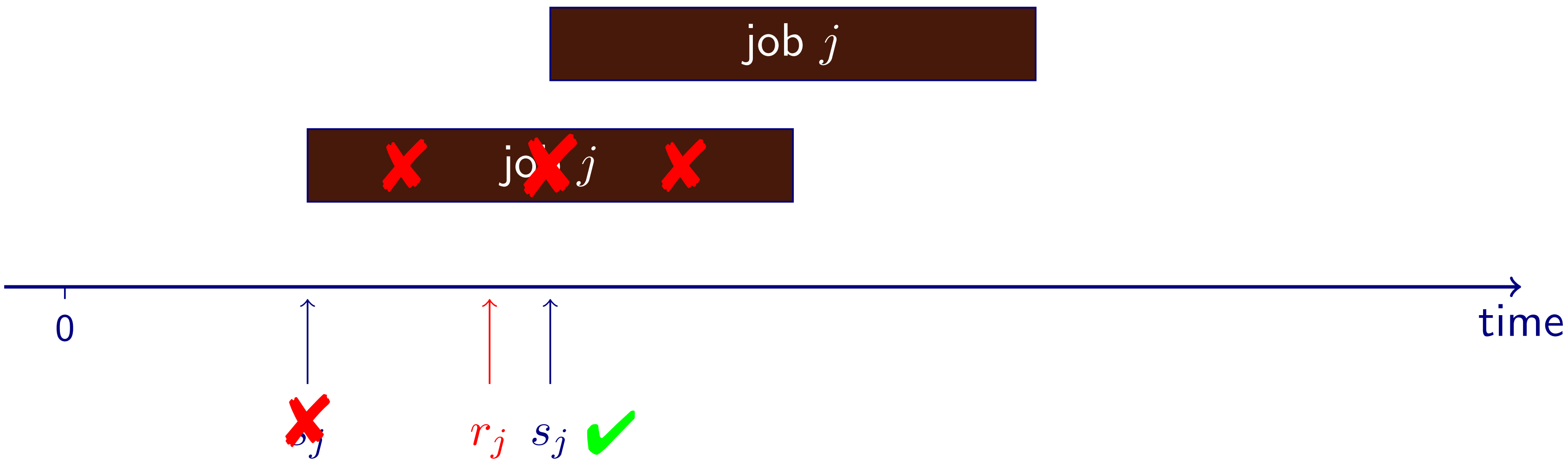
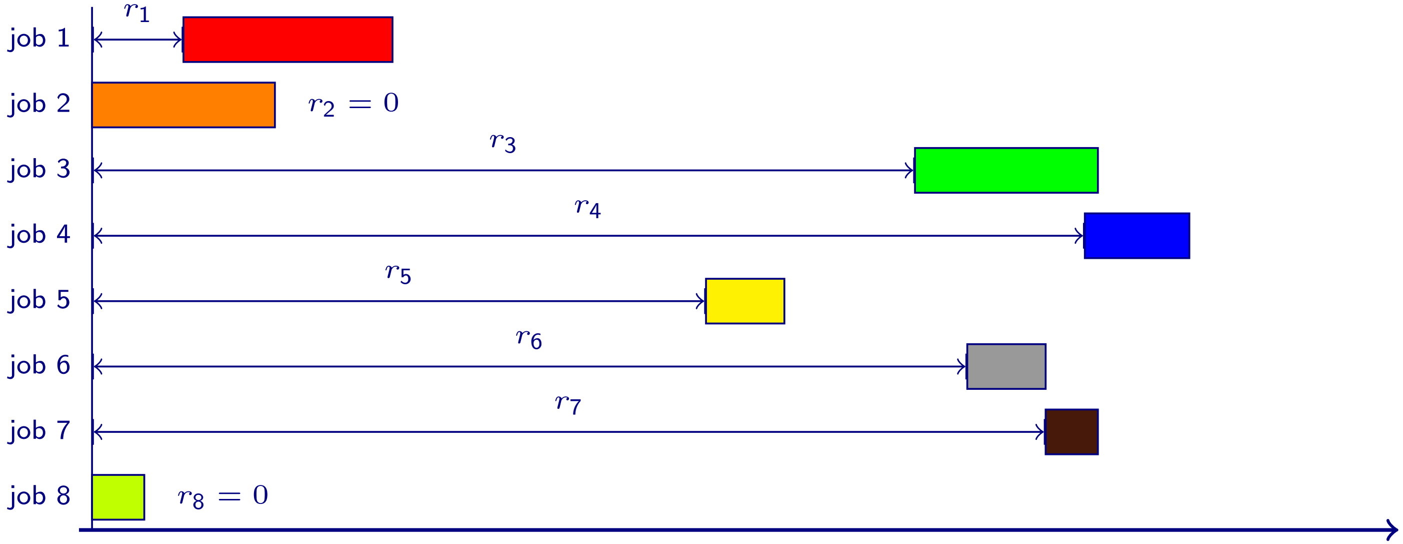
➡ Constraint: $s_j \geq r_j$



- Input now: jobs with release dates



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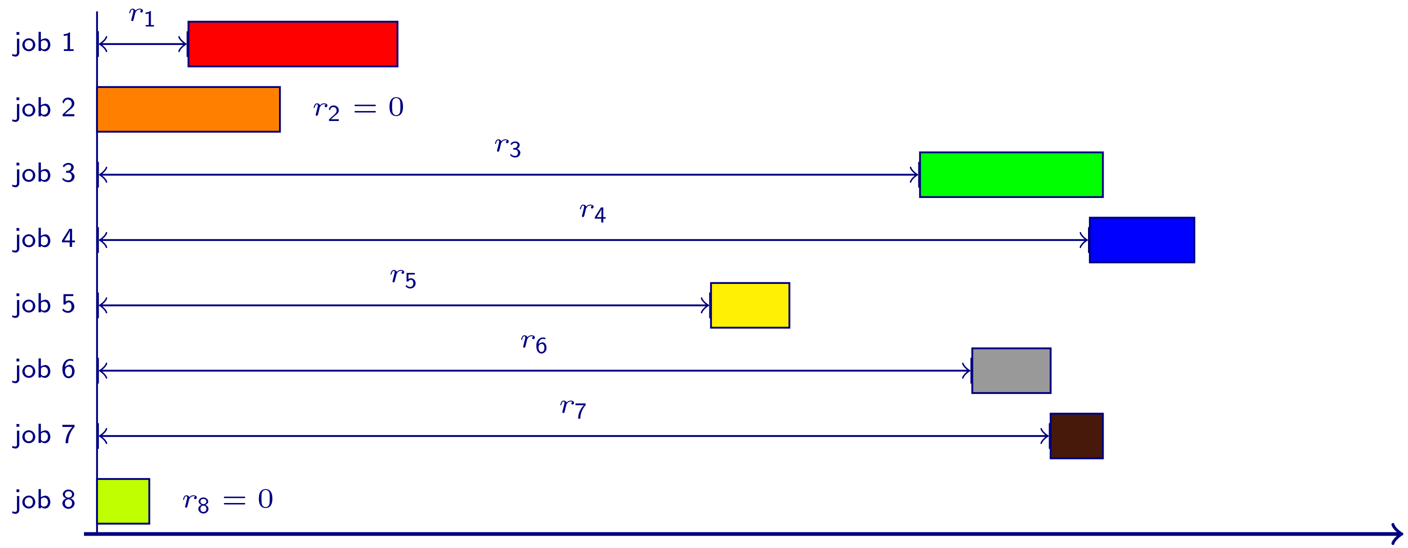

- Minimize makespan

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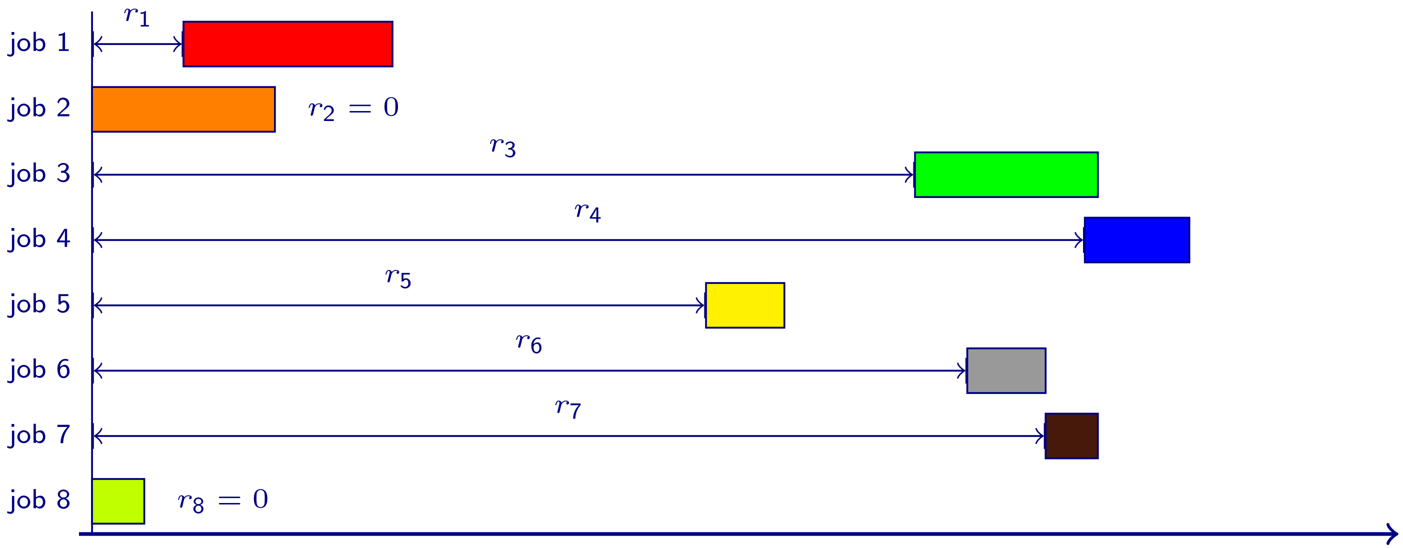

▷ Minimize makespan

➡ Optimal algorithm: schedule jobs in the order of non-decreasing release dates

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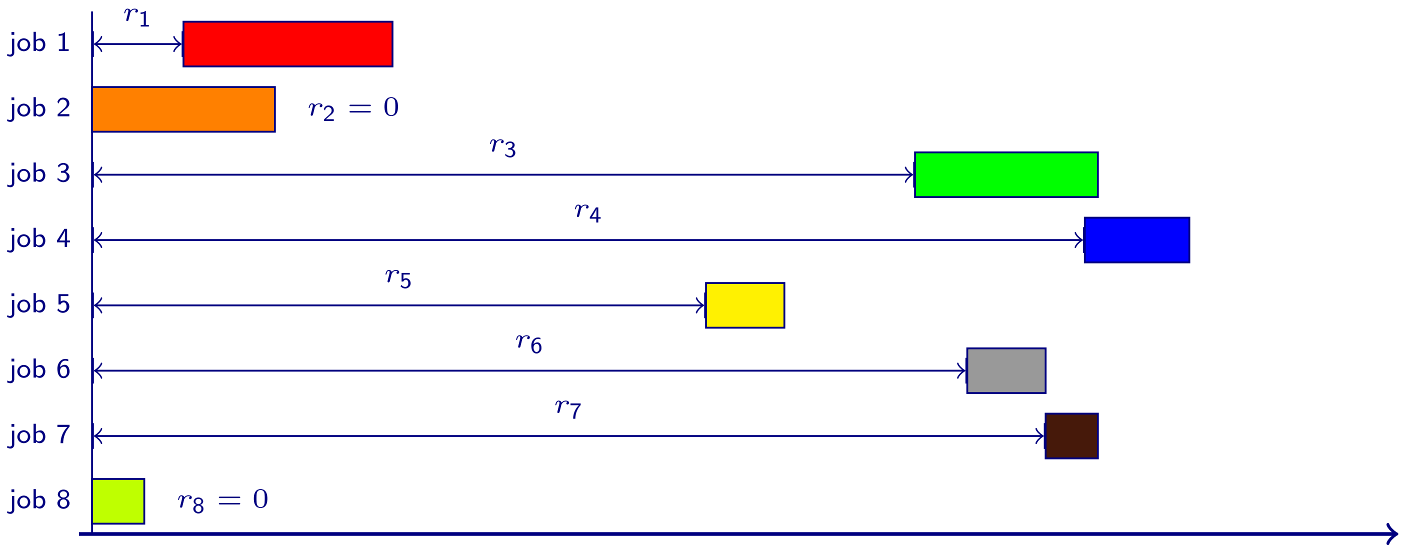

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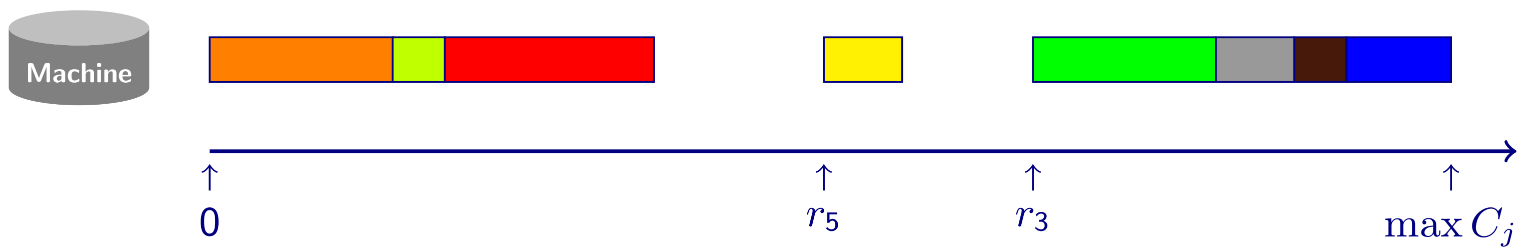

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More variants

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  - ➡ Resource-constrained scheduling

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- ▷ Summary if no interruptions and resources are involved:

	Objective		
	$\sum C_j$	$\max C_j$	lateness
no release dates	non-decreasing process times	trivial	non-decreasing due dates
with release dates	$\mathcal{NP}$-hard	non-decreasing release dates	$\mathcal{NP}$-hard

▷ Example:

job j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
process time p_j	5	6	9	12	7	12	10	6	10	9	7	8	7	5

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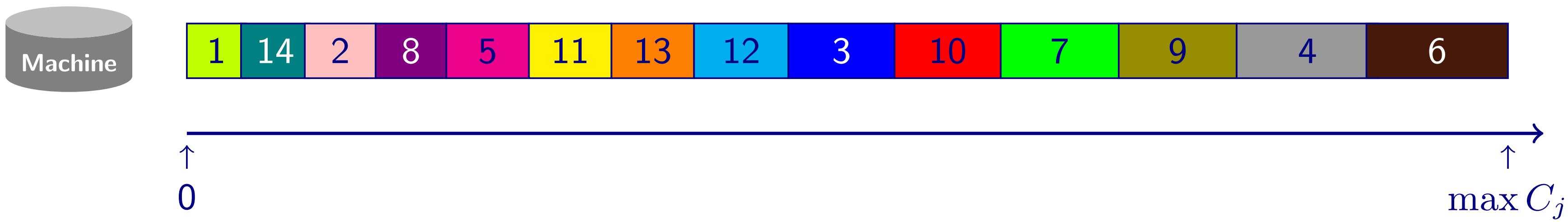

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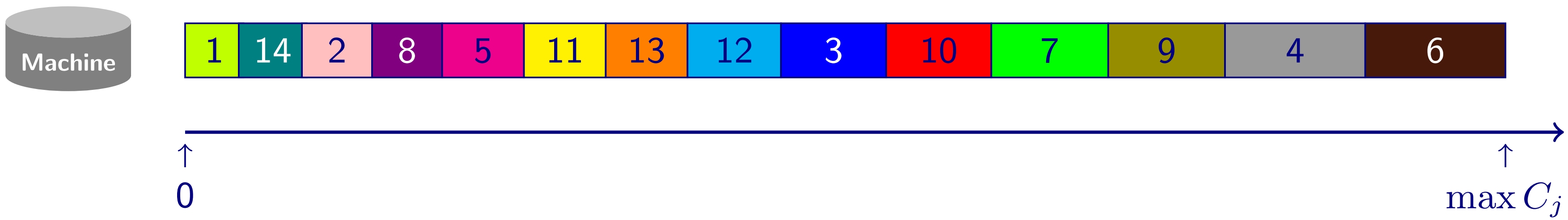
preceded by	–	1	1	2	3	3	4	5,6	5,6	7	8,9	10,11	11	12,13



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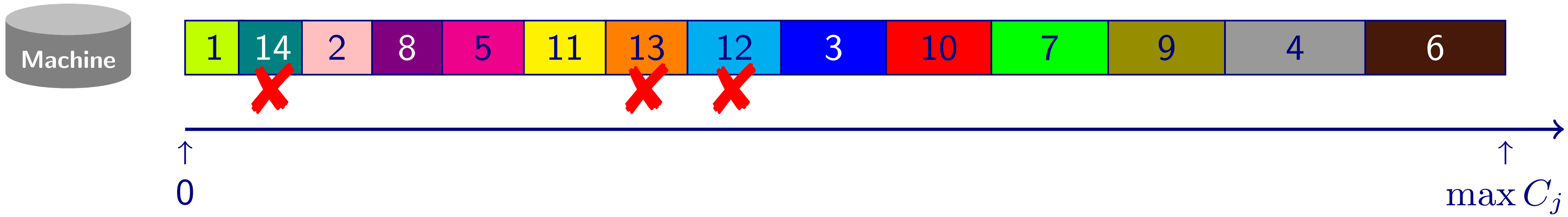

Jobs with precedence constraints

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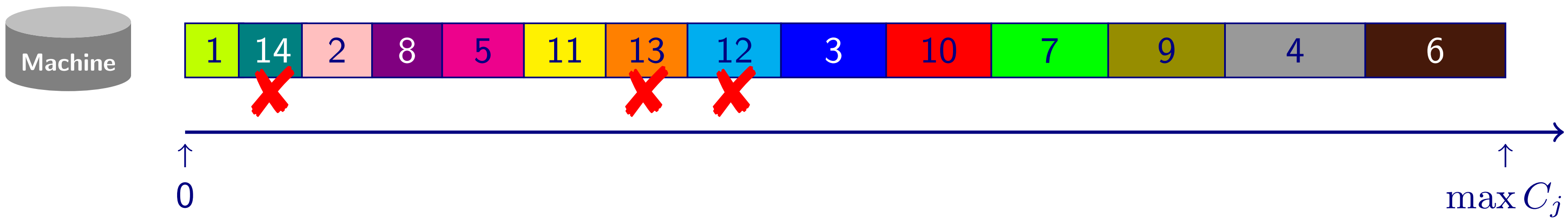

➡ Schedule is infeasible!

▷ Example: with **precedence constraints** ➡ **Project scheduling**

job j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
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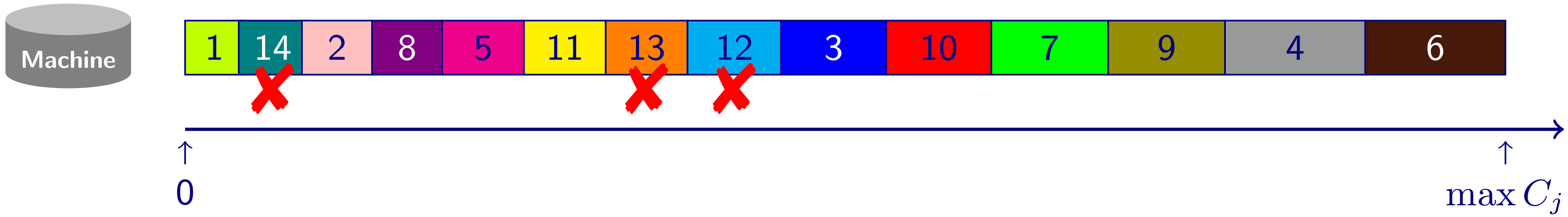

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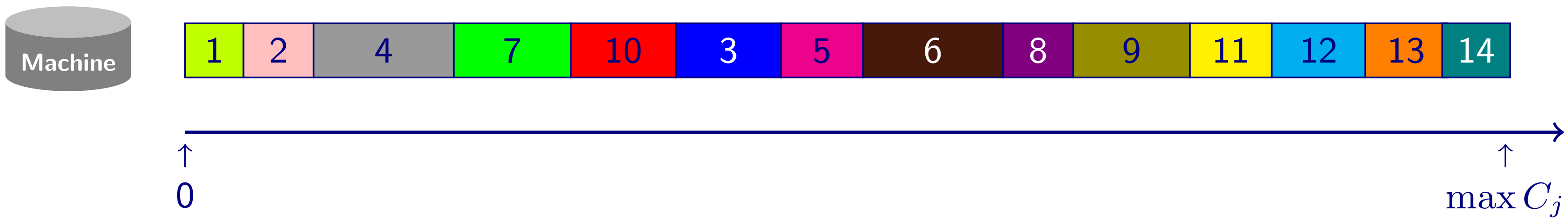

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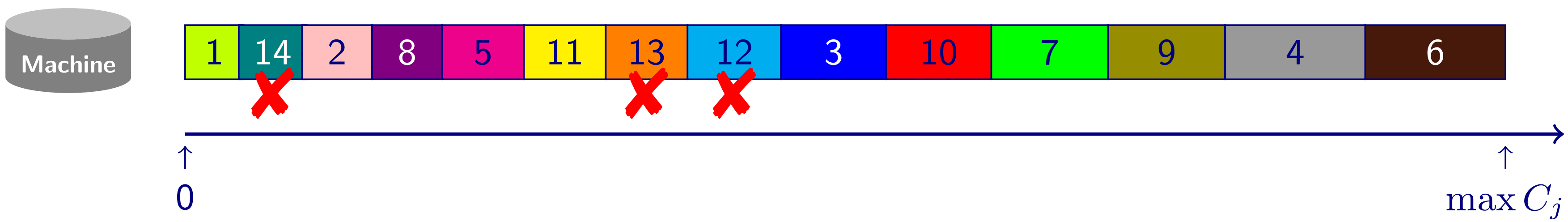

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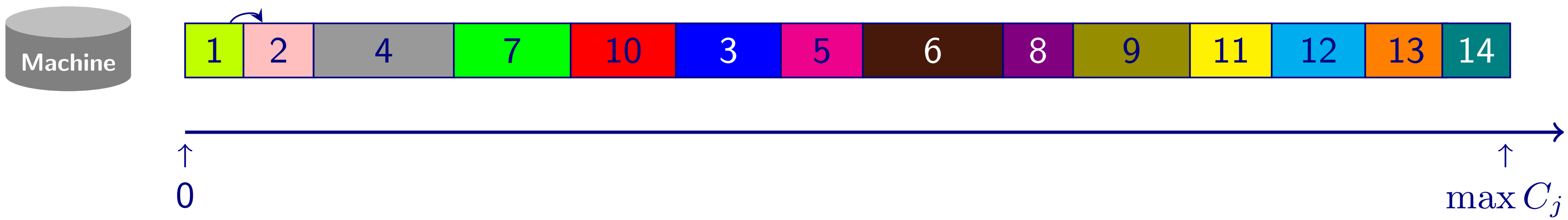

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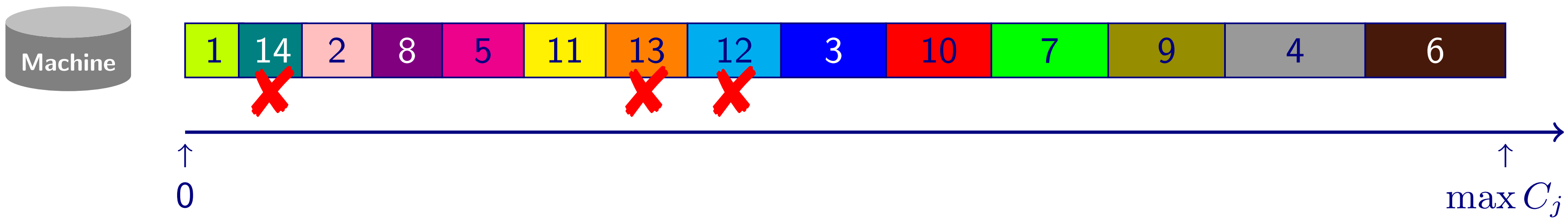

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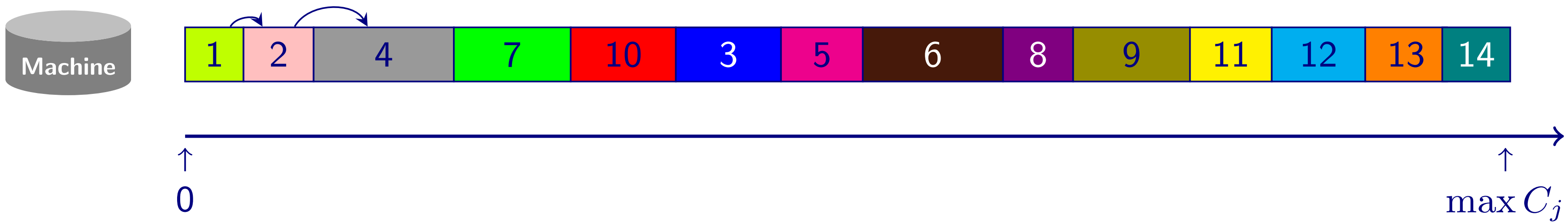

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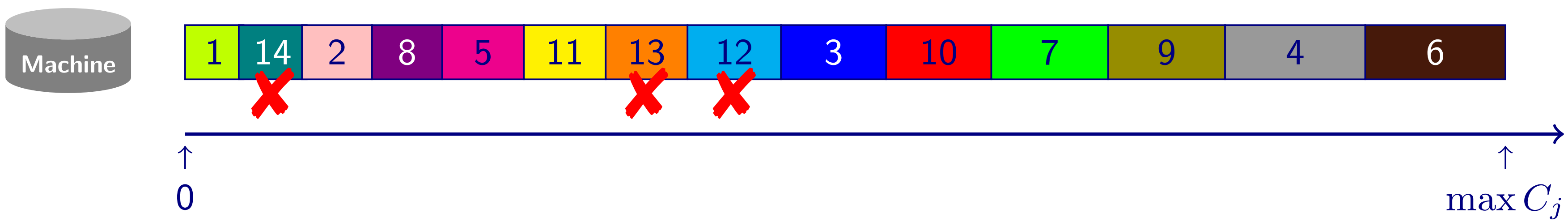

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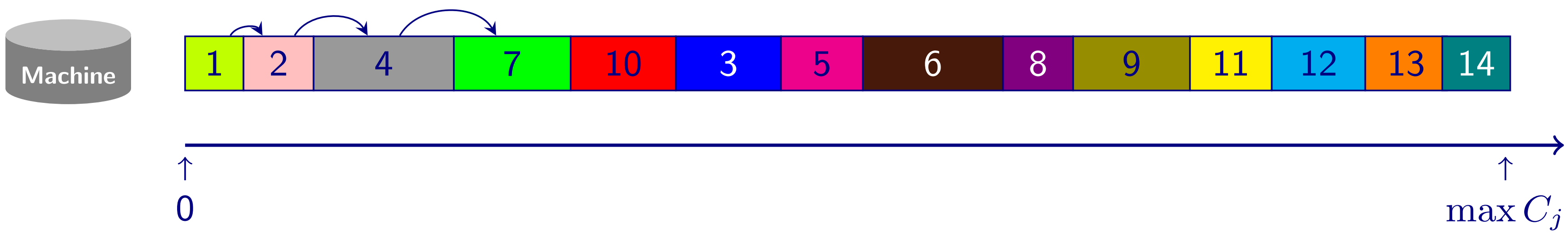

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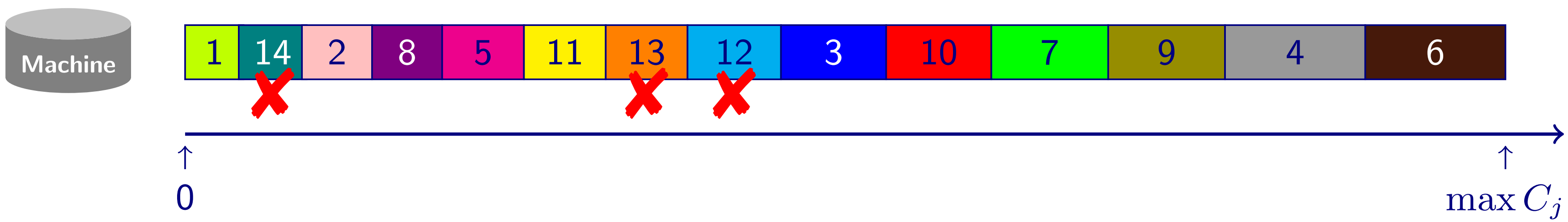

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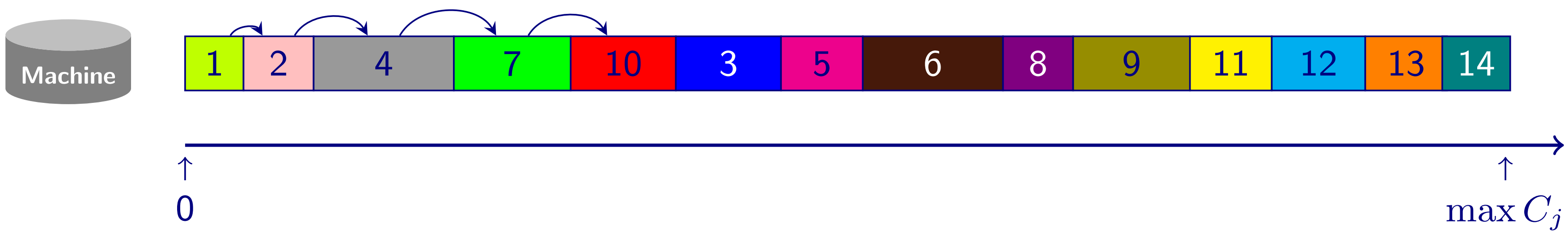

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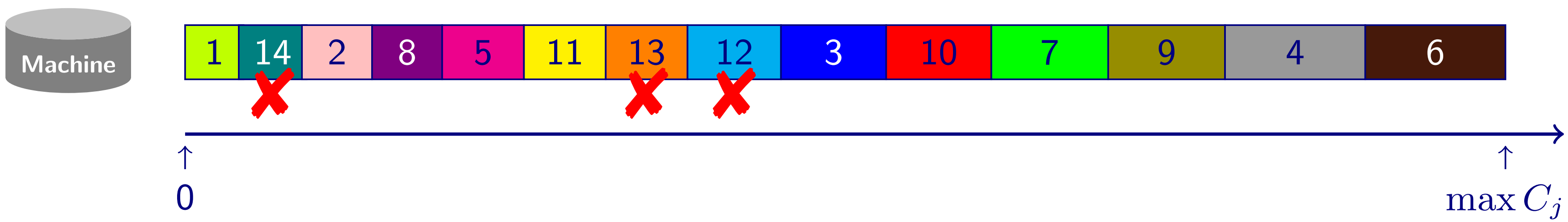

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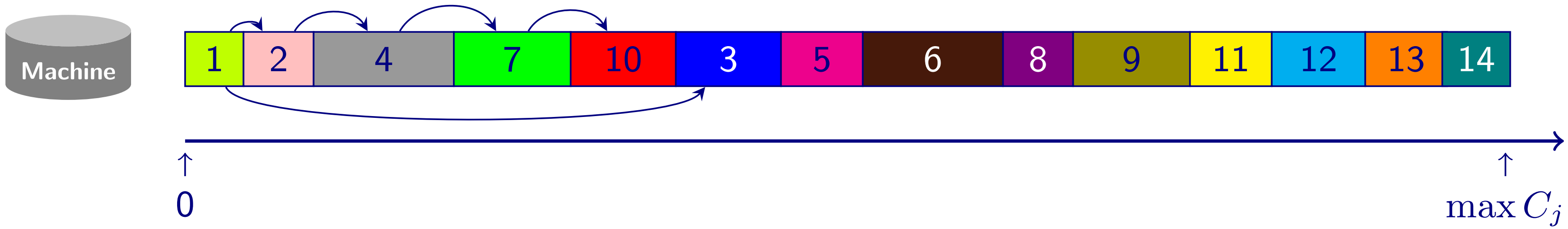

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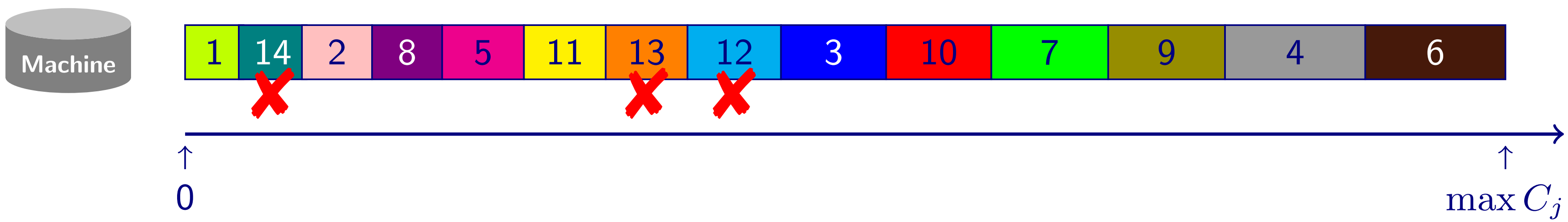

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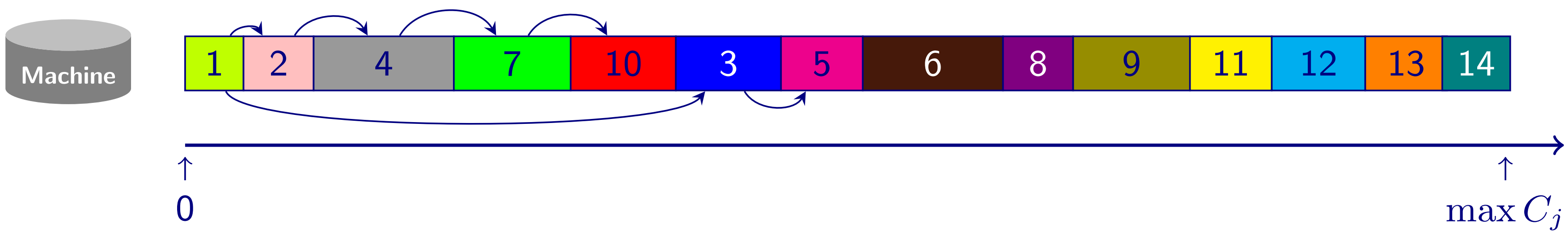

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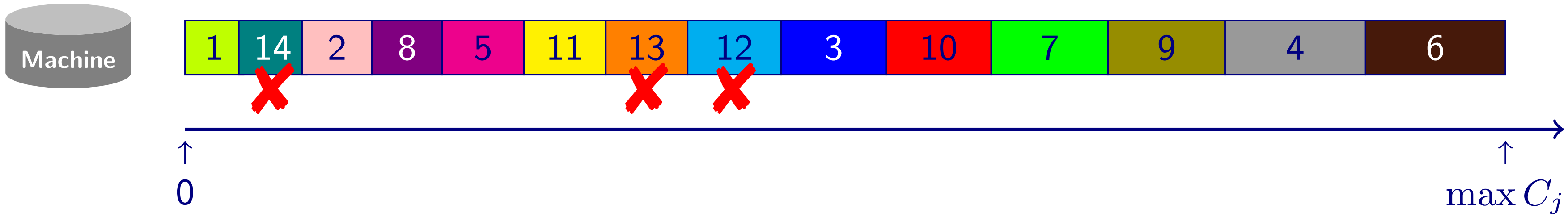

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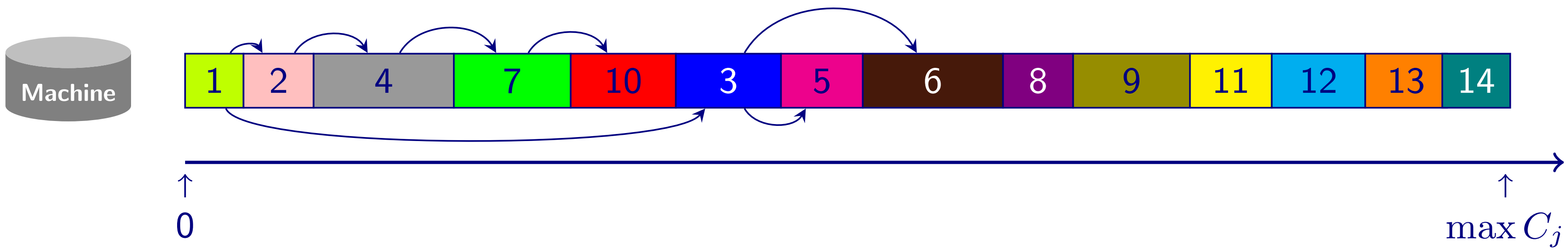

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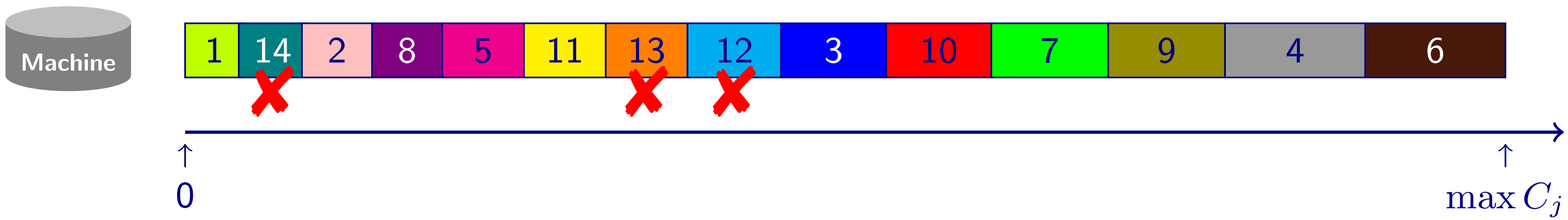

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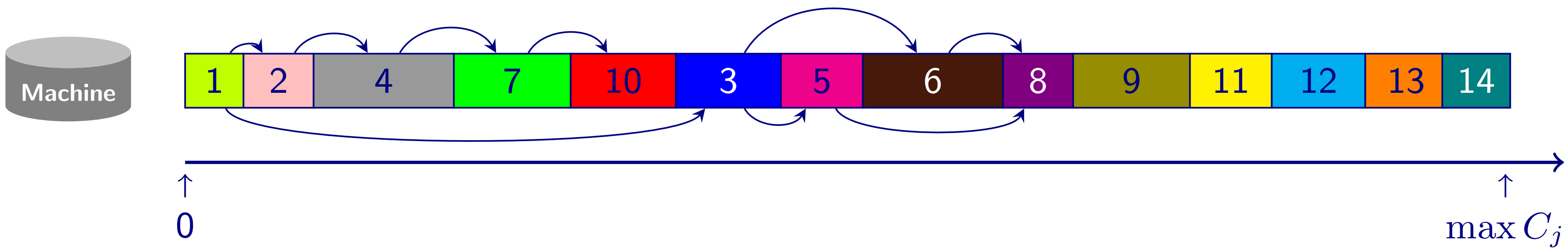

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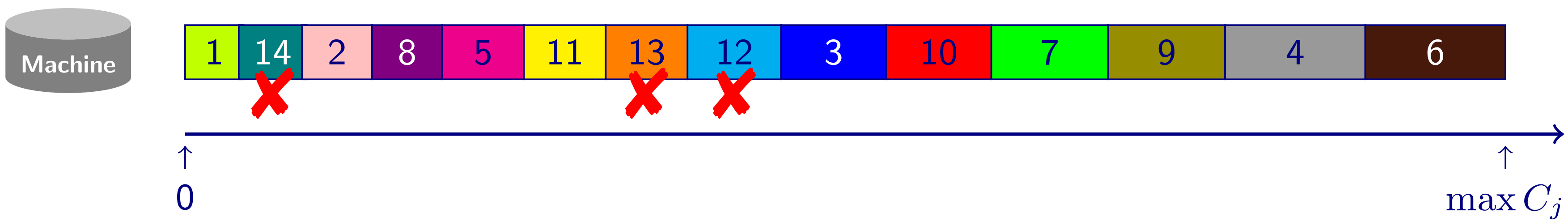

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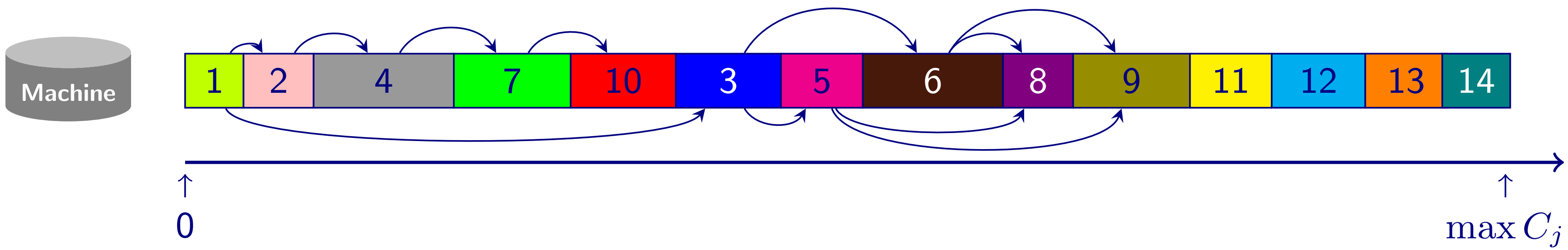

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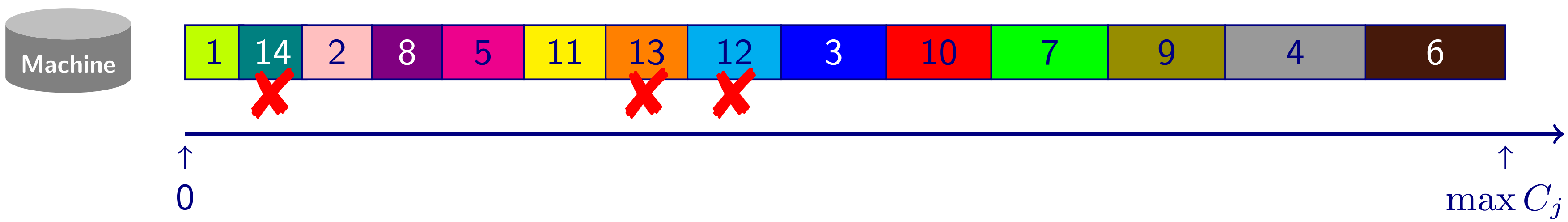

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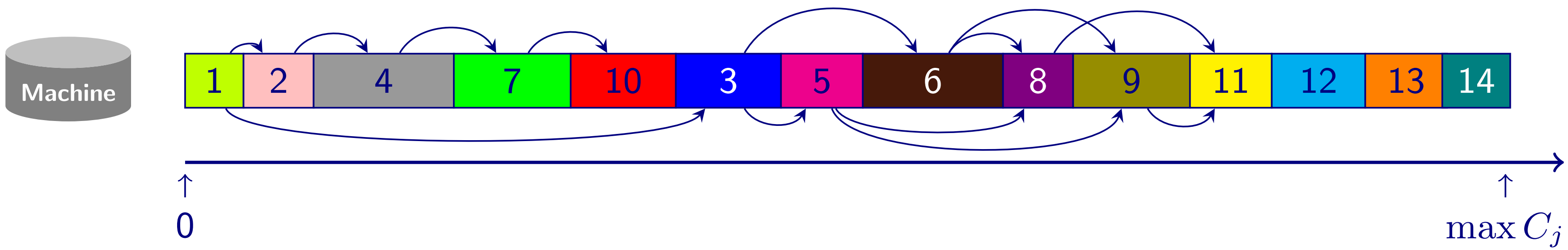

- Example: with **precedence constraints** ➡ **Project scheduling**

job j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
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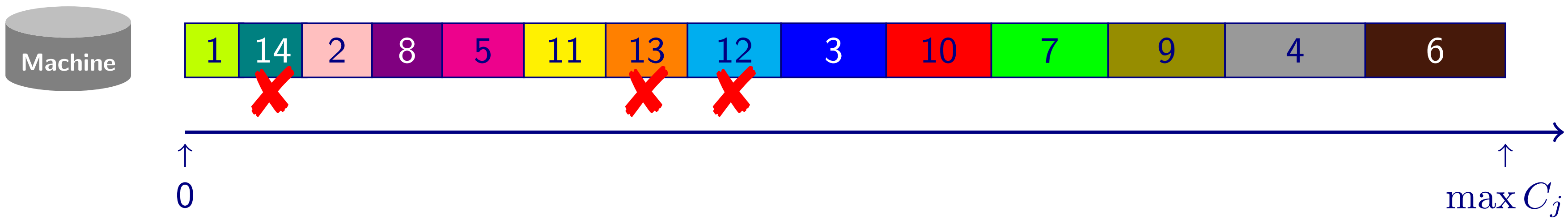
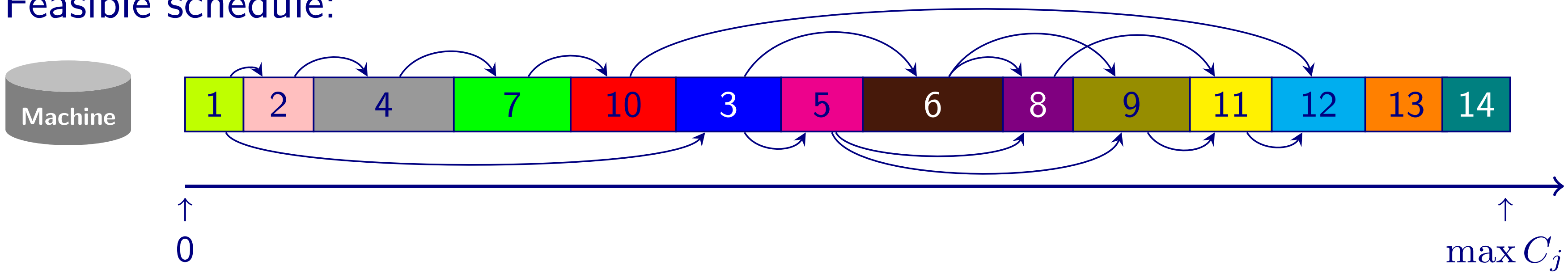


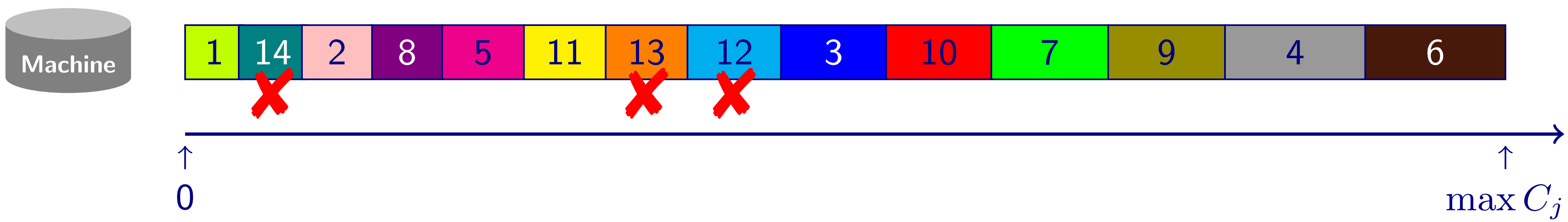
➡ Schedule is infeasible!

- Feasible schedule:



▷ Example: with precedence constraints ➡ Project scheduling

job j	1	2	3	4	5	6	7	8	9	10	11	12	13	14
process time p_j	5	6	9	12	7	12	10	6	10	9	7	8	7	5
preceded by	–	1	1	2	3	3	4	5,6	5,6	7	8,9	10,11	11	12,13



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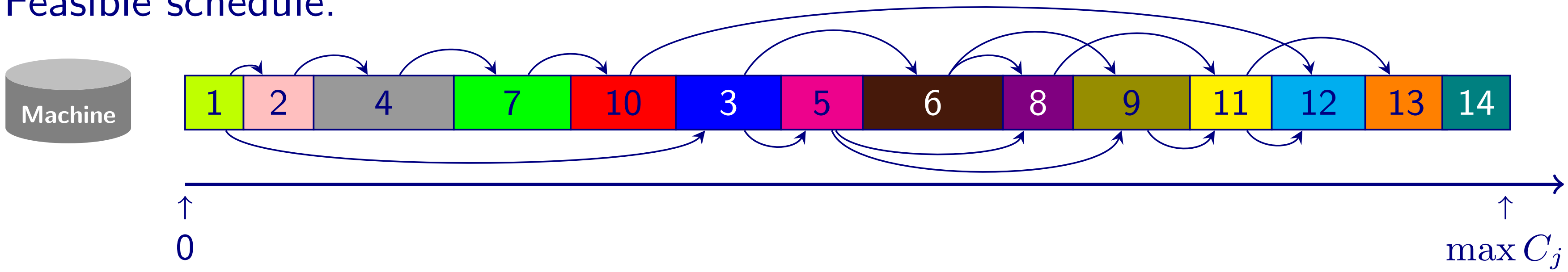

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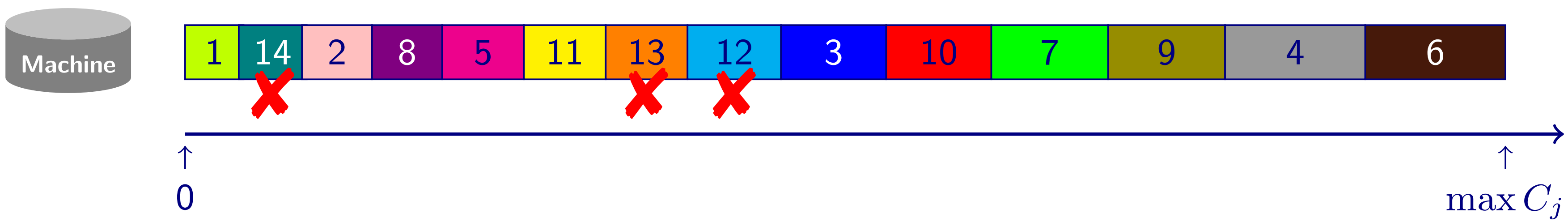

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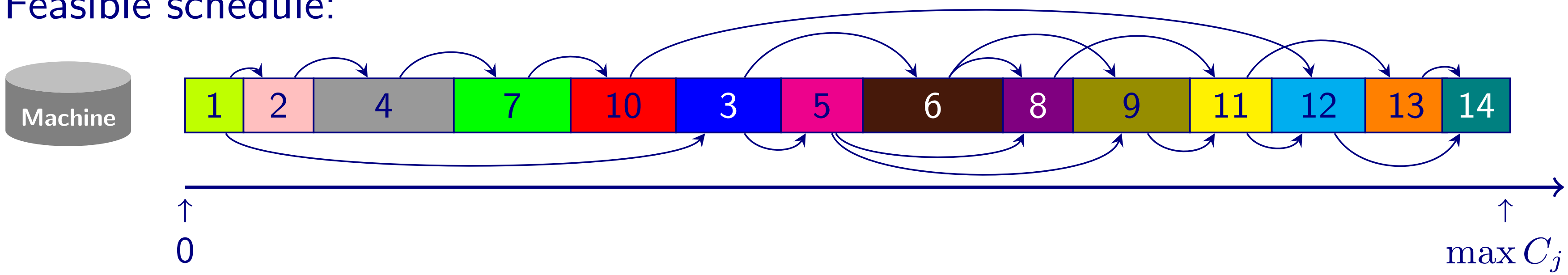

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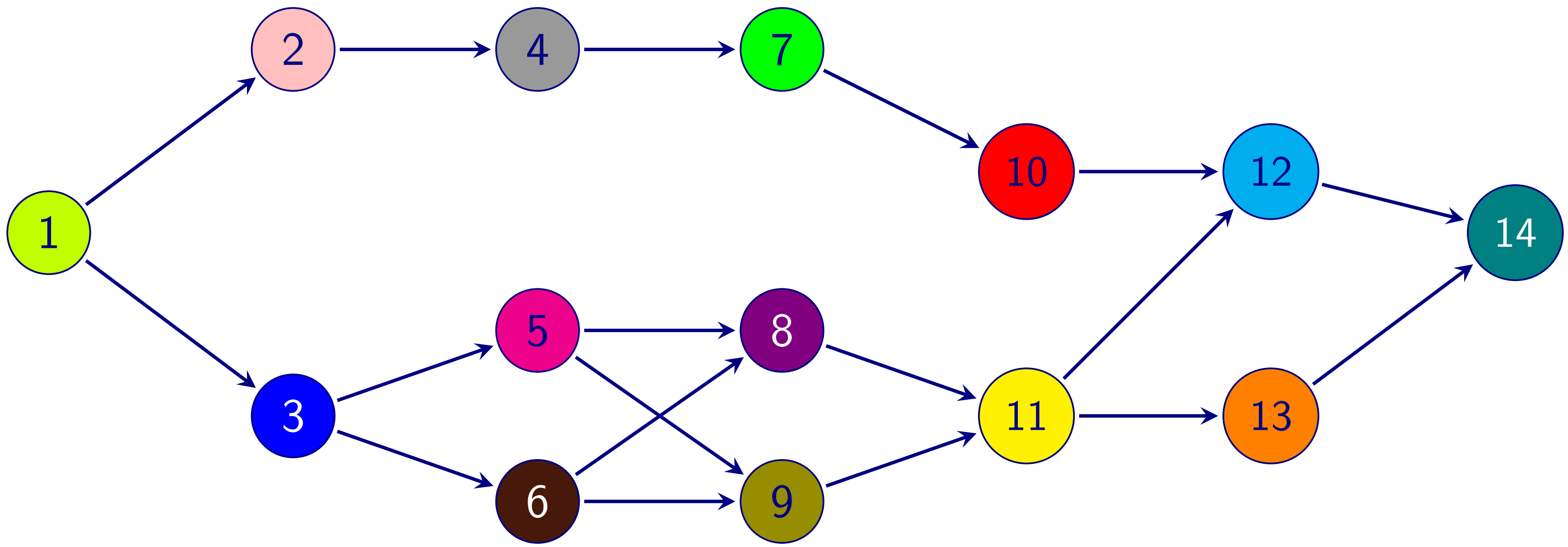

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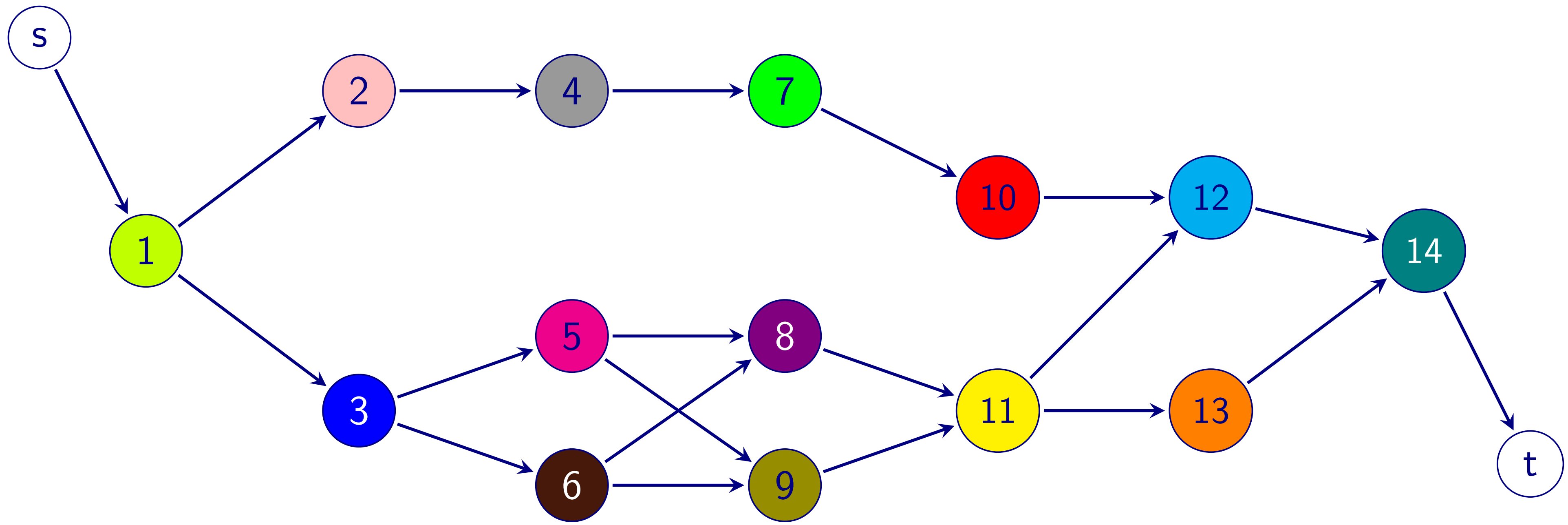

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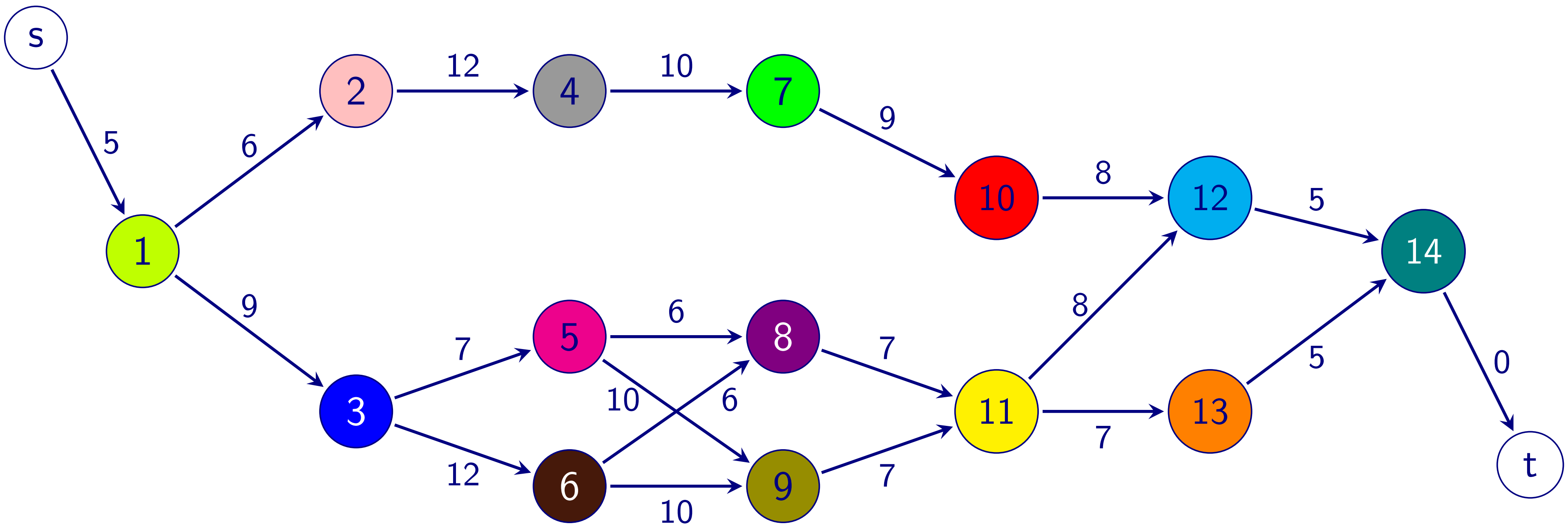
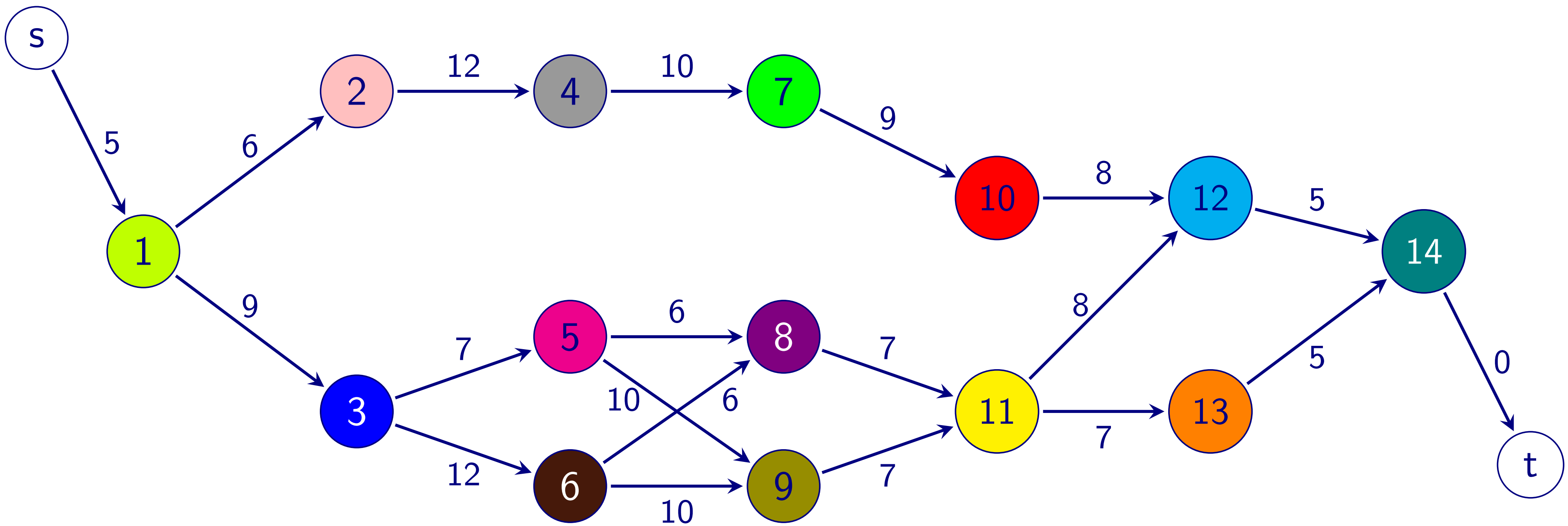


- Greedy strategy: schedule an arbitrary job next with already fulfilled precedences



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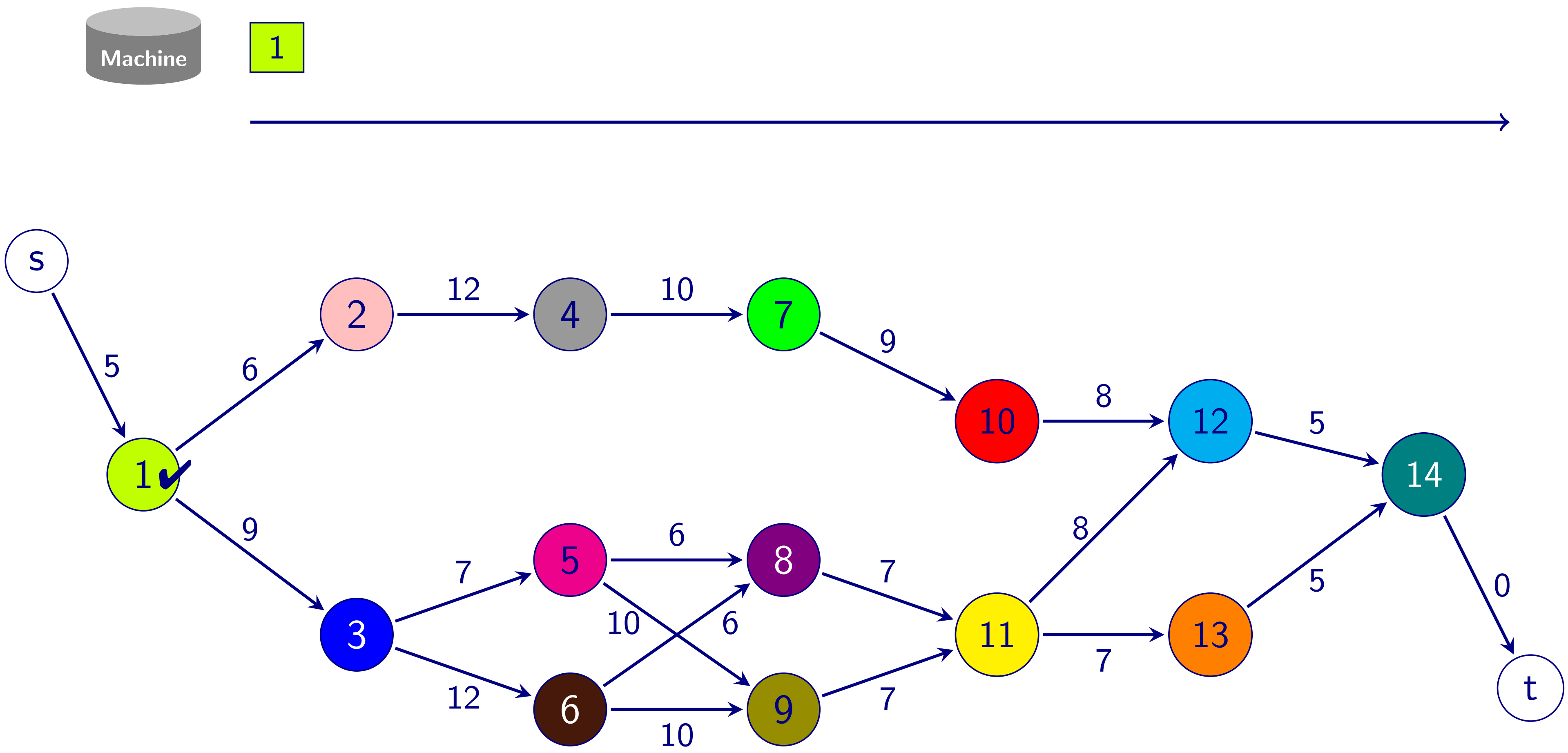

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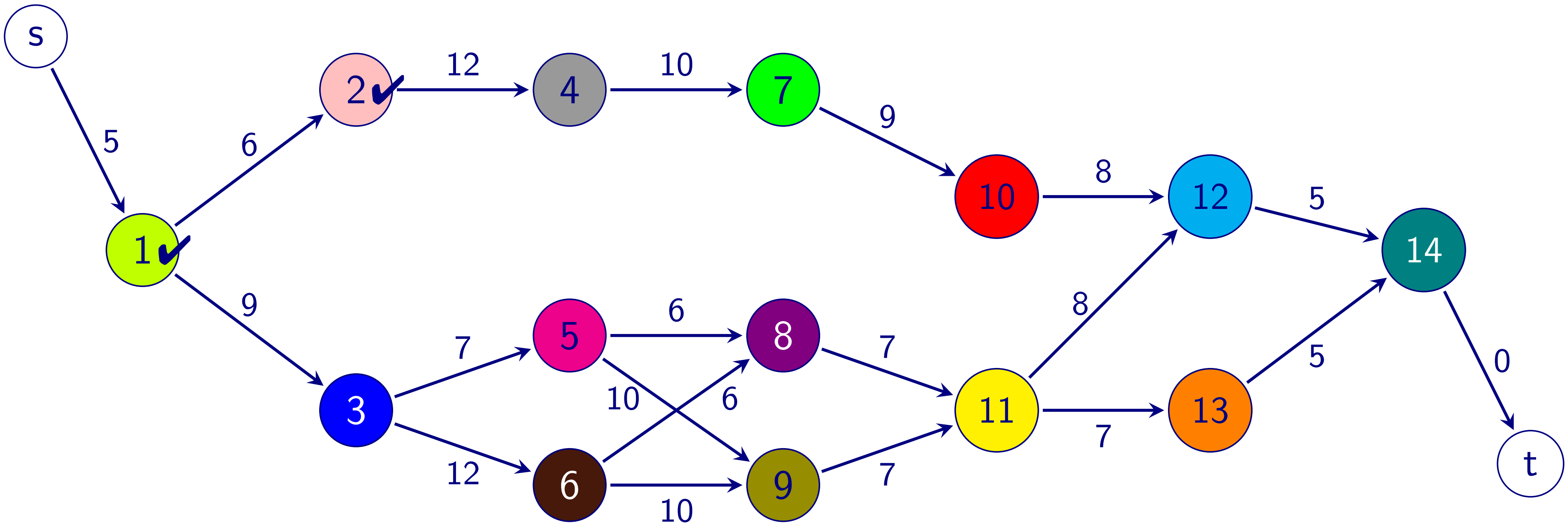

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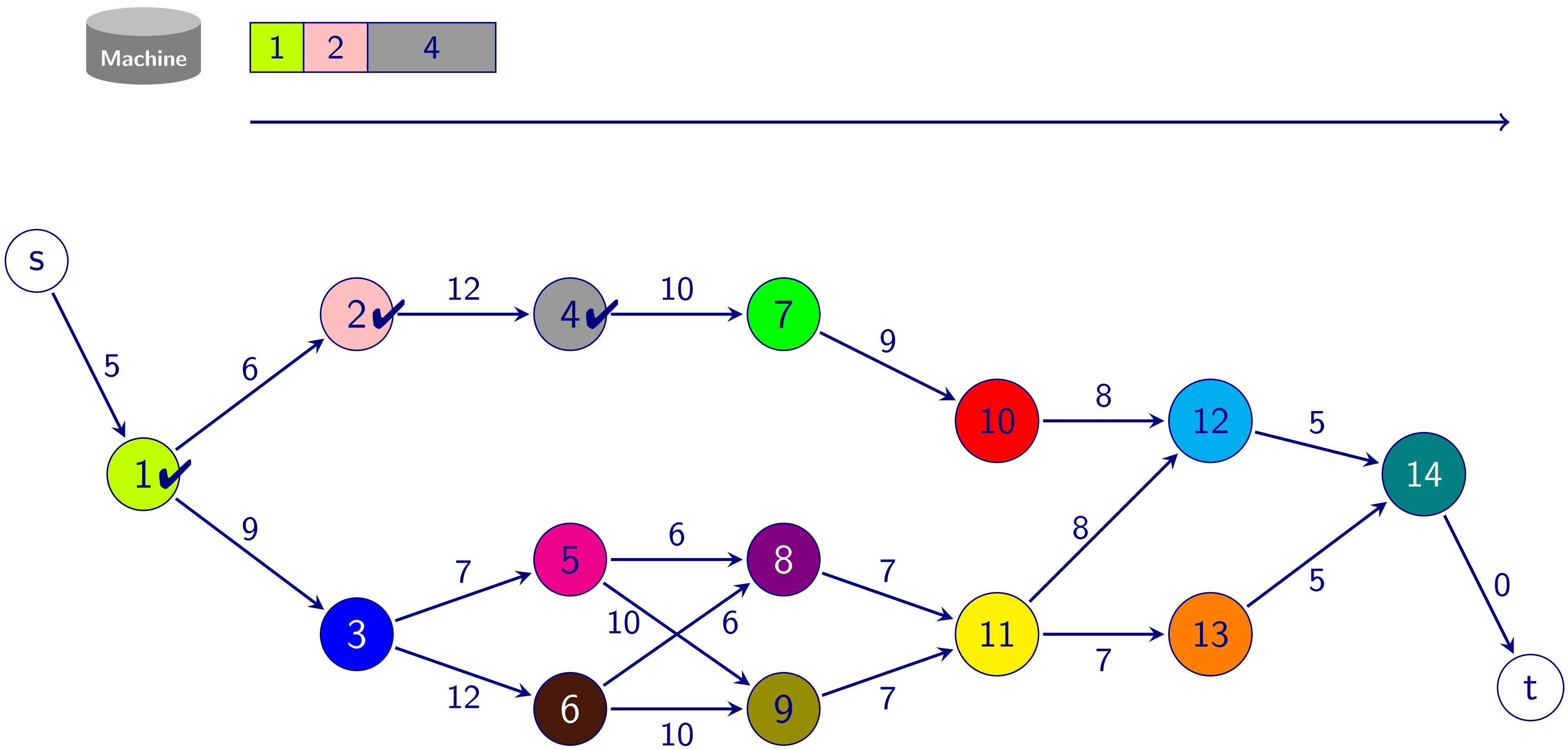

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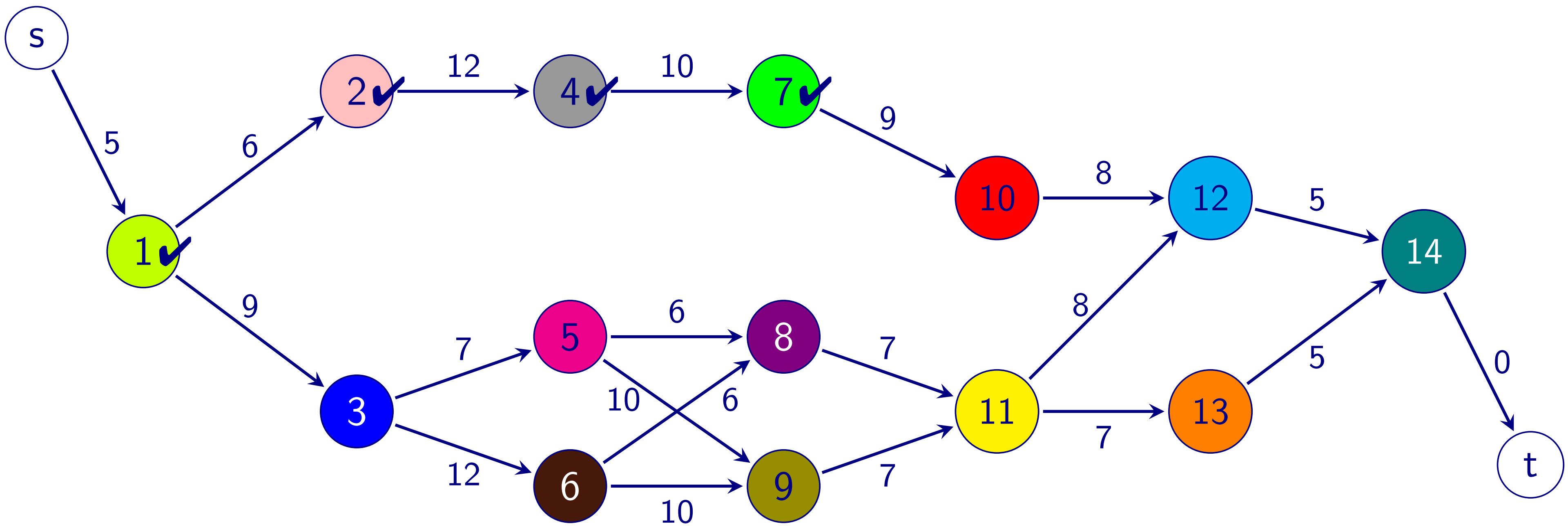

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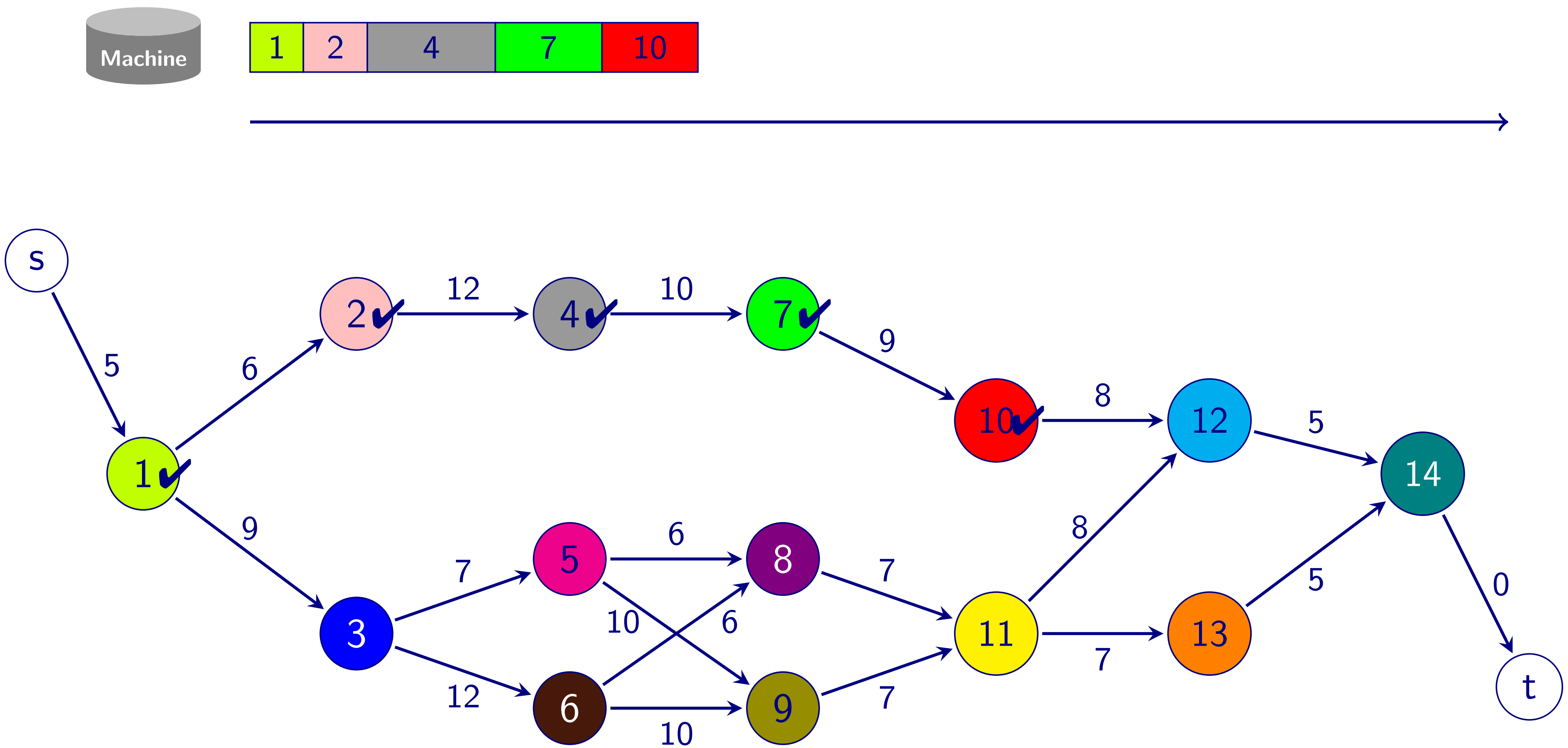

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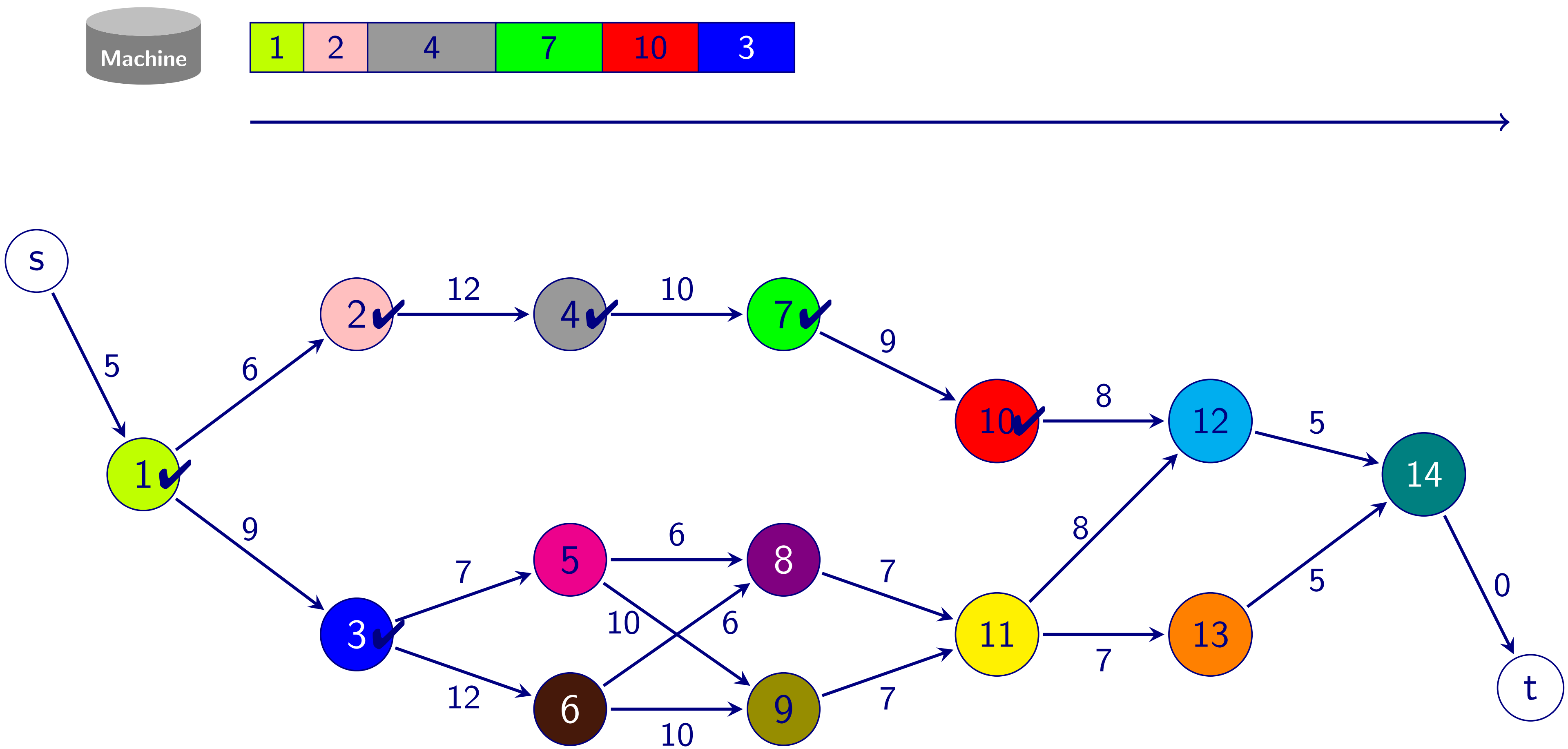

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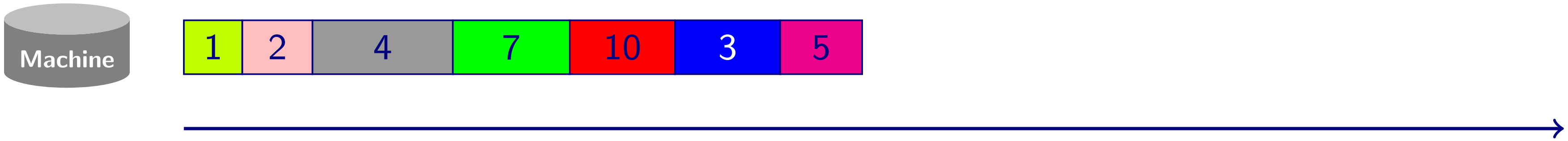

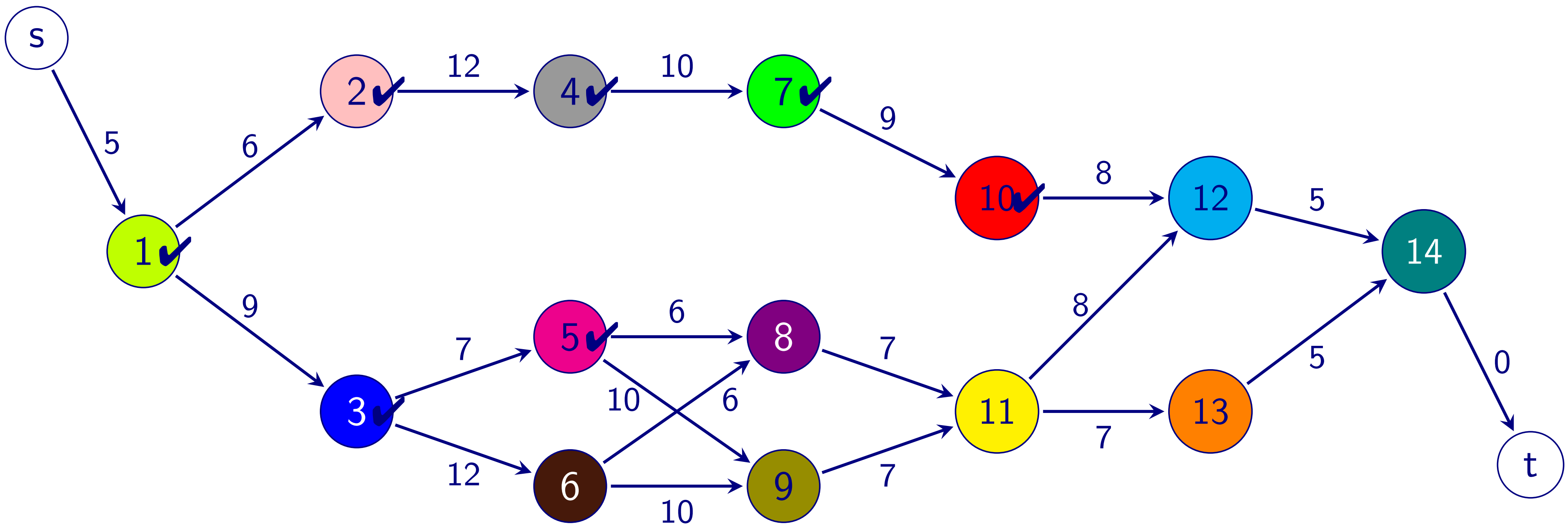

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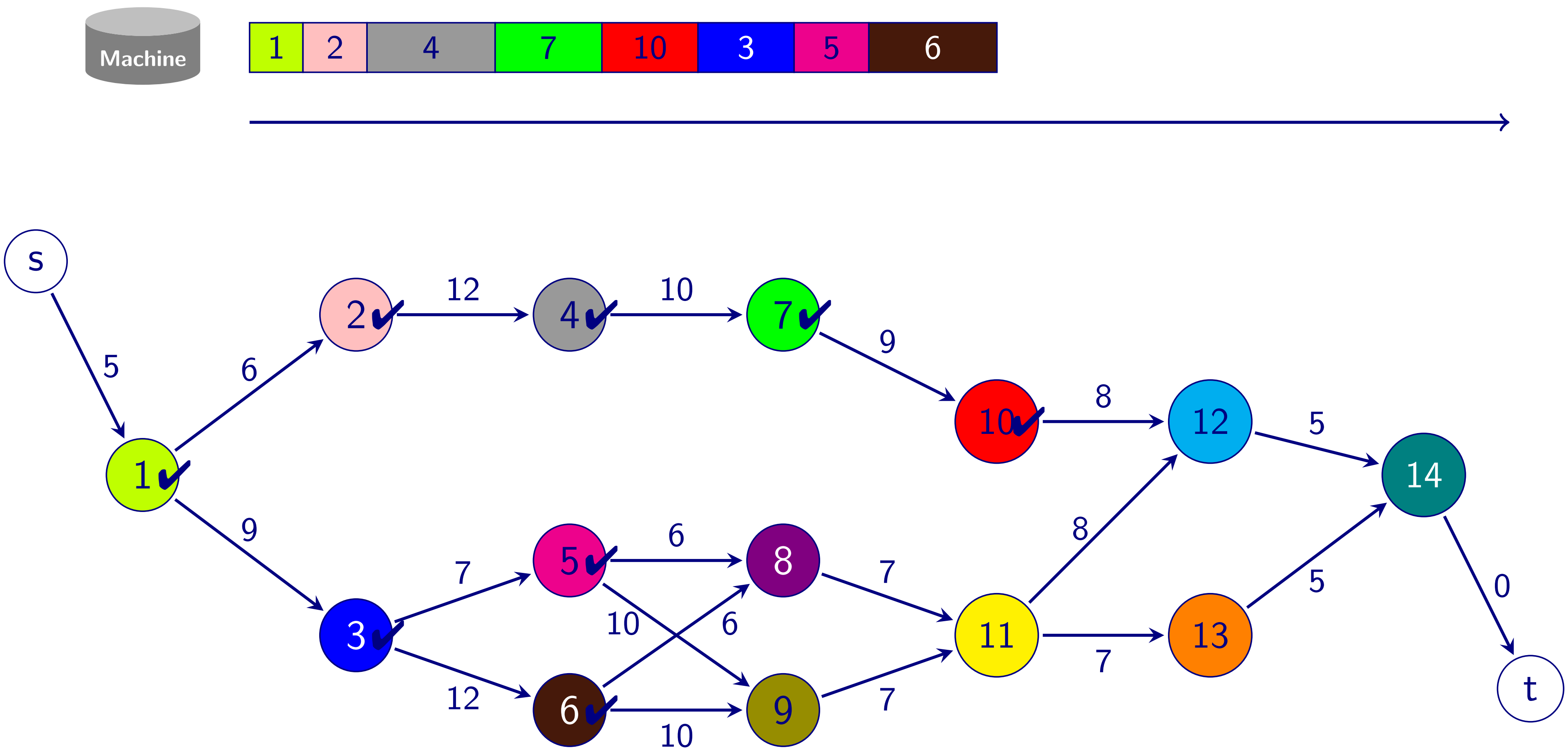

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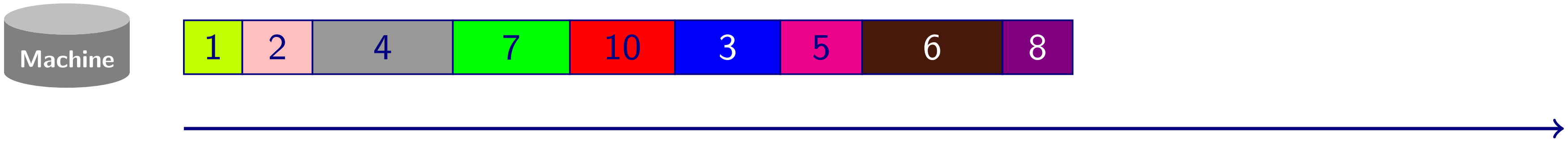

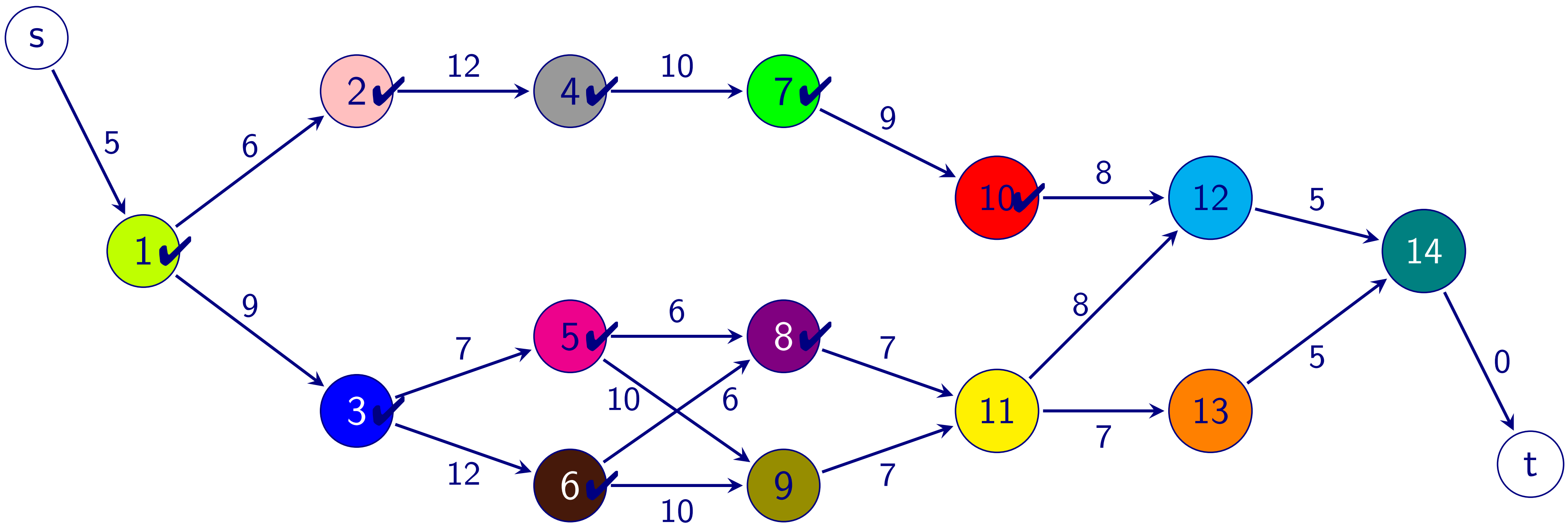

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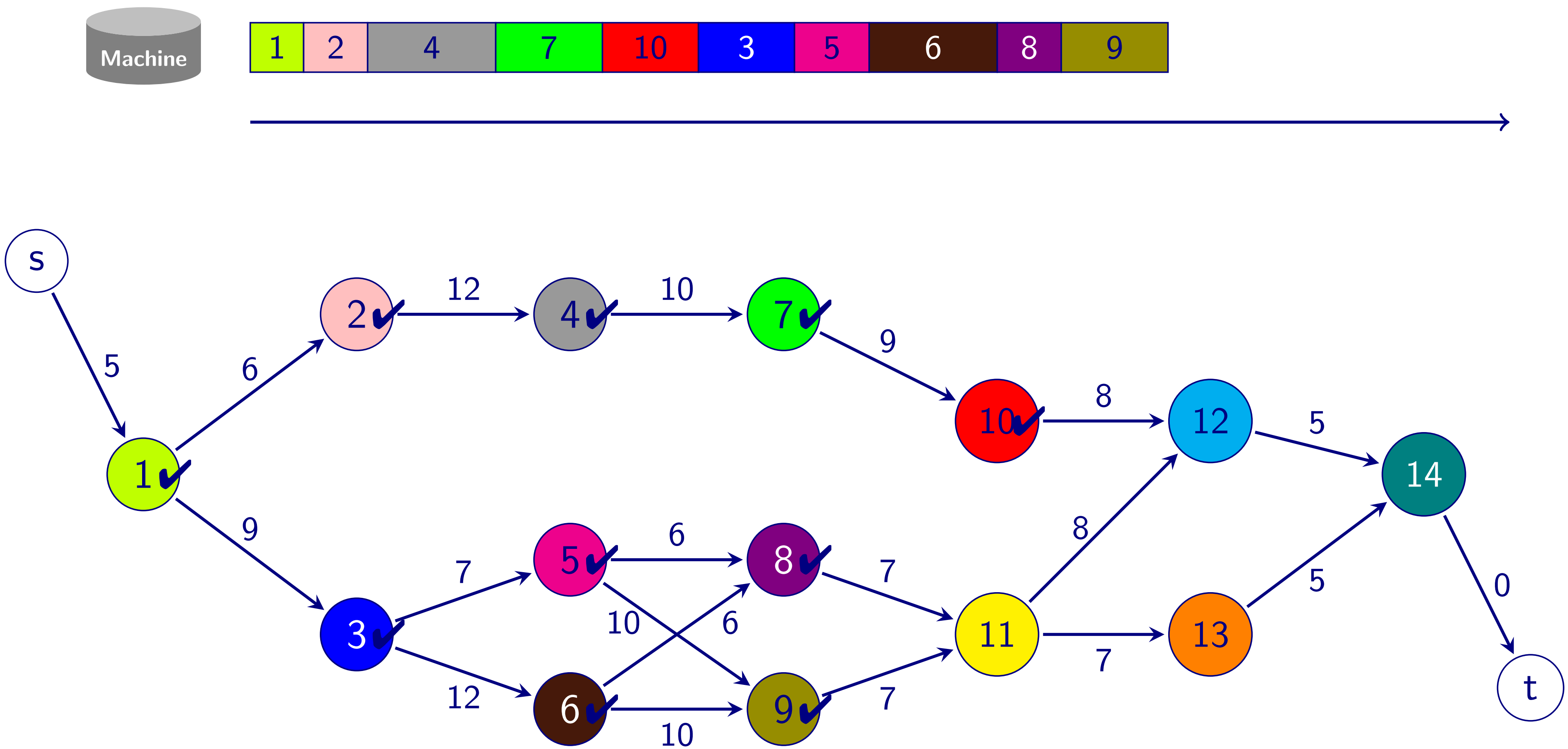

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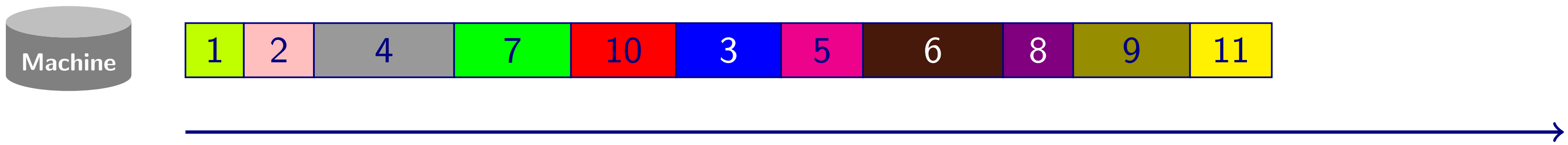

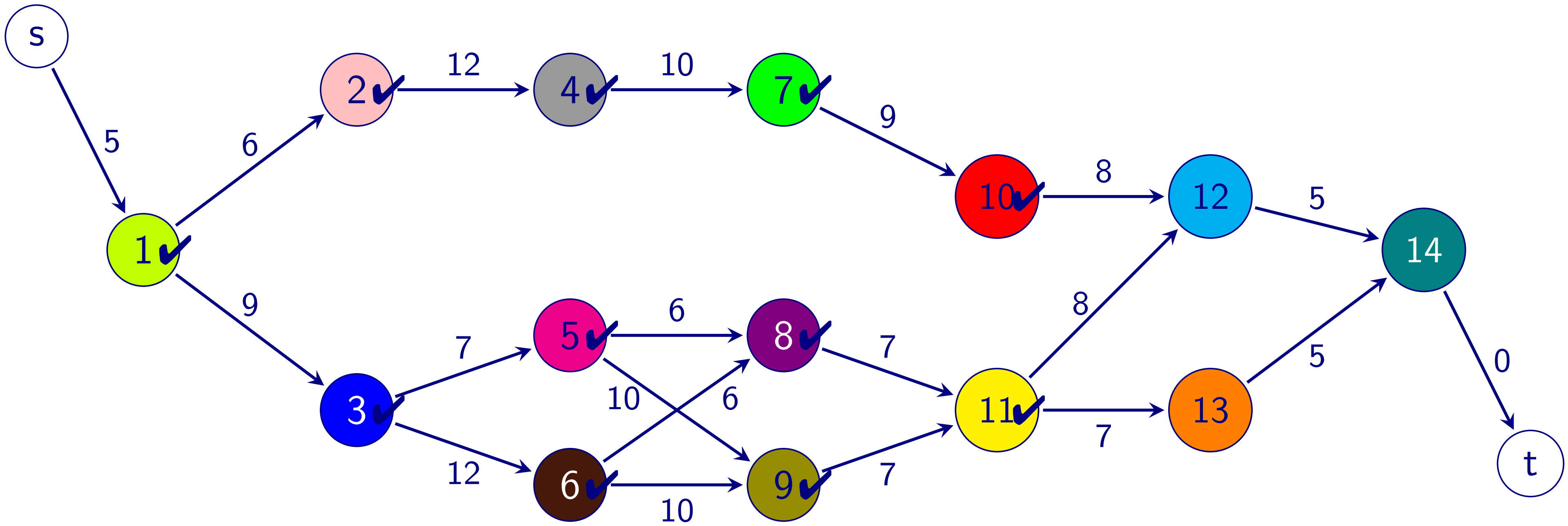

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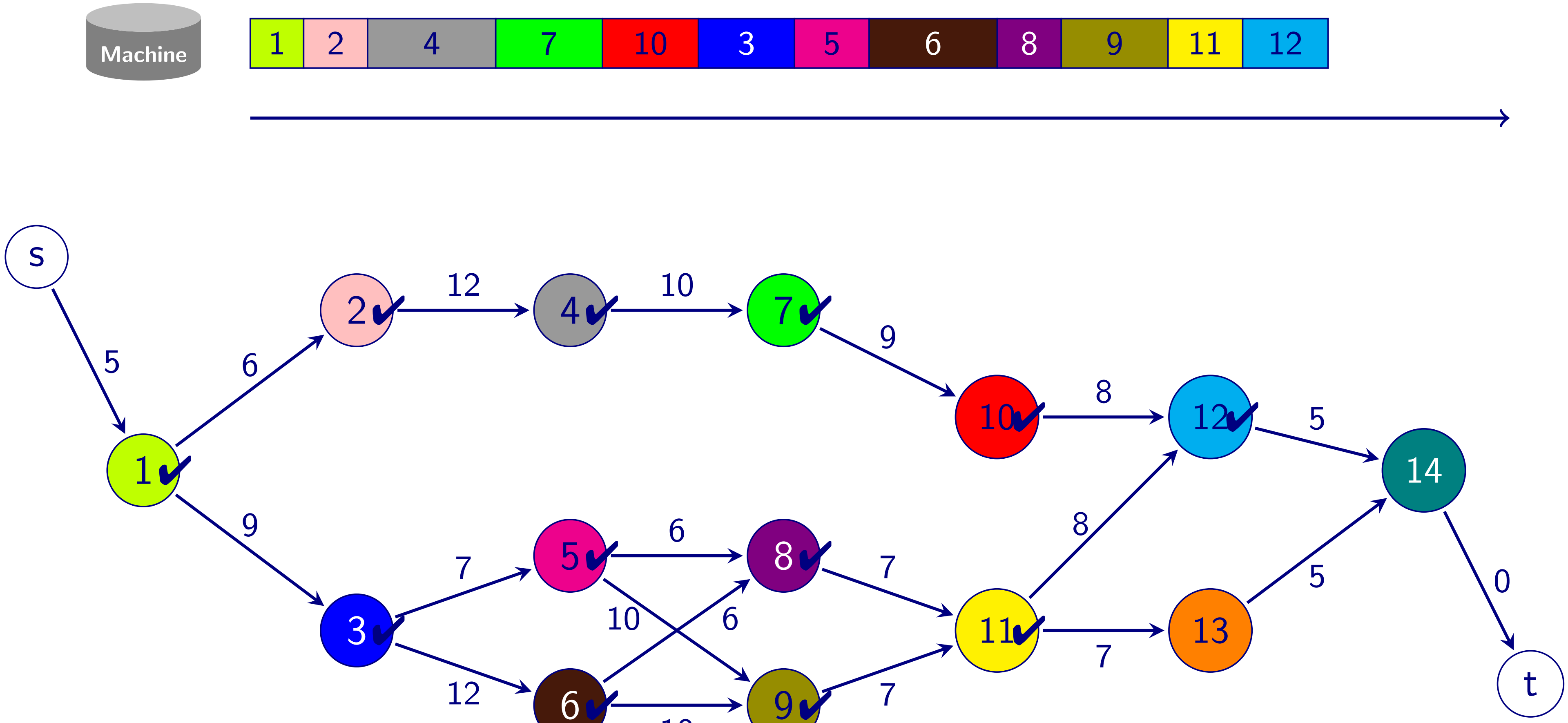

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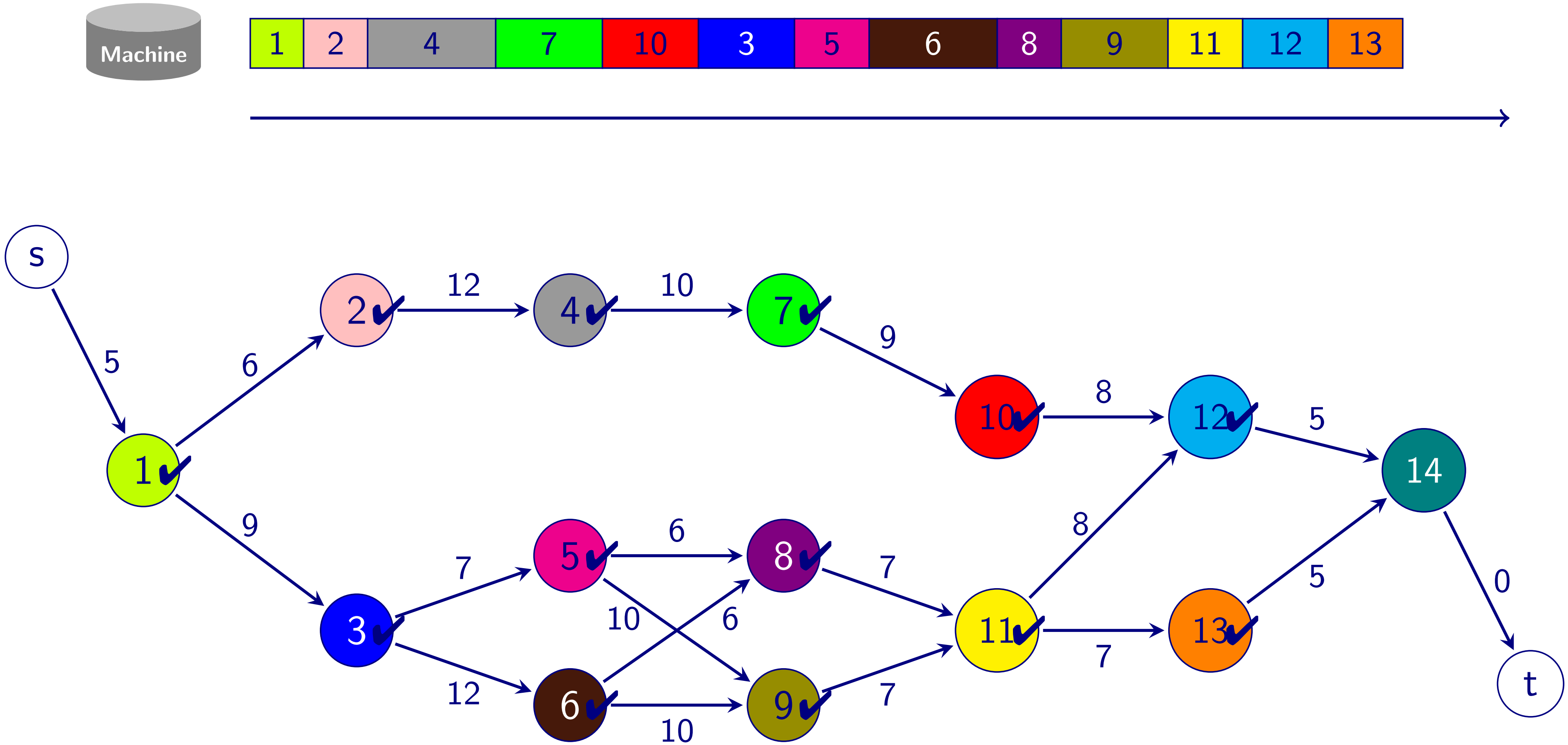

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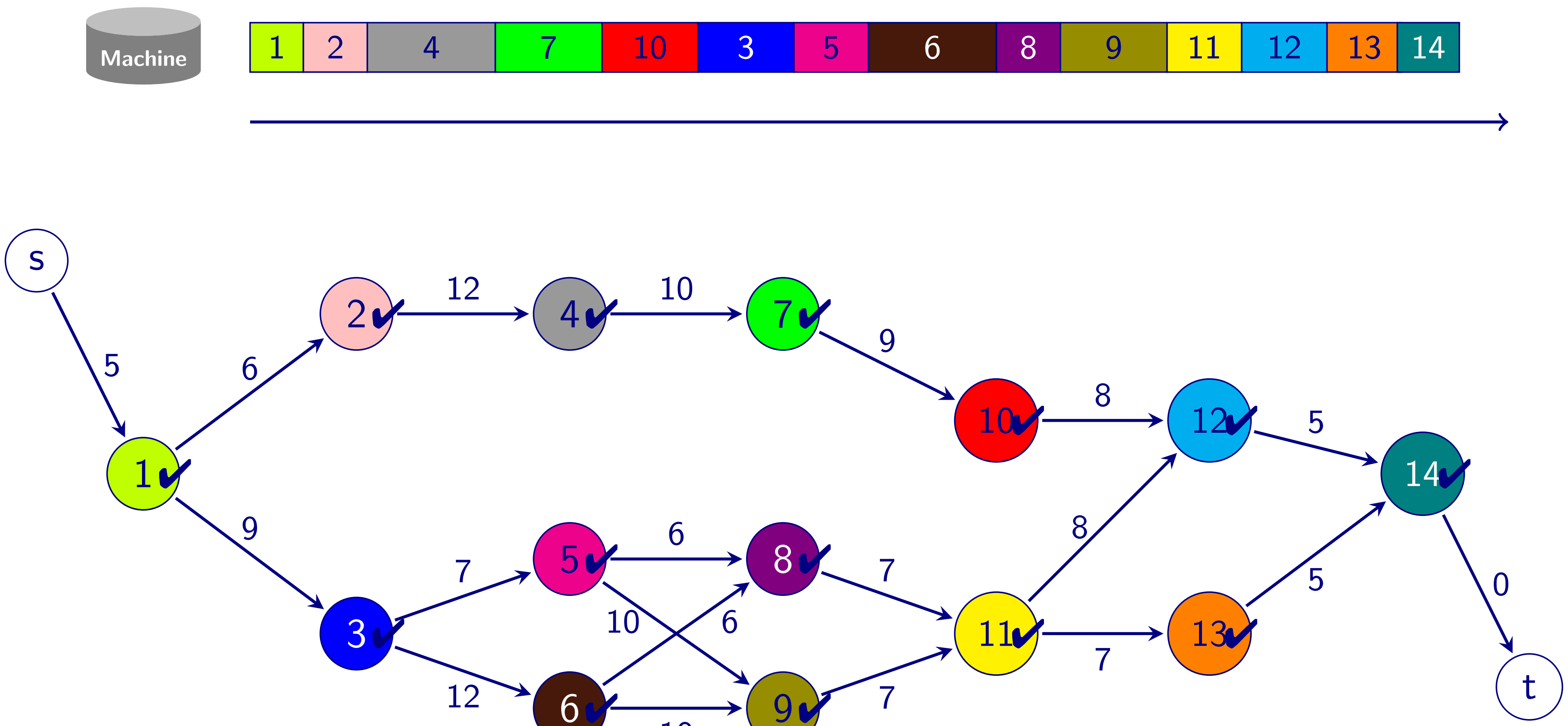
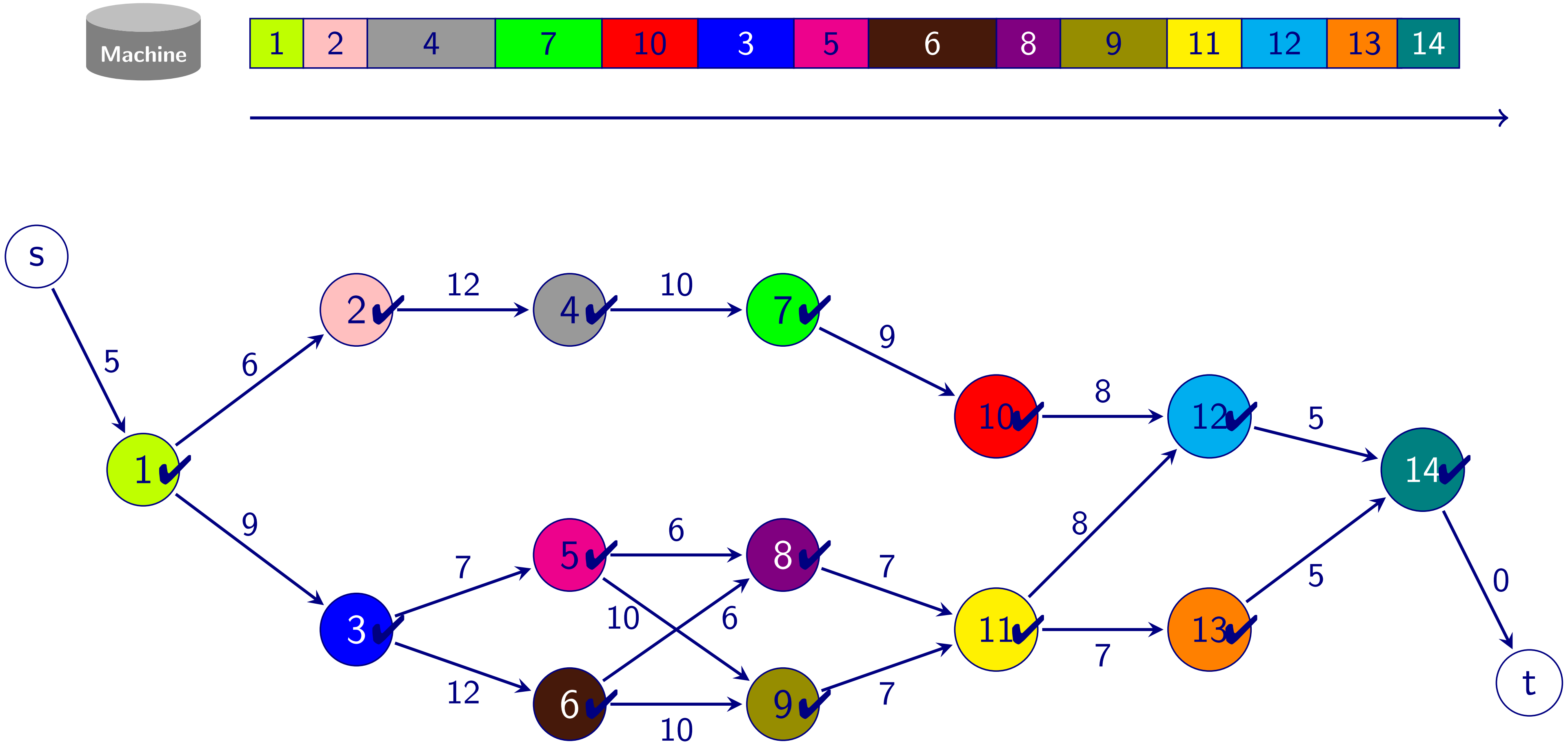
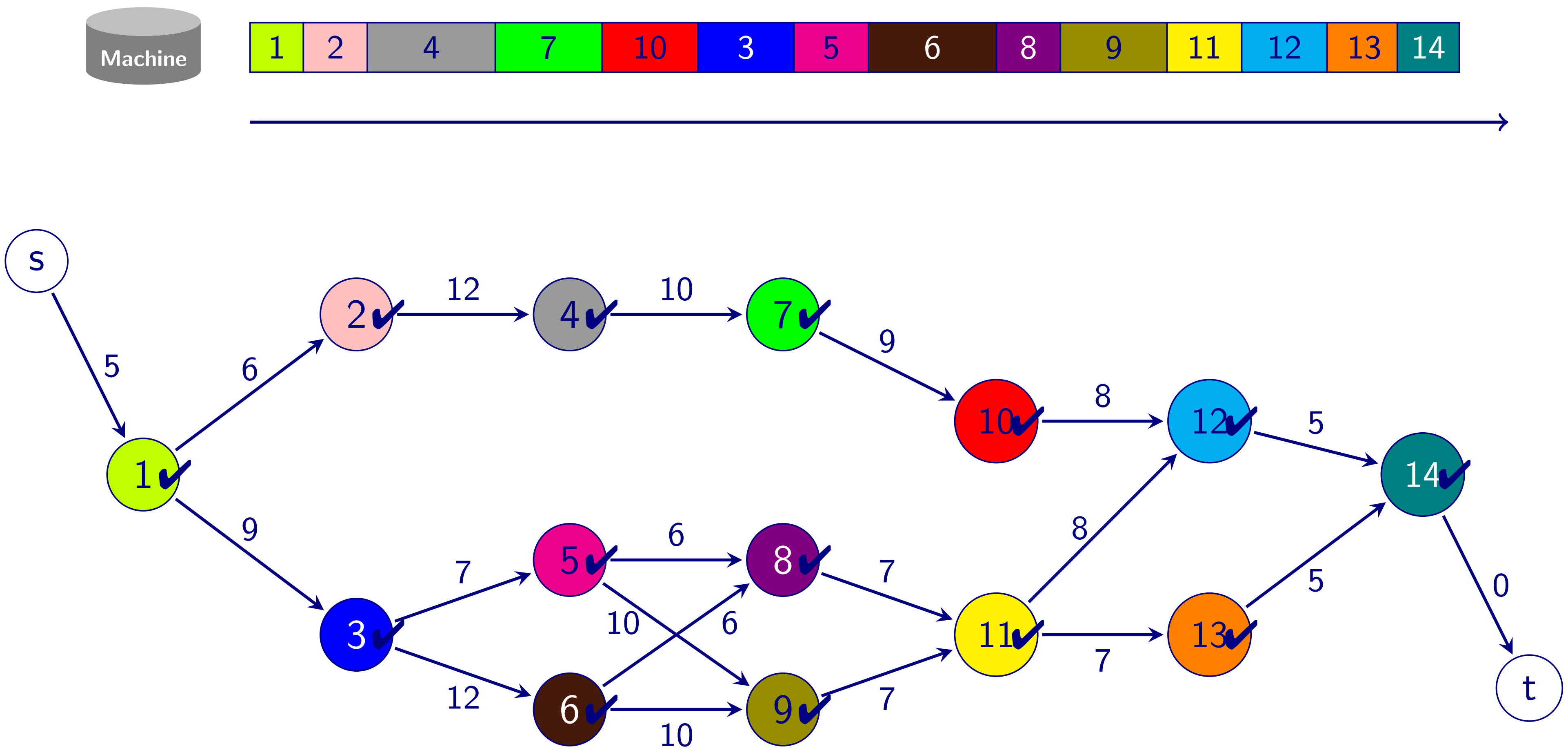


- Greedy strategy: schedule an arbitrary job next with already fulfilled precedences
  - ➡ Polynomial runtime



- Greedy strategy: schedule an arbitrary job next with already fulfilled precedences
  - ➡ Polynomial runtime ➡ Efficient algorithm!



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- Example: Project scheduling on construction site

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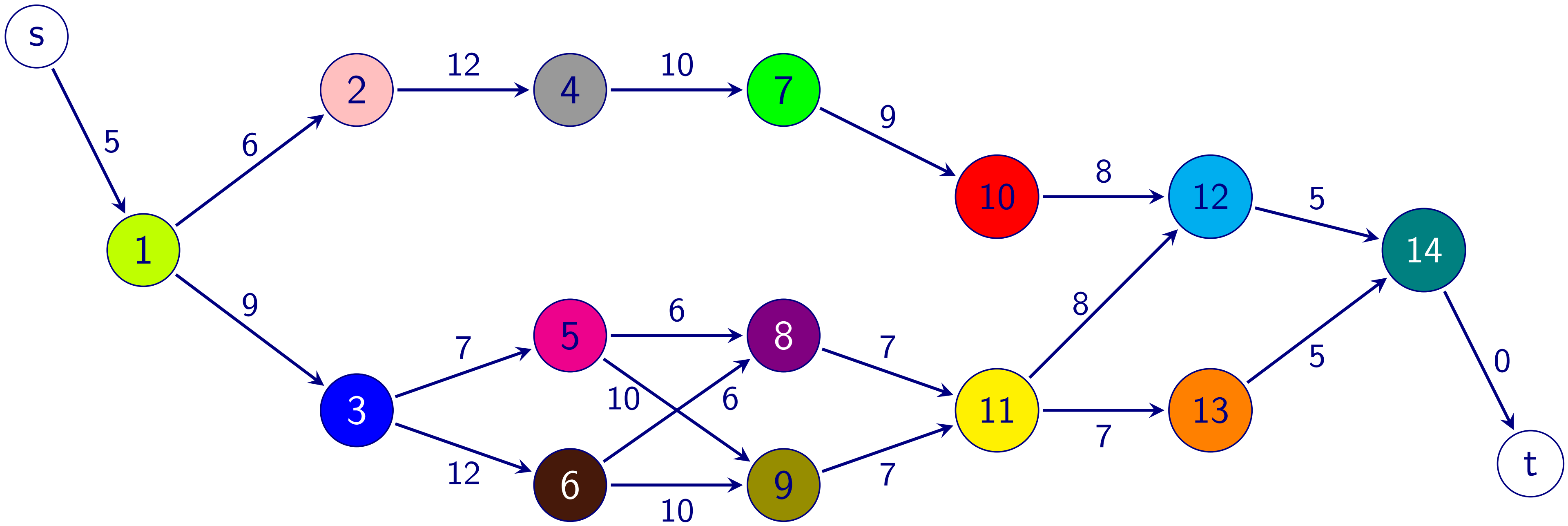
- ▷ Example: Project scheduling on construction site
  - ➡ Different tasks done by different contractors: Concrete builder, stonemasonry, house painter, glazier, ...
  - ➡ can provide as many workers as necessary to carry out each task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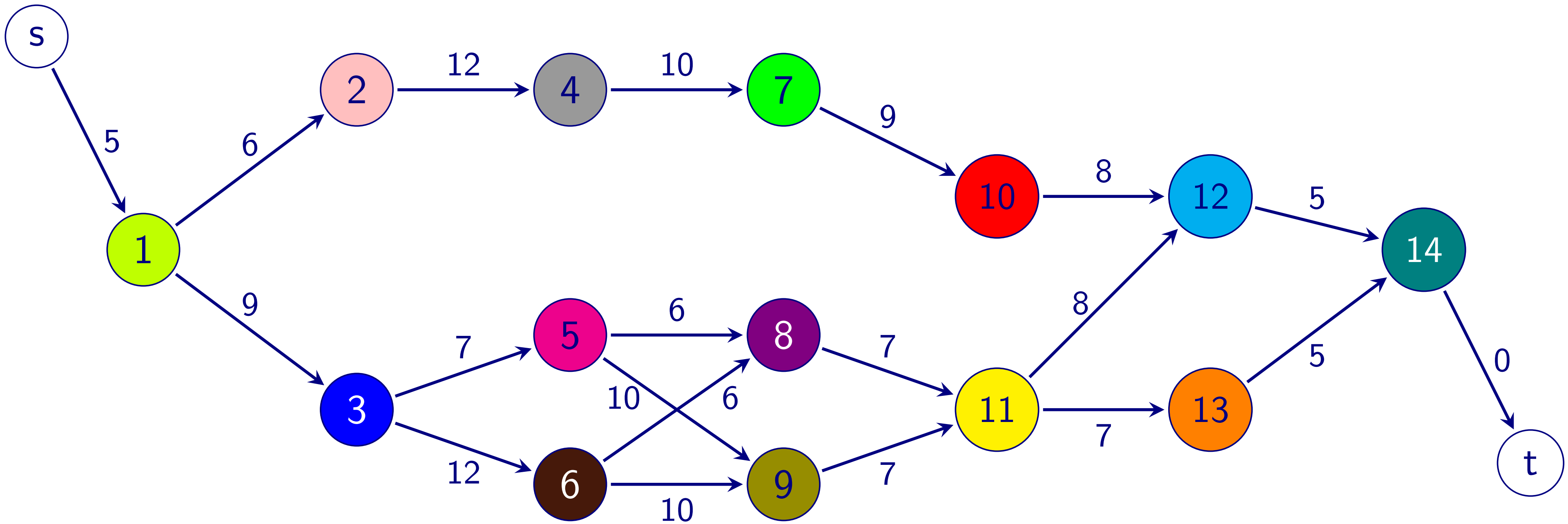
▷ Forward procedure: compute earliest possible completion times for all jobs



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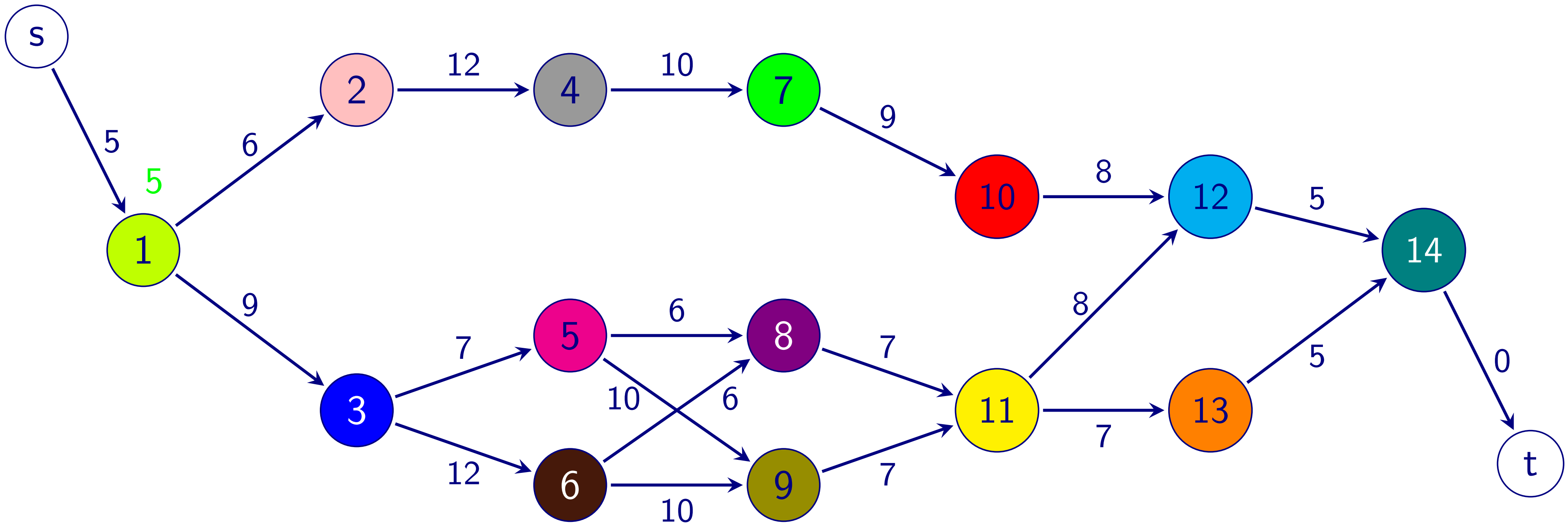

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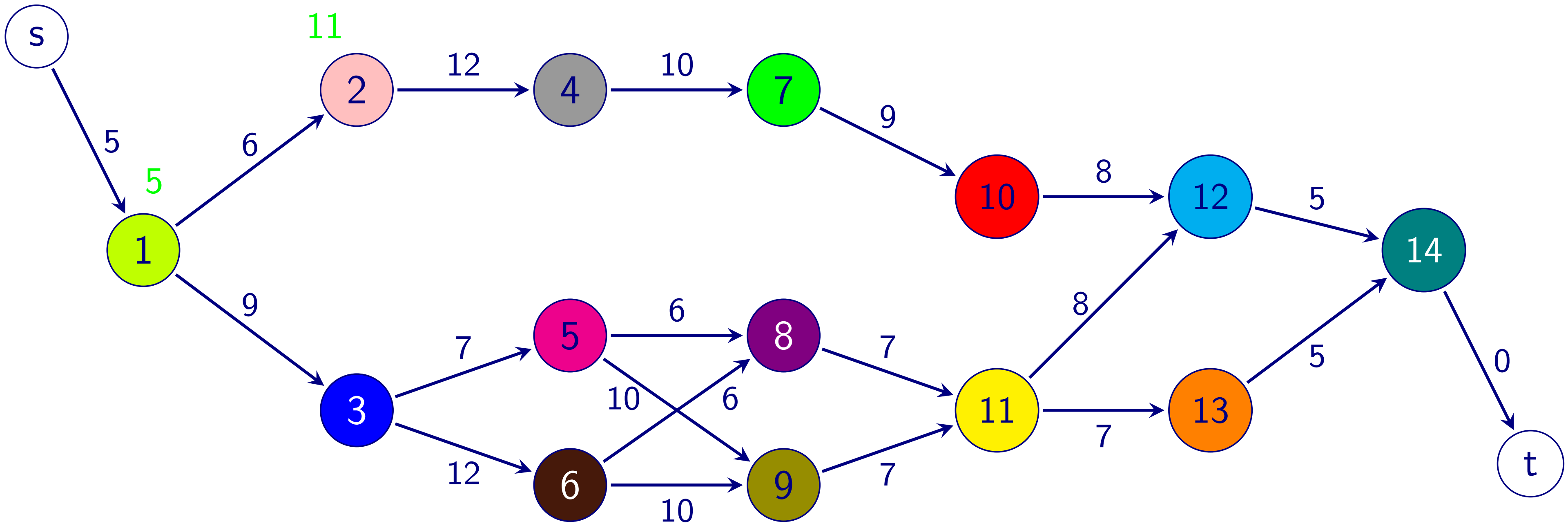

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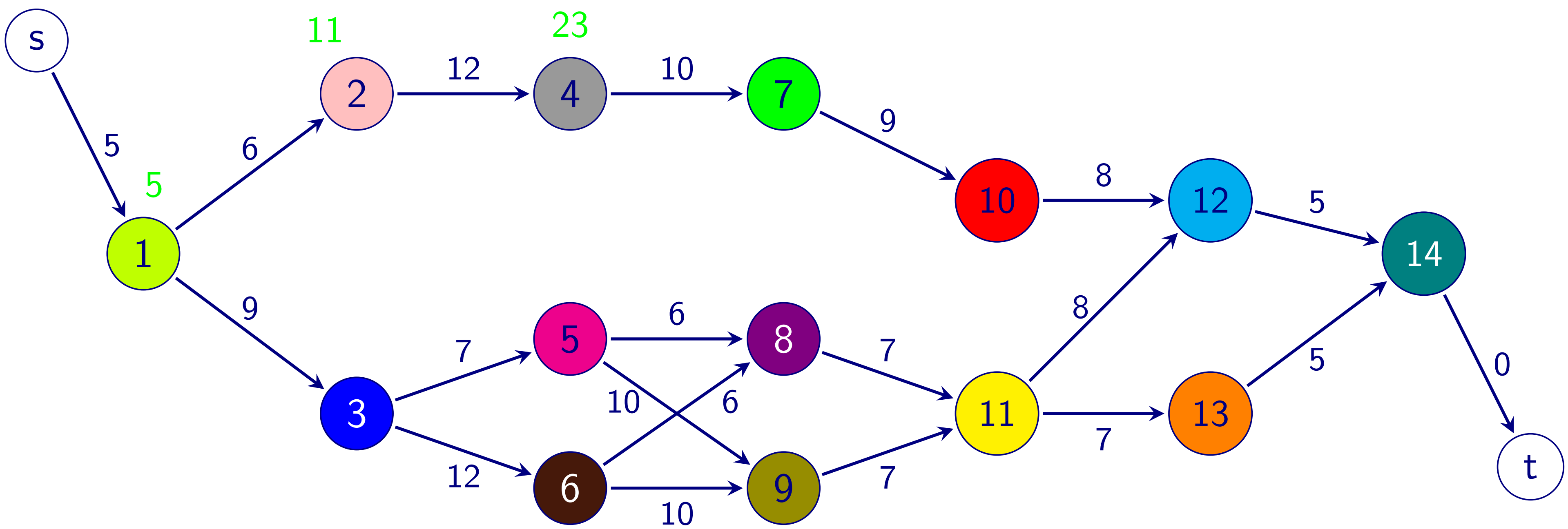

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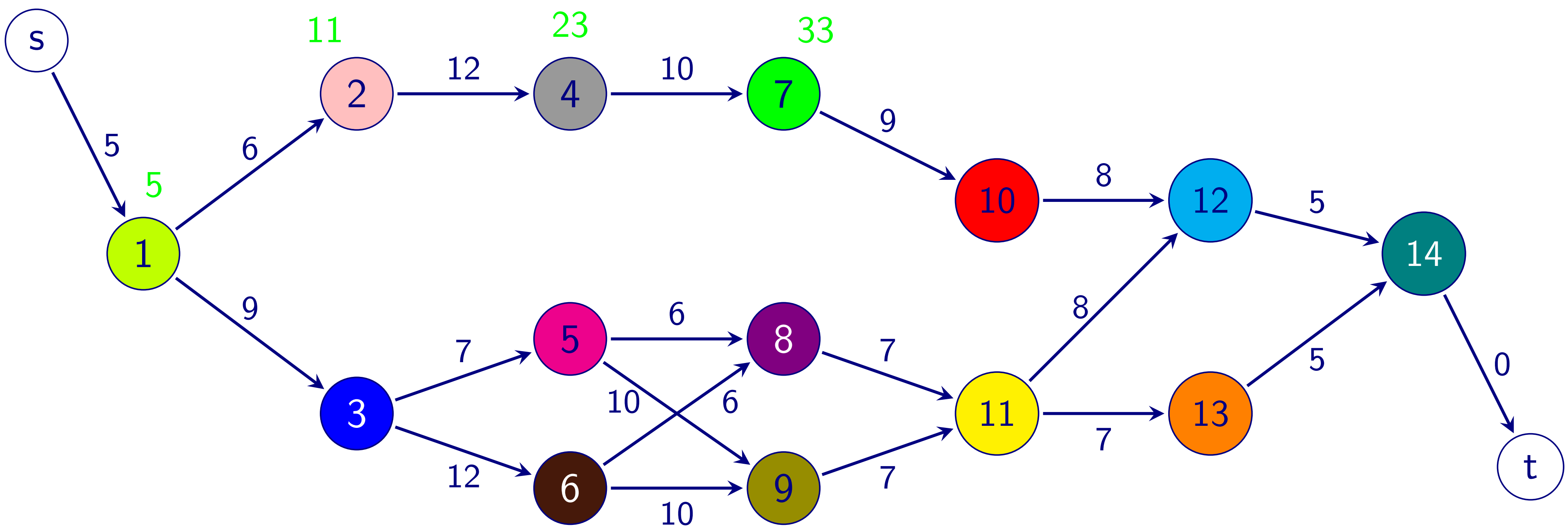

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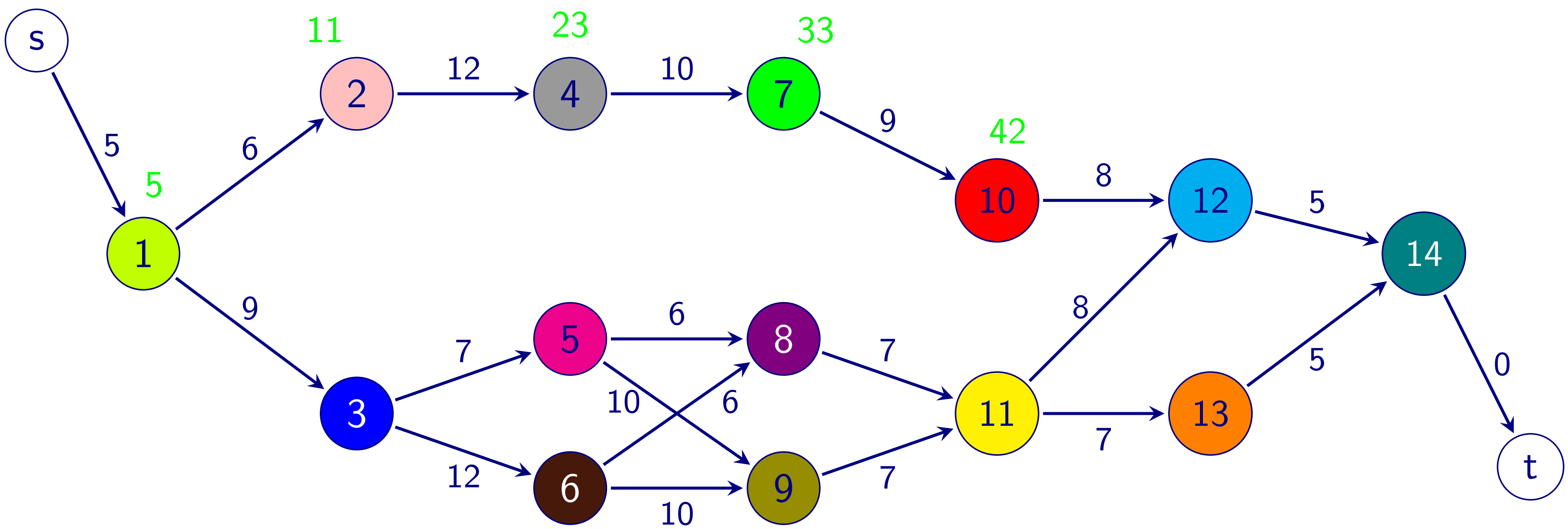

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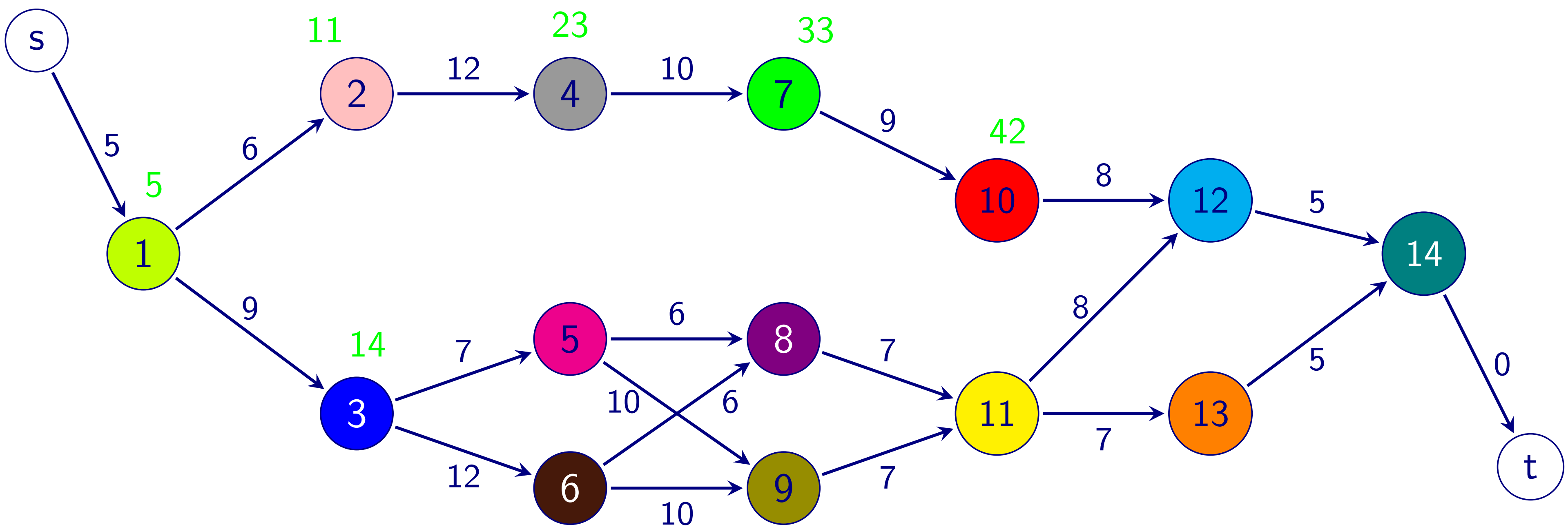

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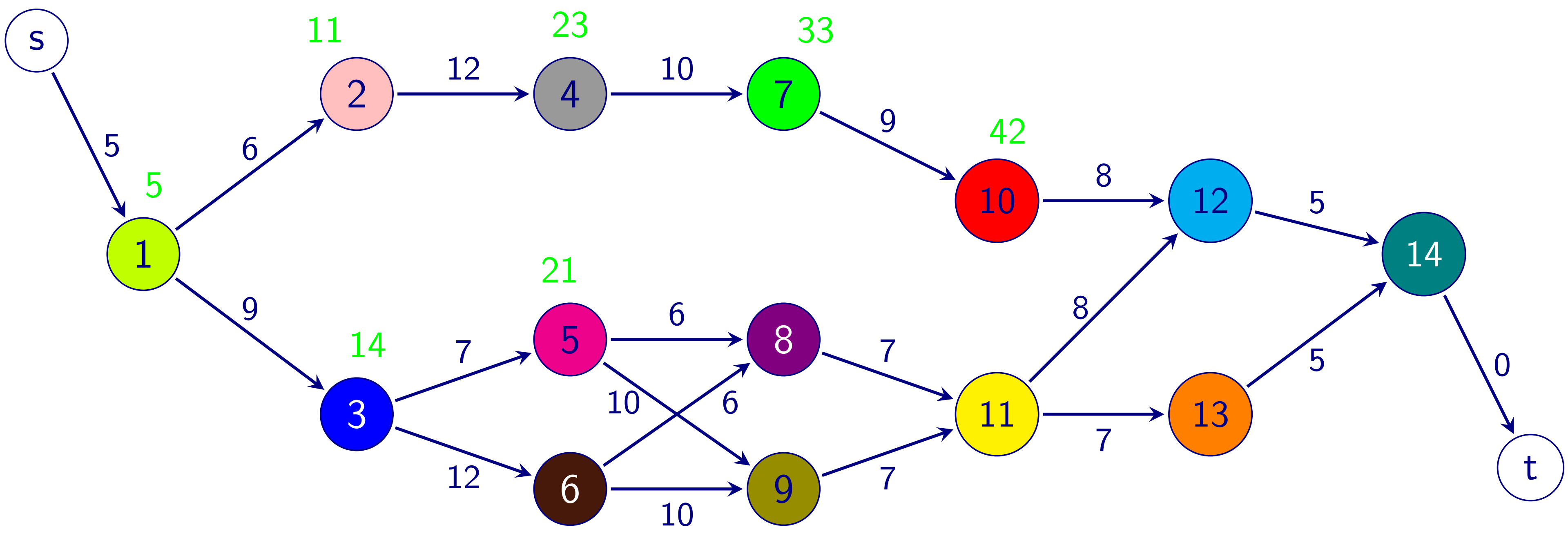

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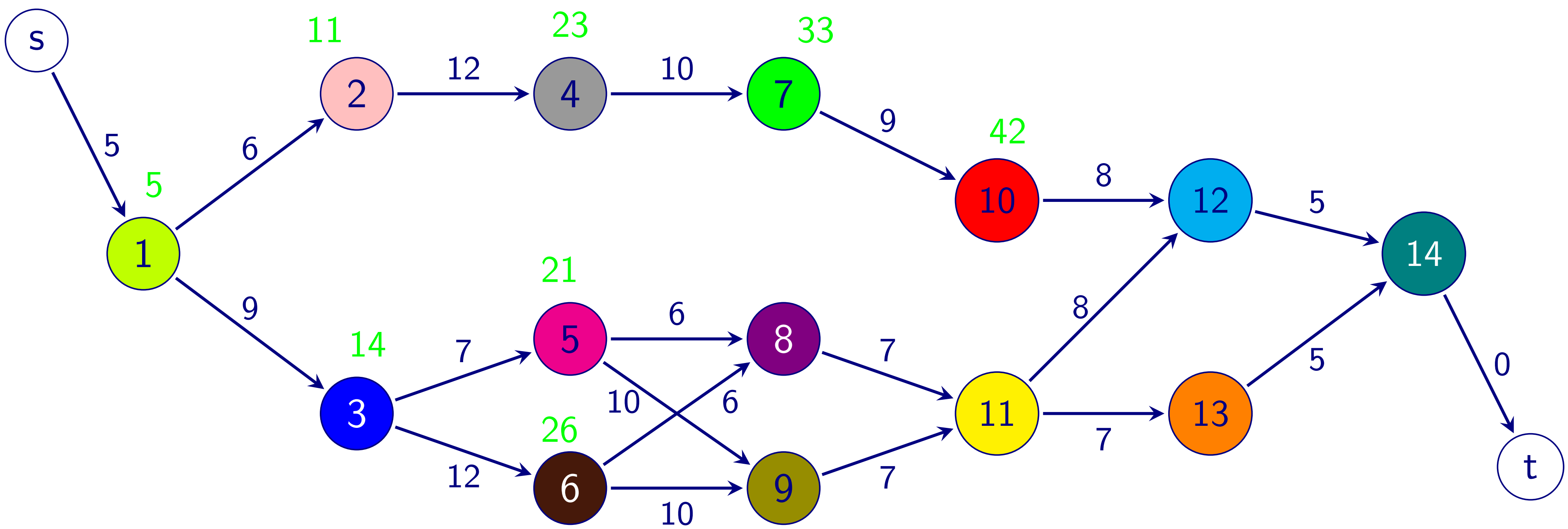

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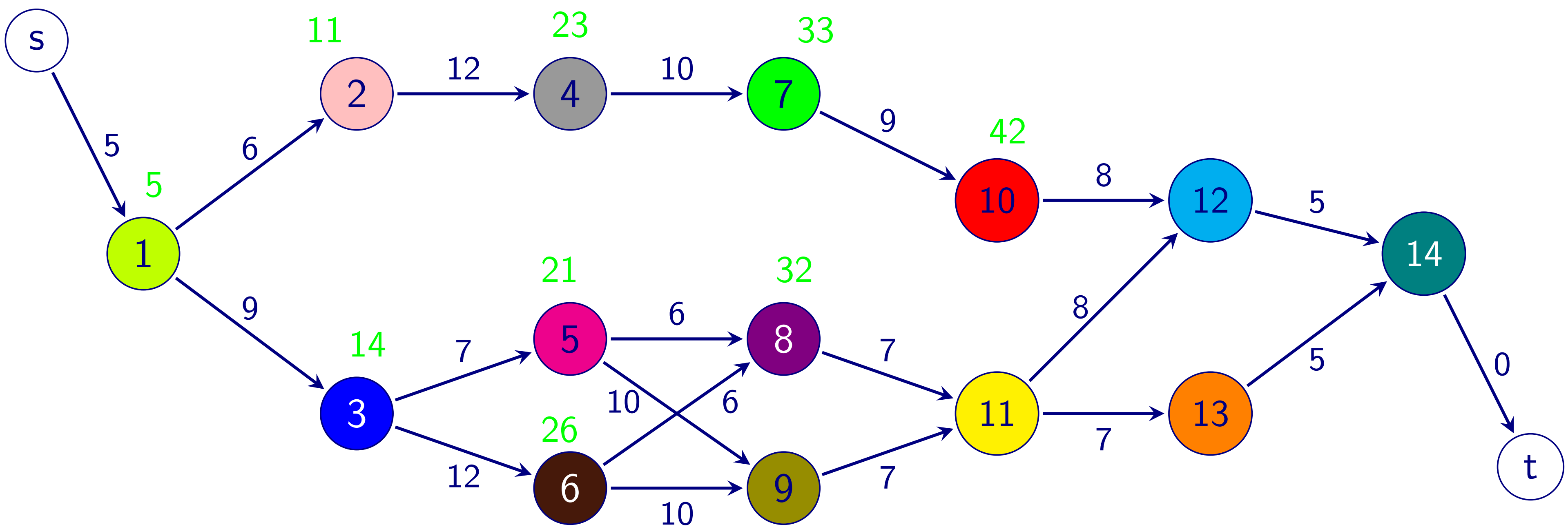

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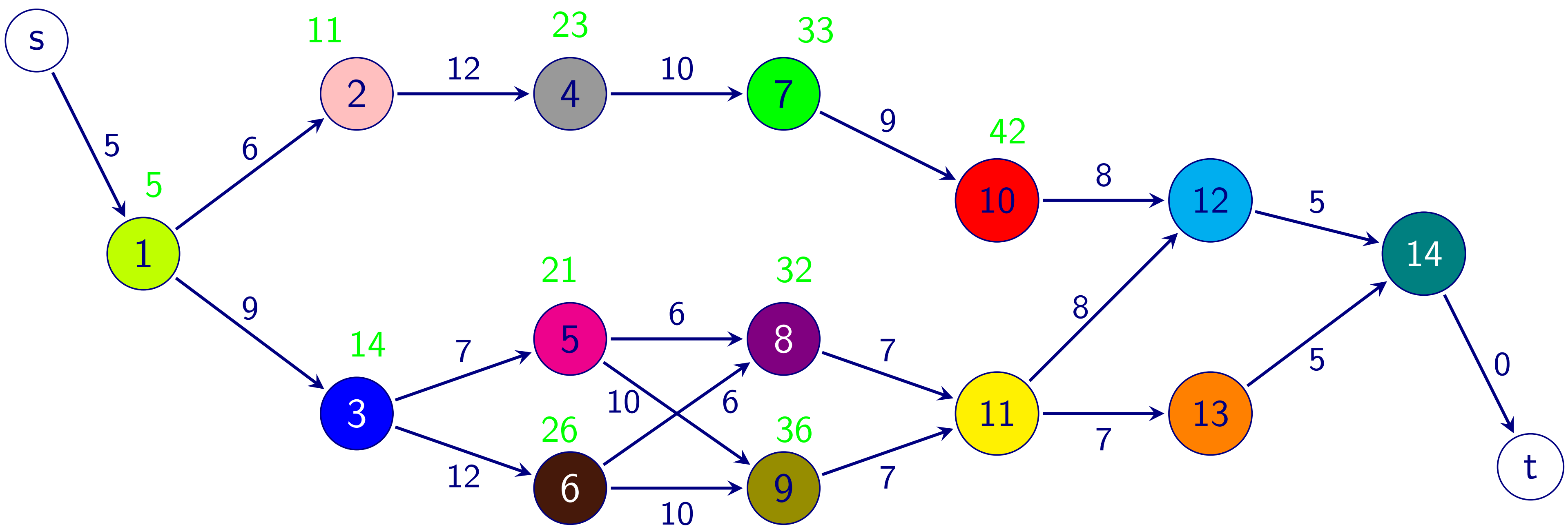

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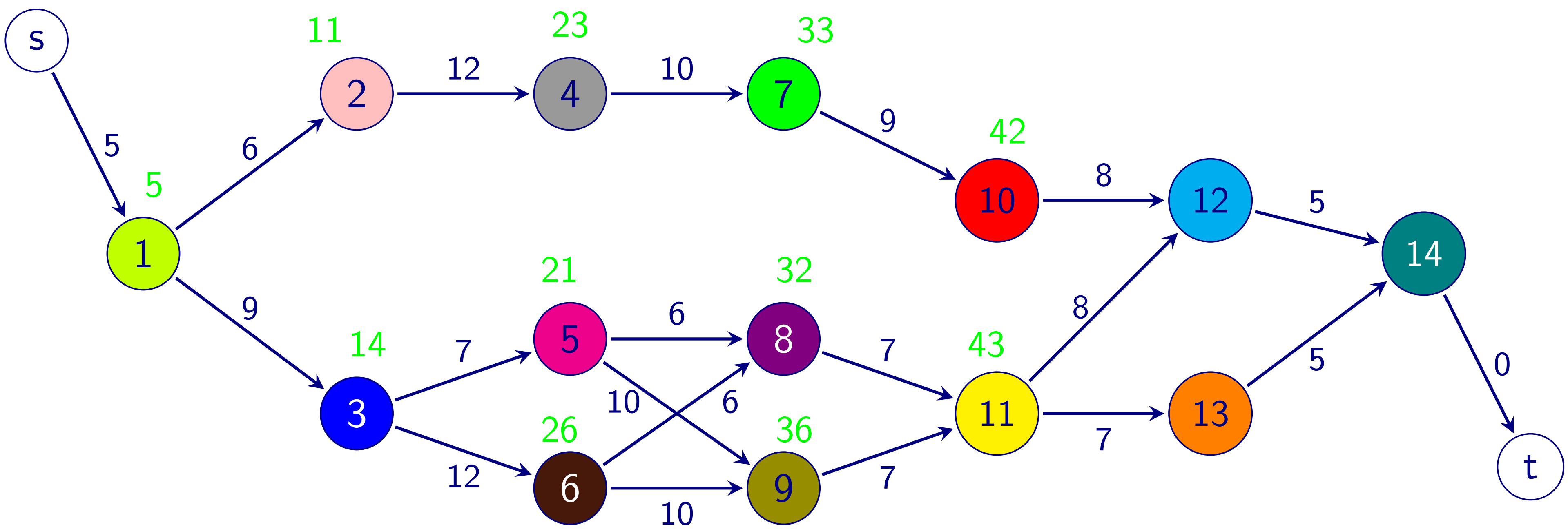

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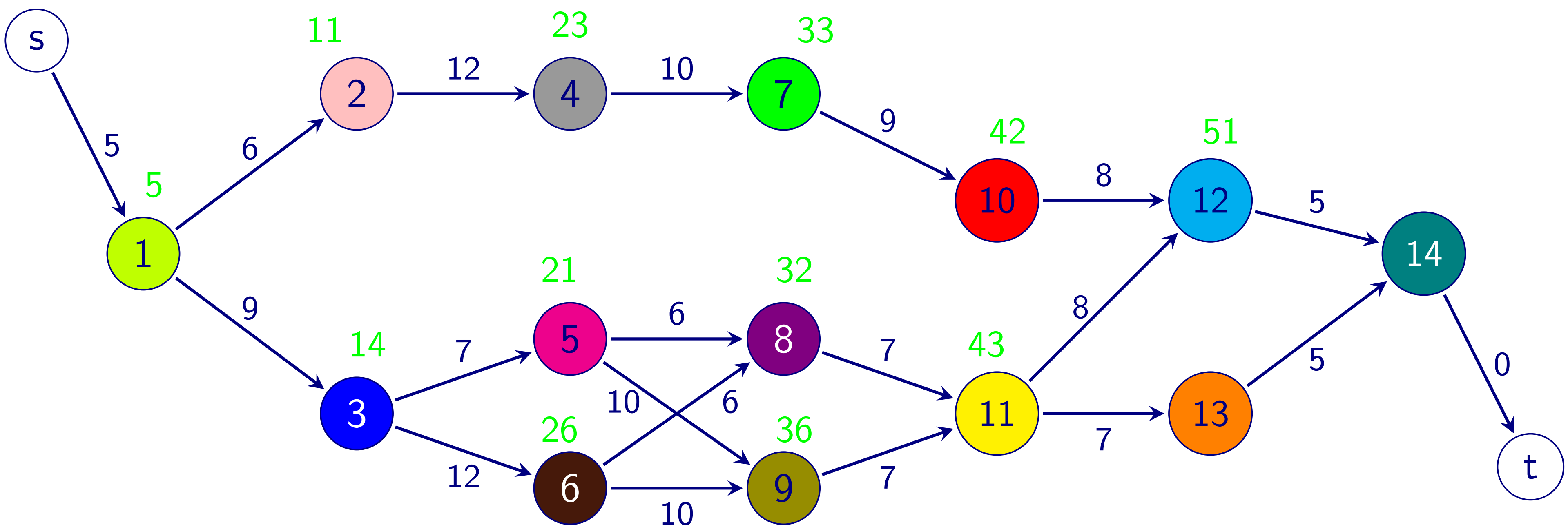

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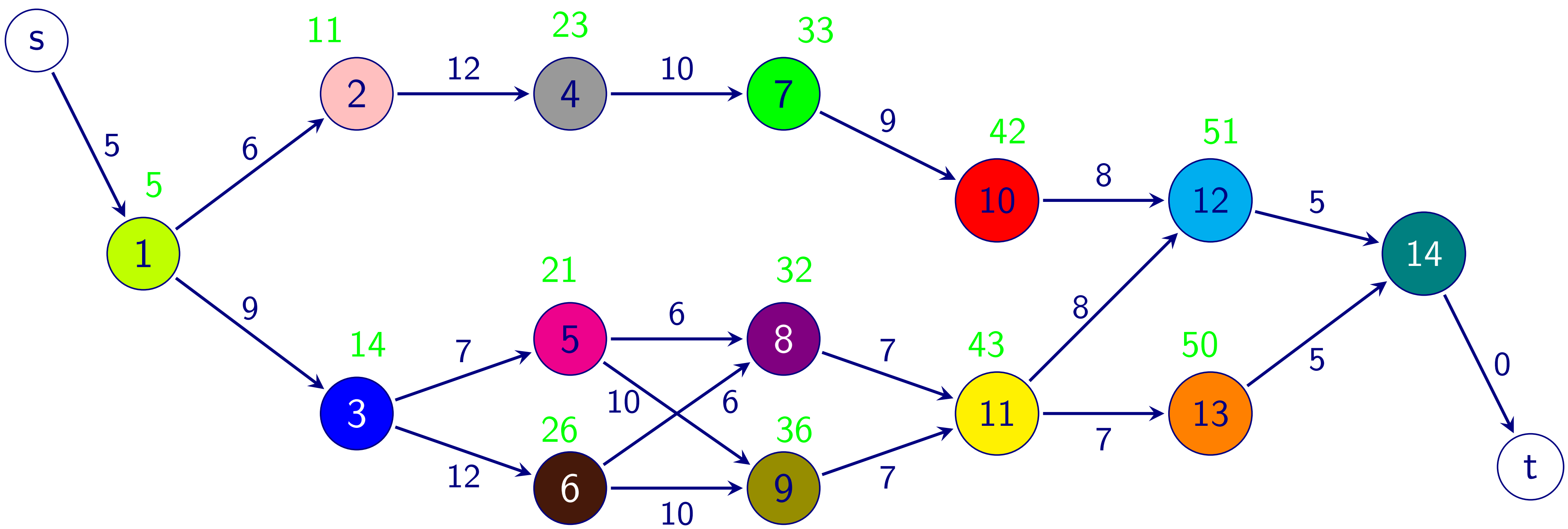

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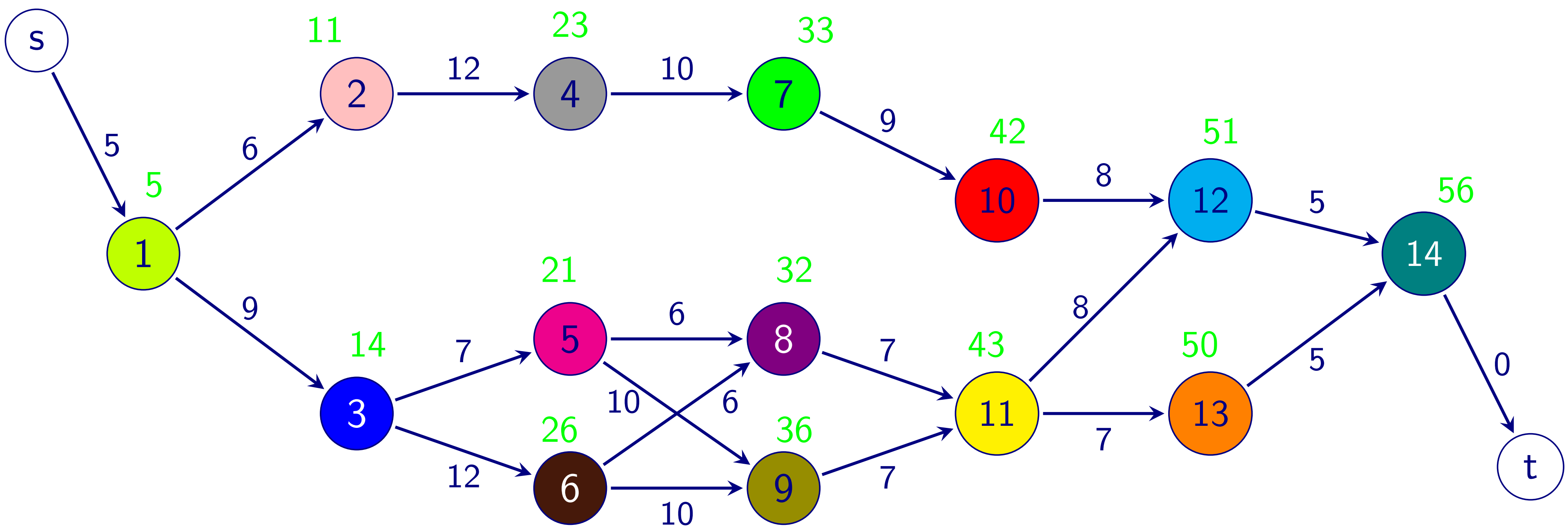

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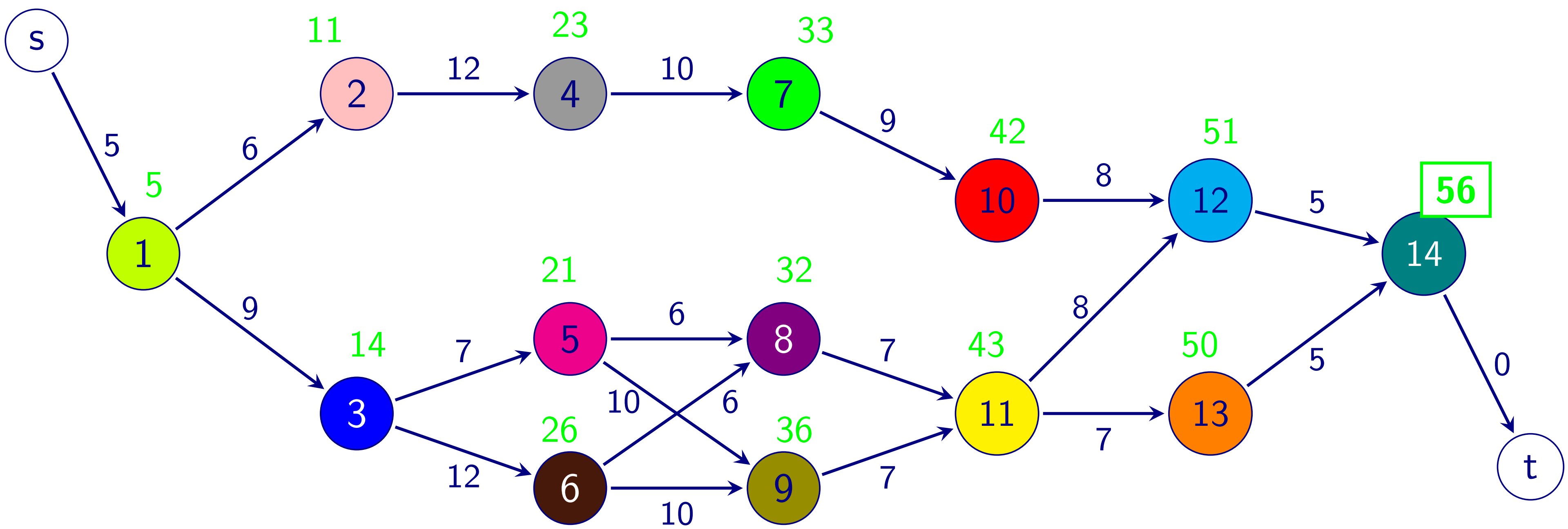

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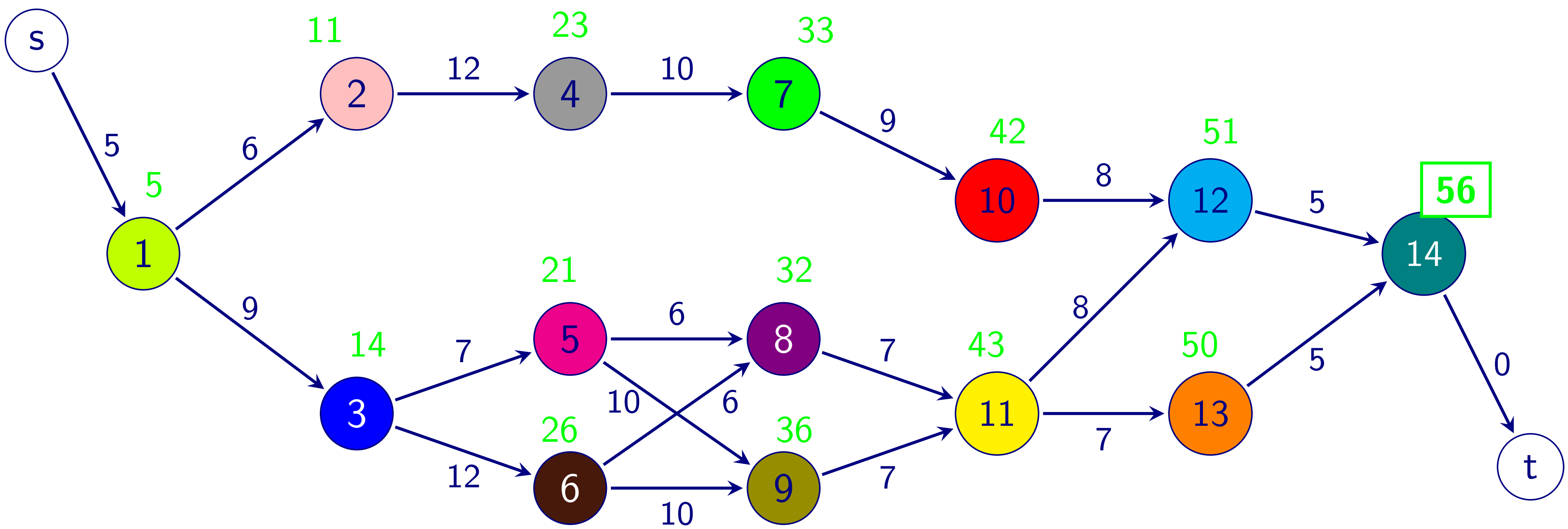
  - ➡ Makespan is the maximal earliest possible completion time computed



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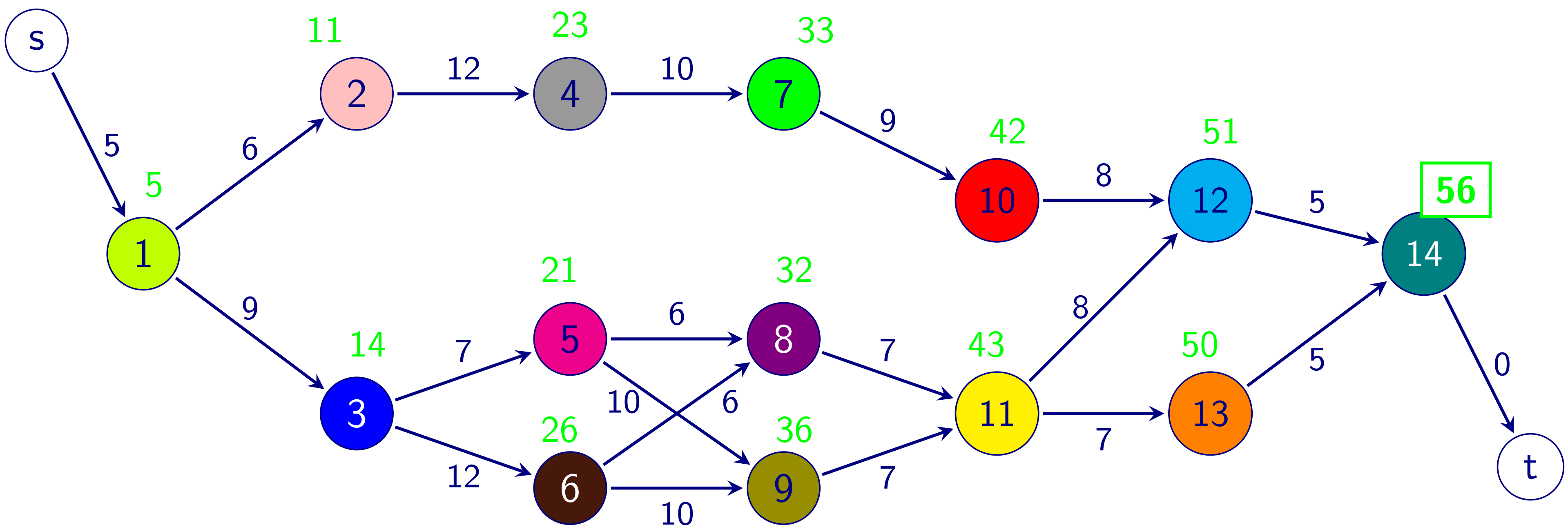

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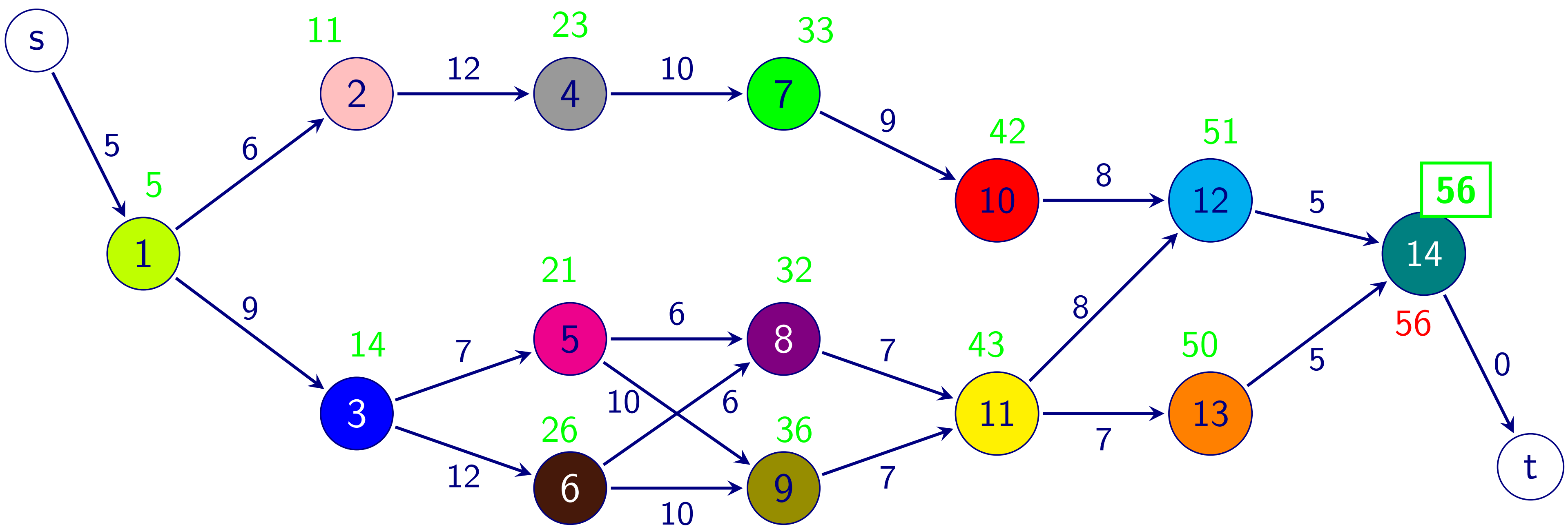

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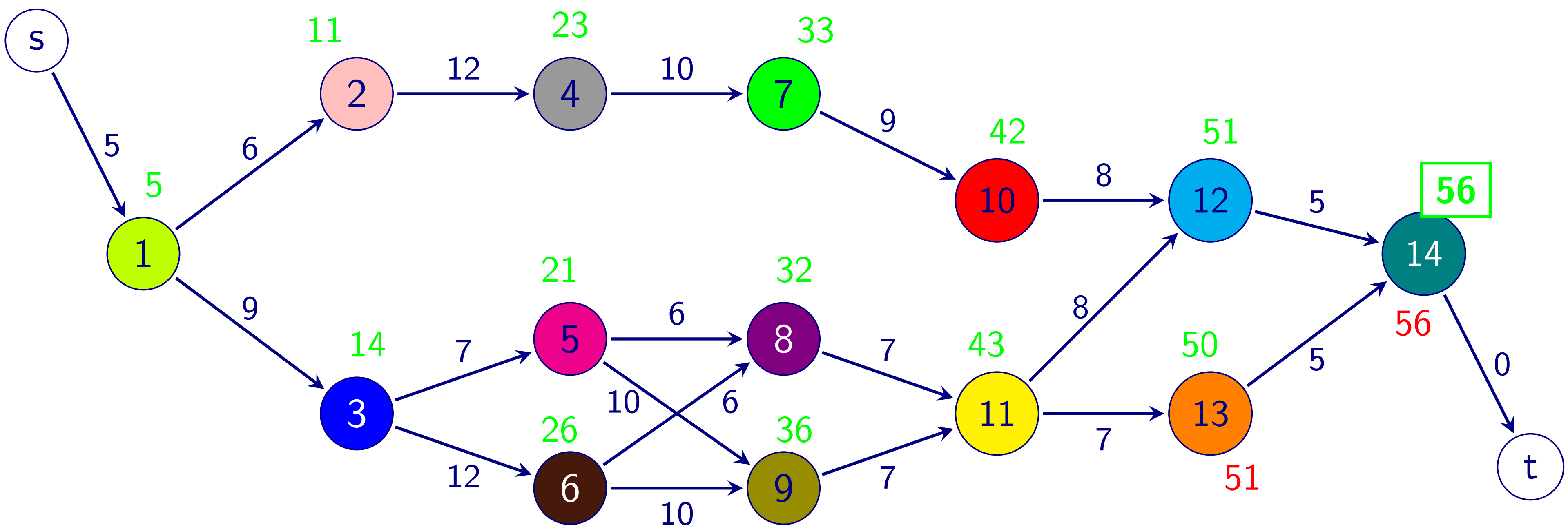

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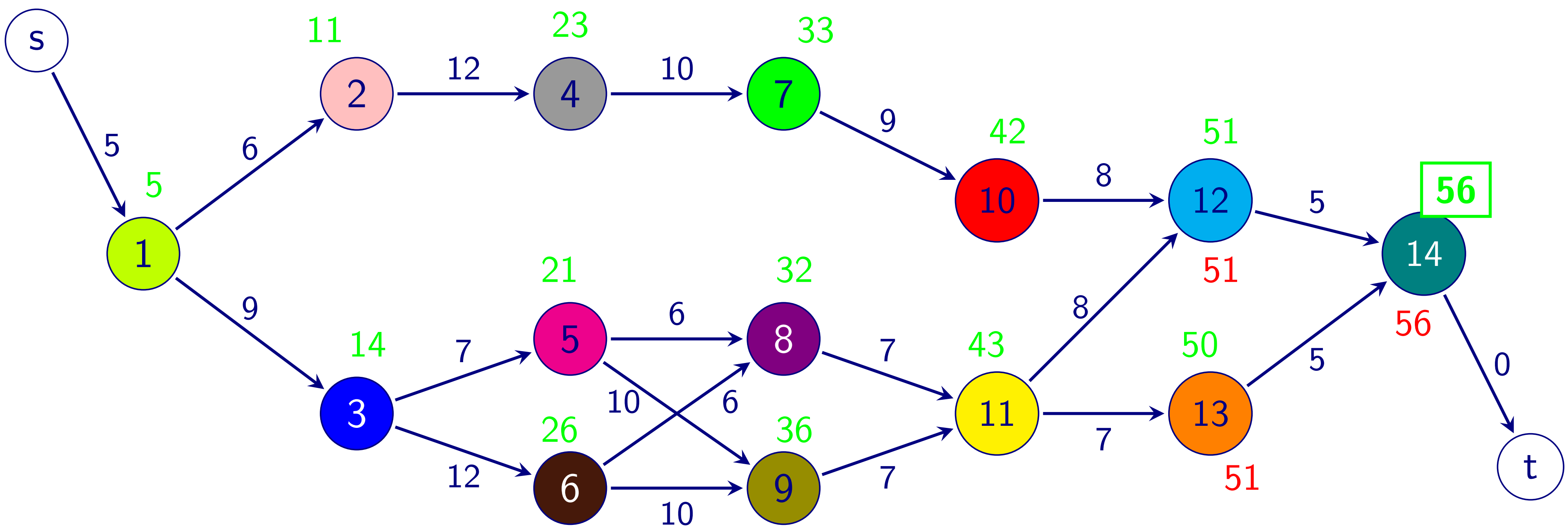

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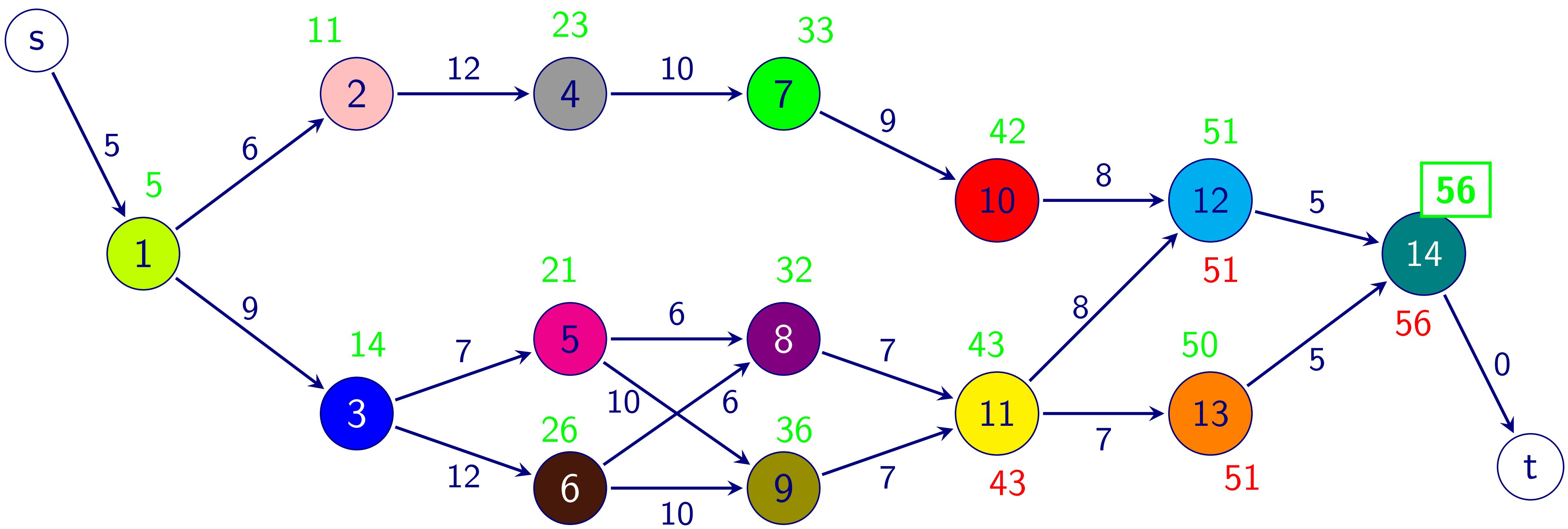

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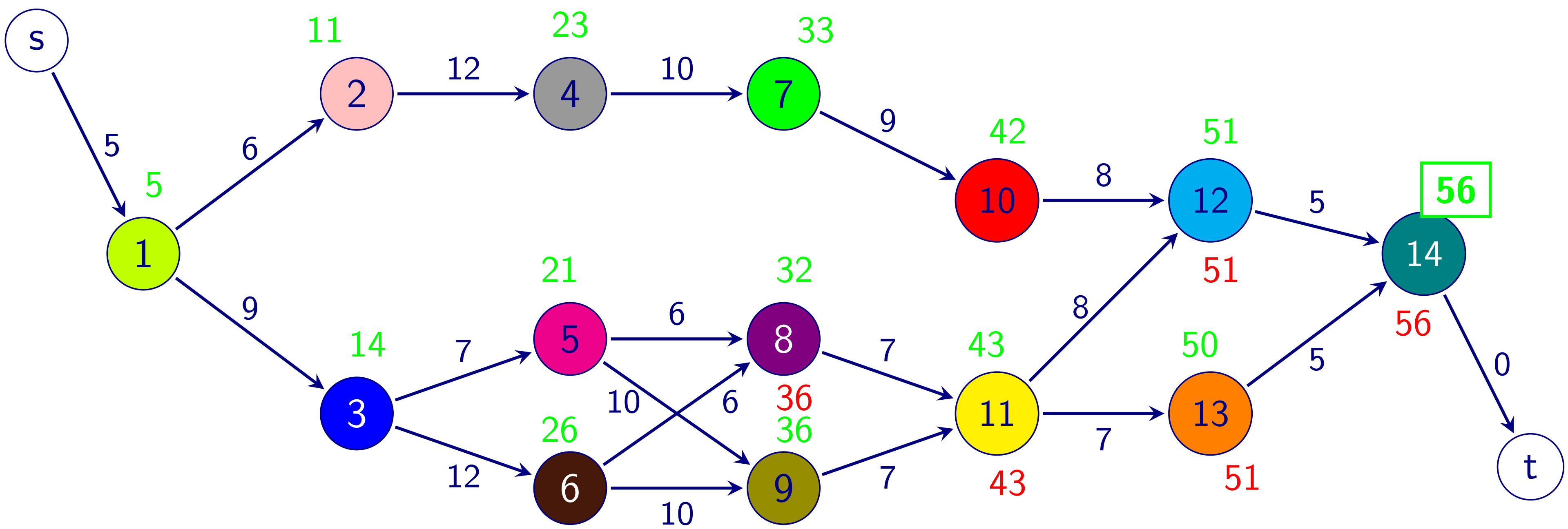

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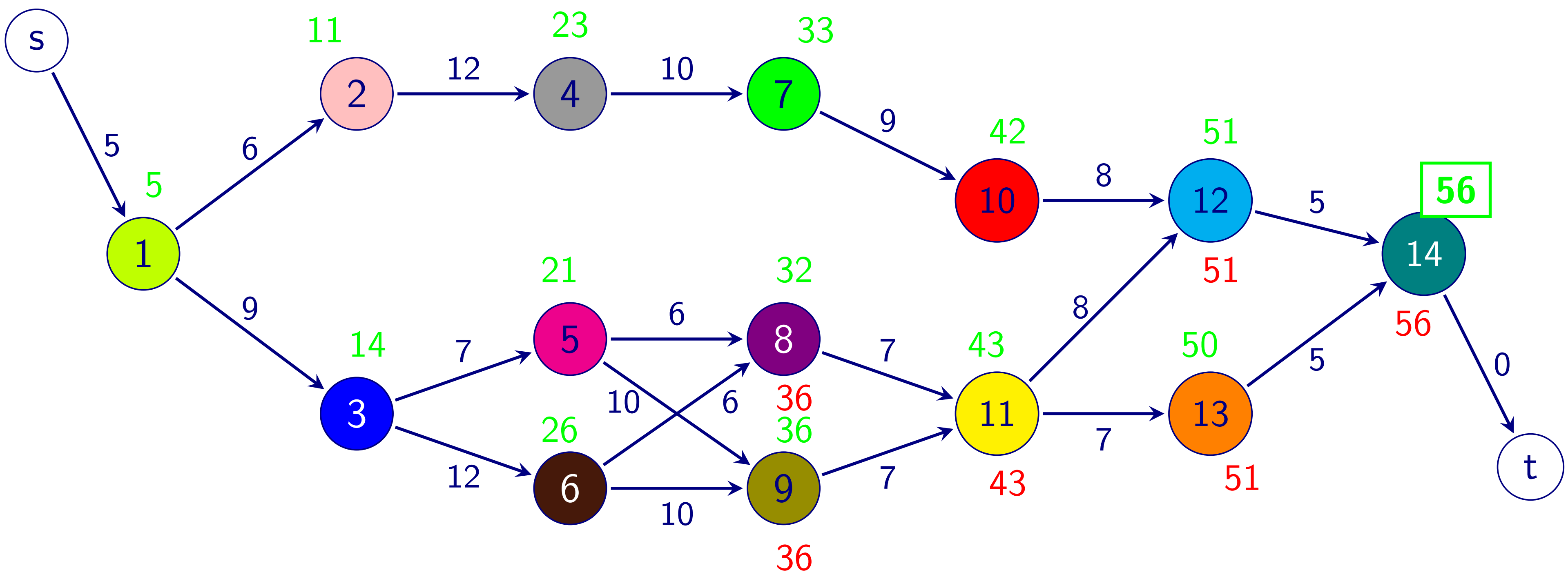

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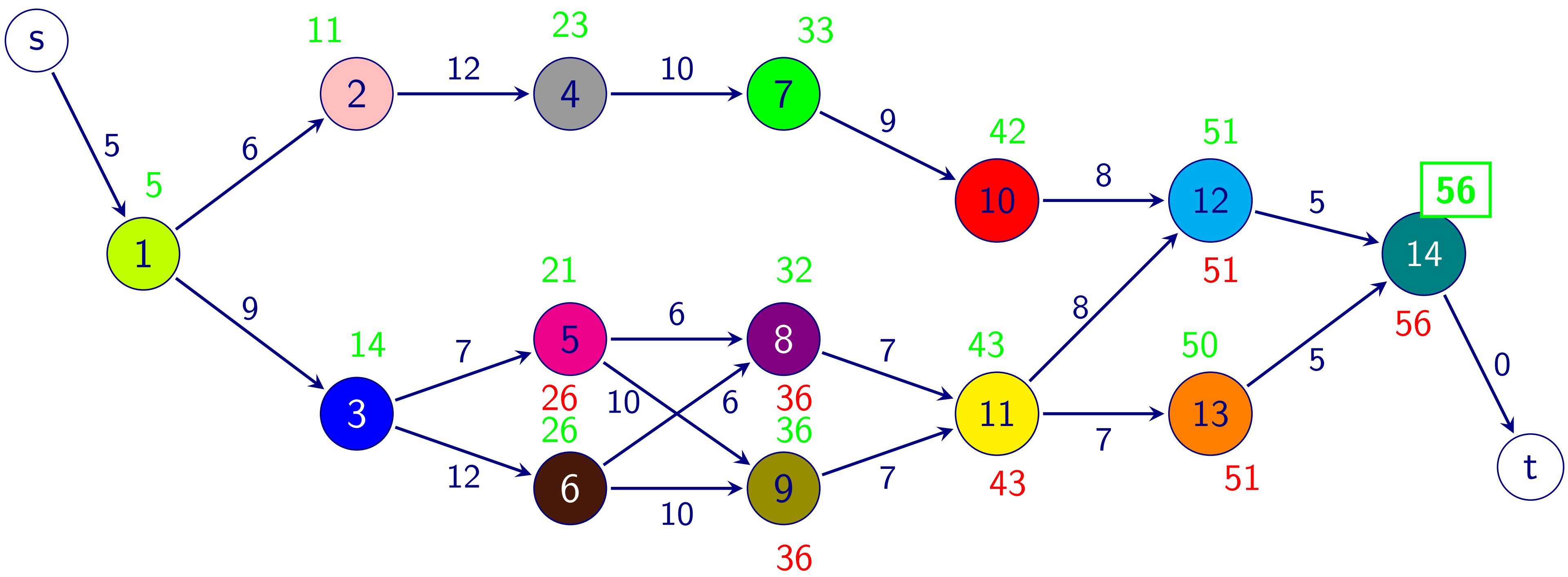

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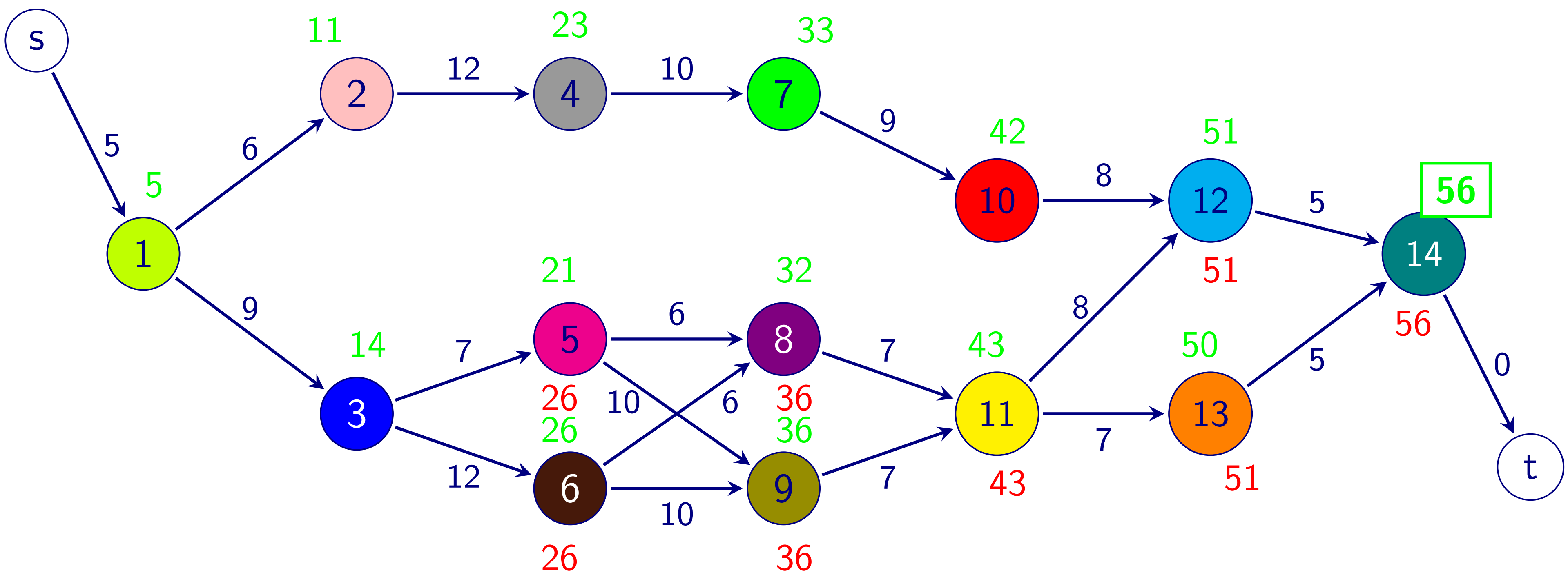

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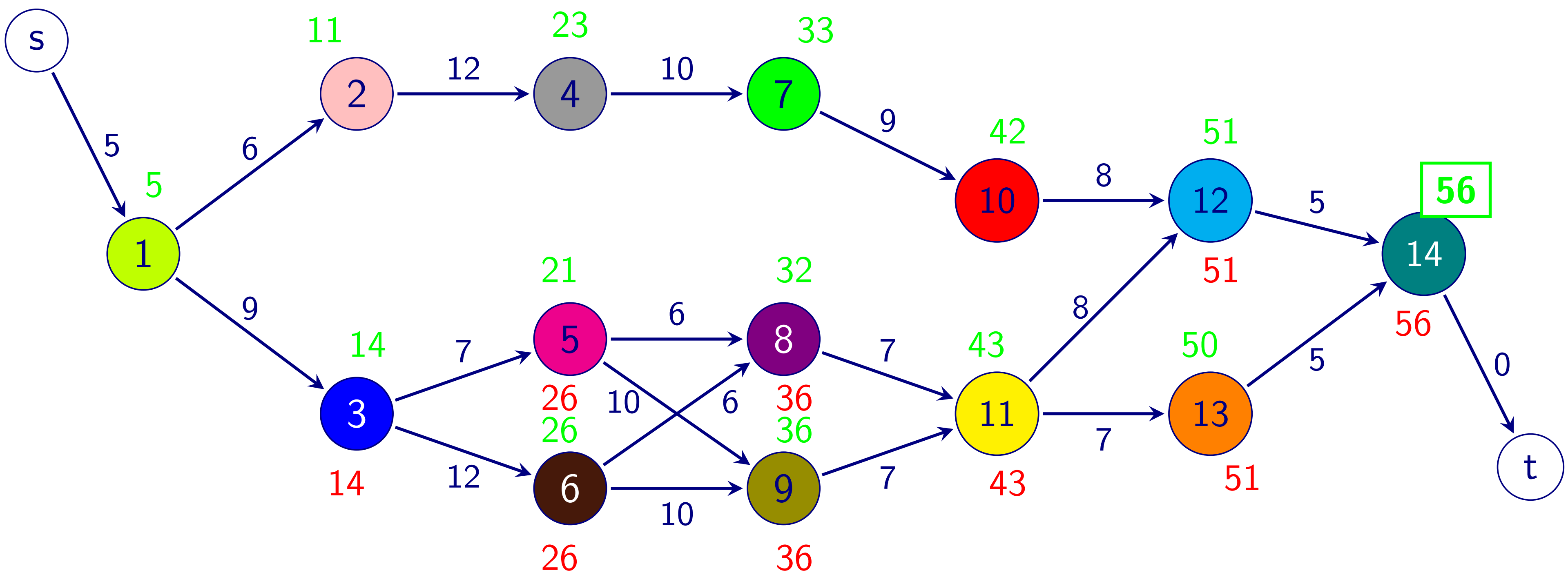

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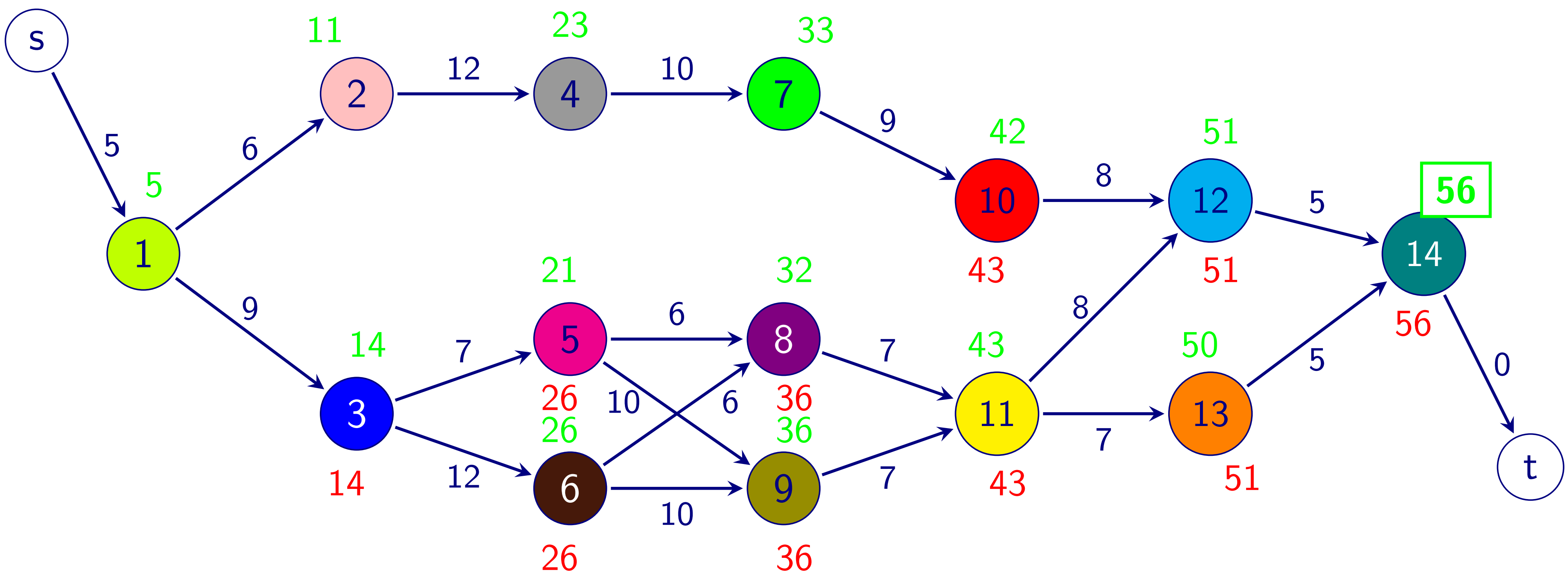

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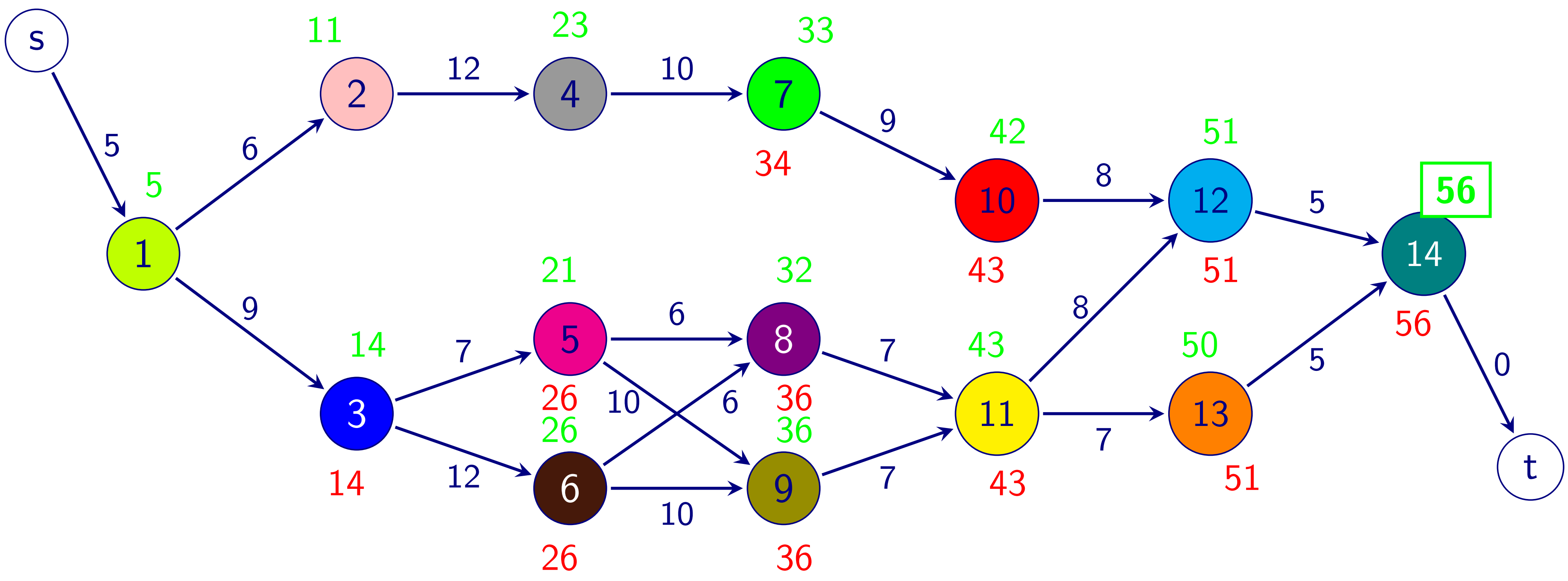

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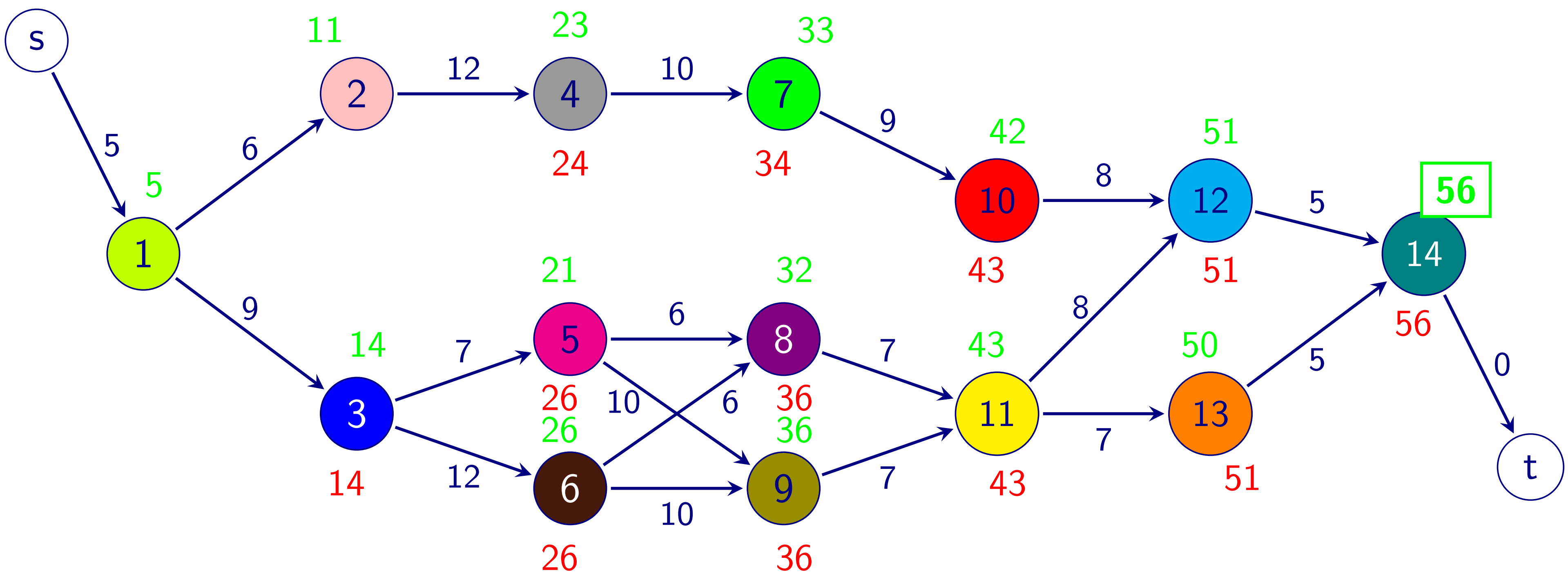

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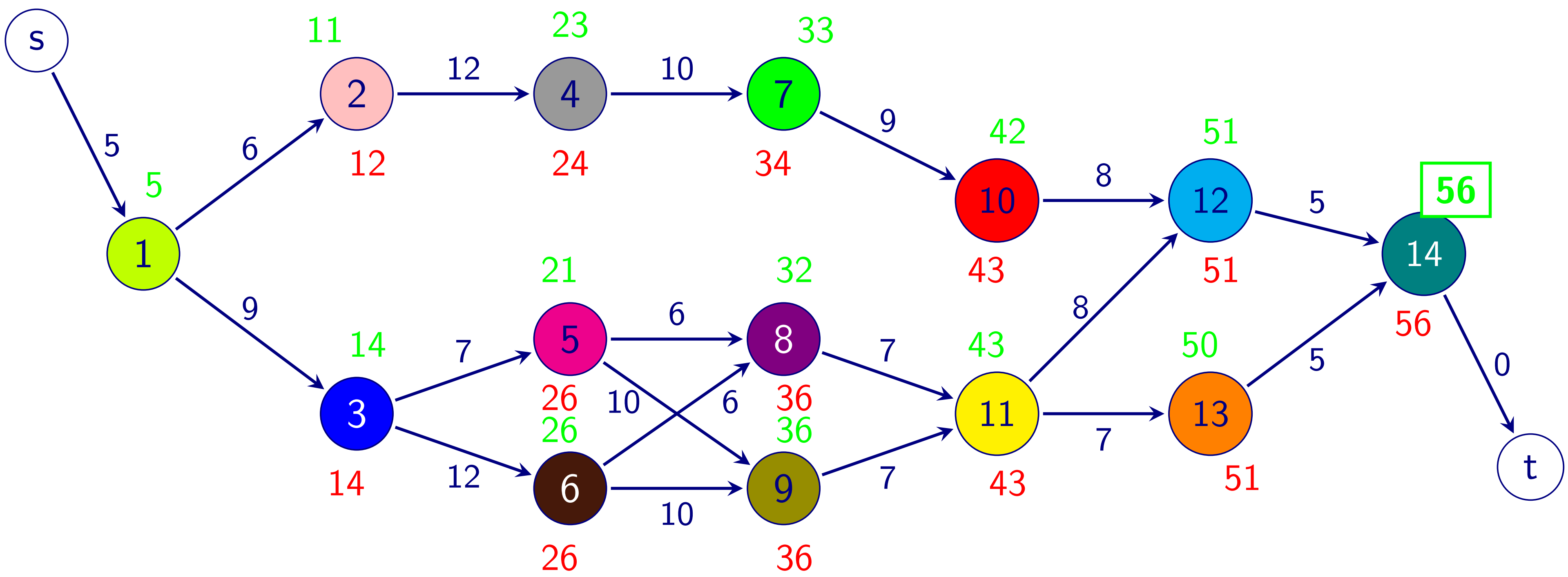

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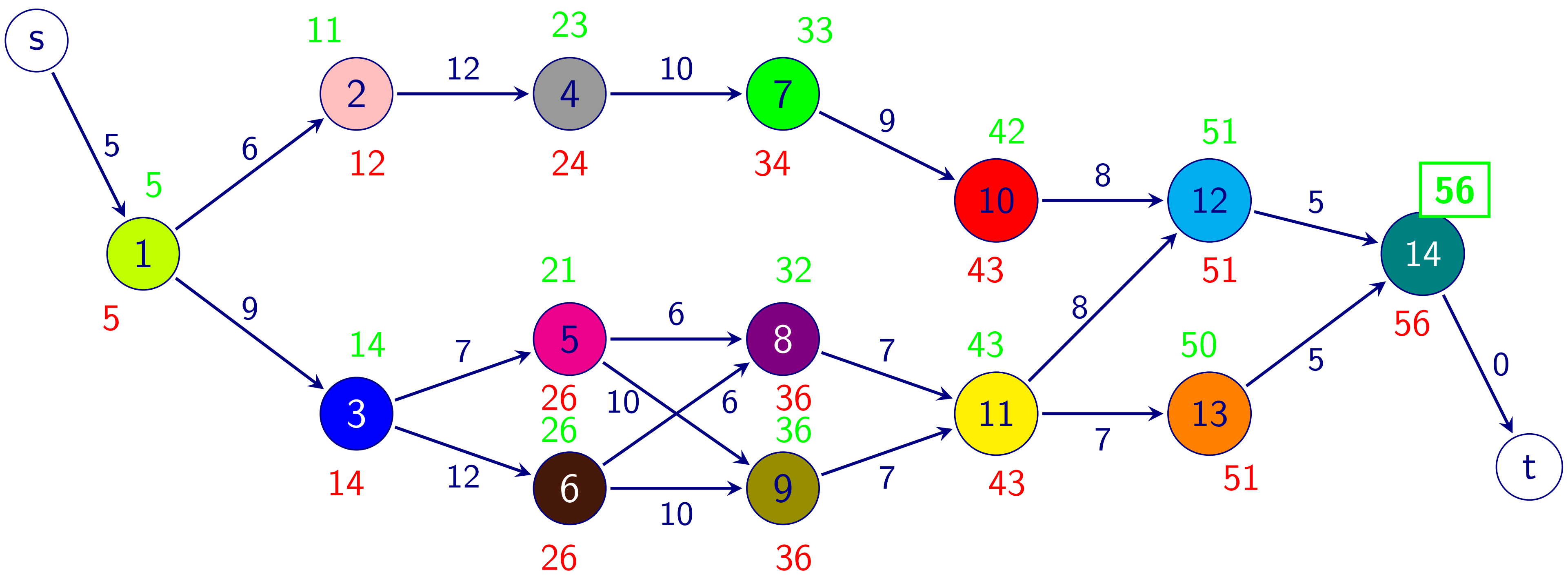

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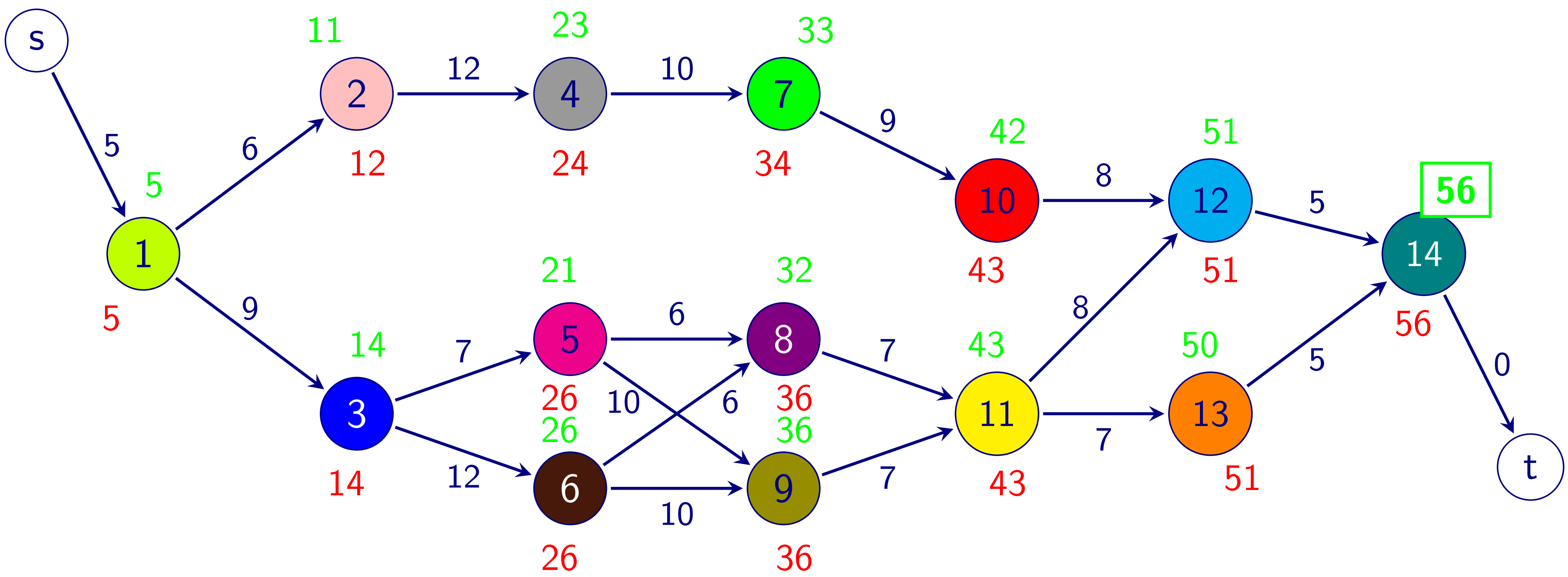

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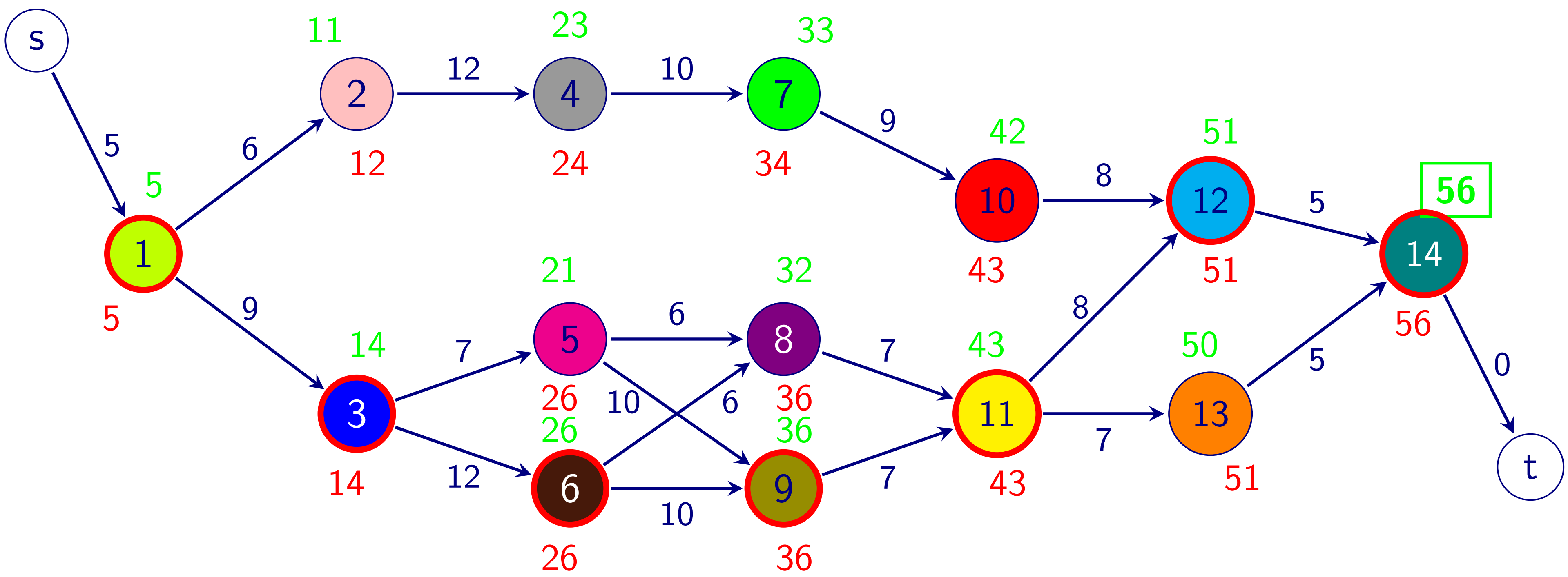

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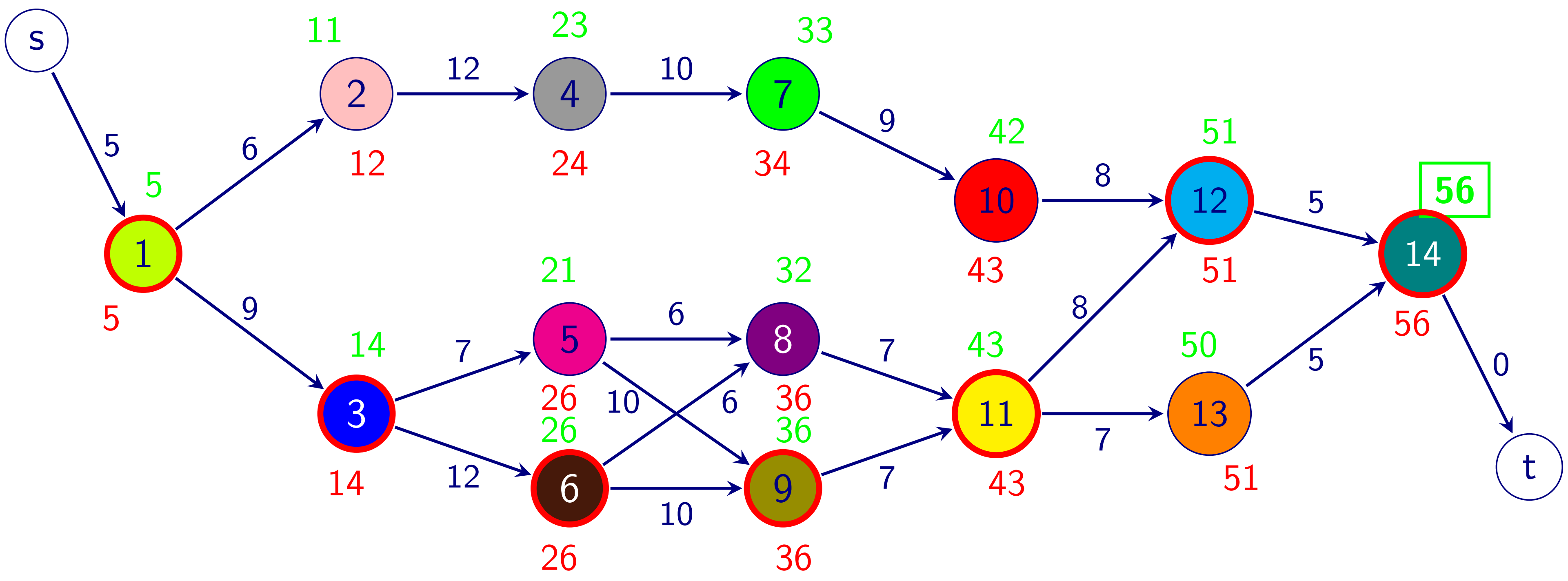
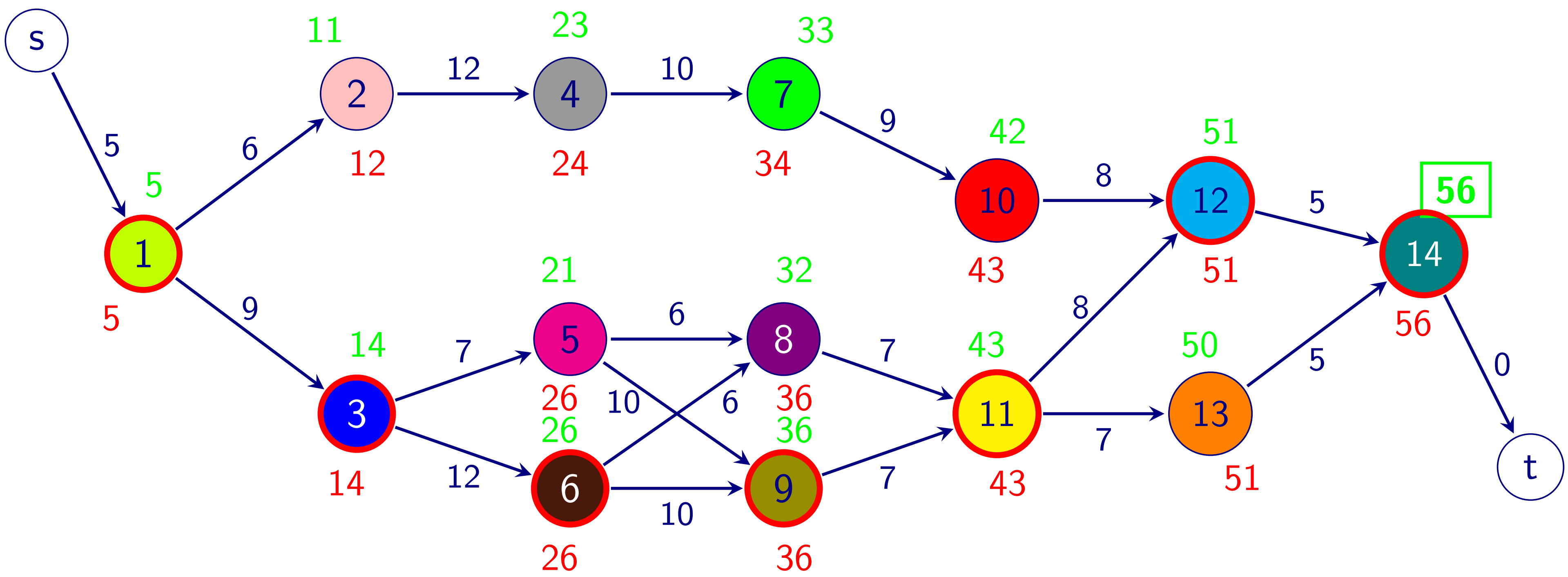


▷ A critical path is a chain of critical jobs, starting at time 0 and ending at the makespan



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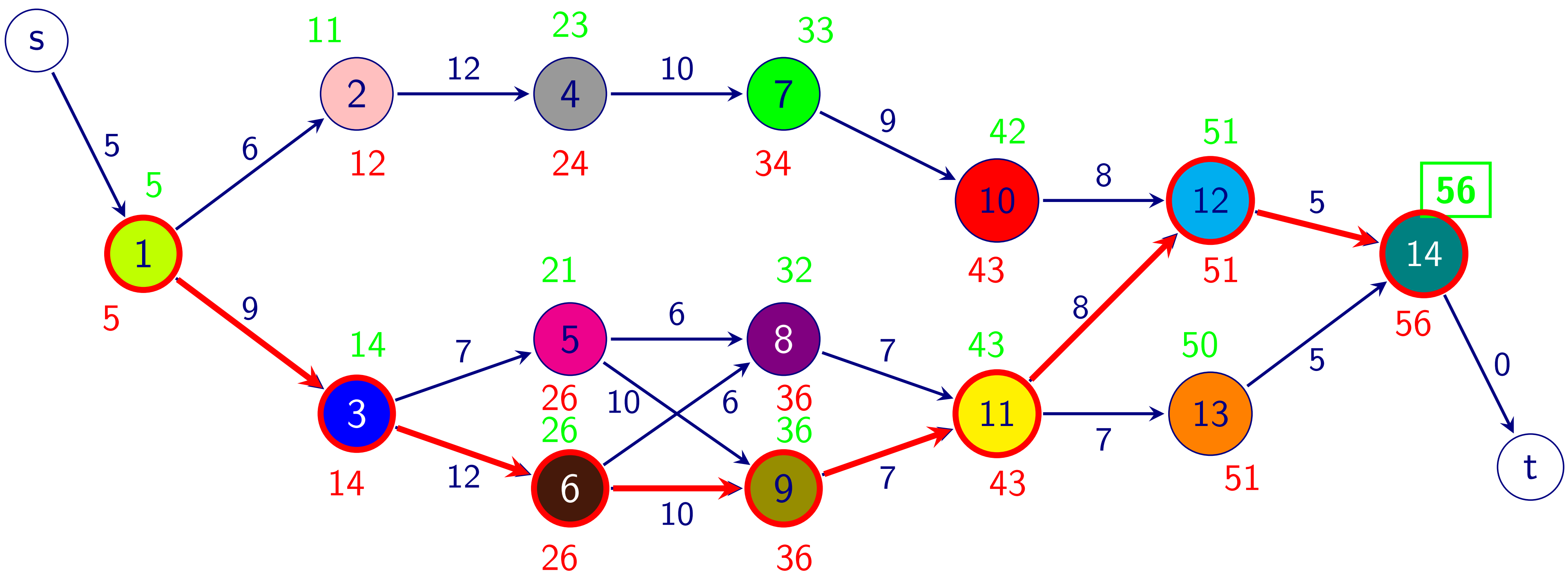

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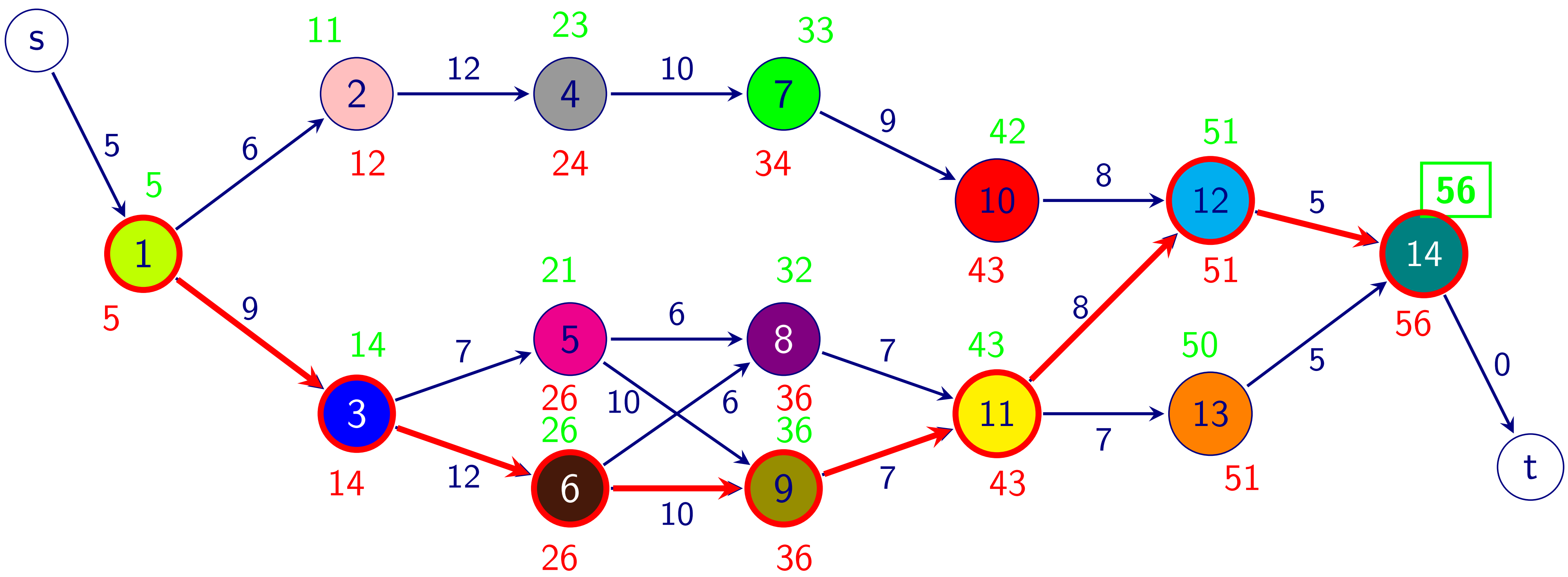

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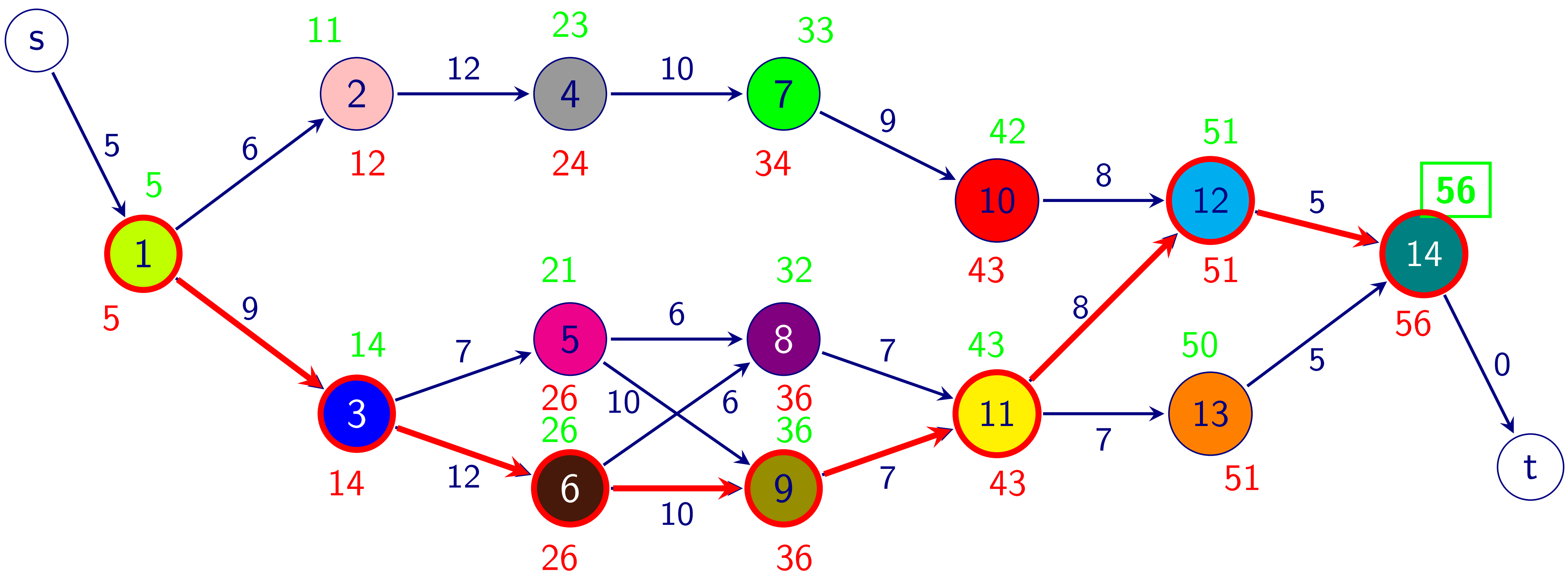

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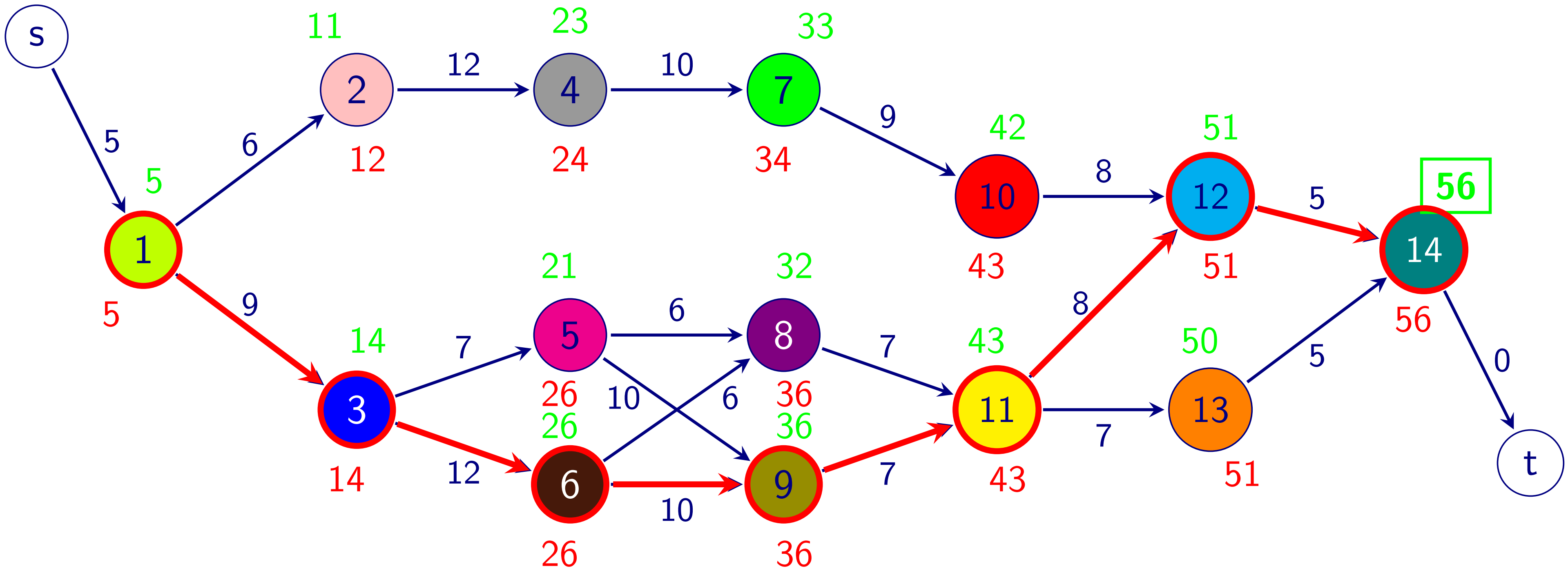

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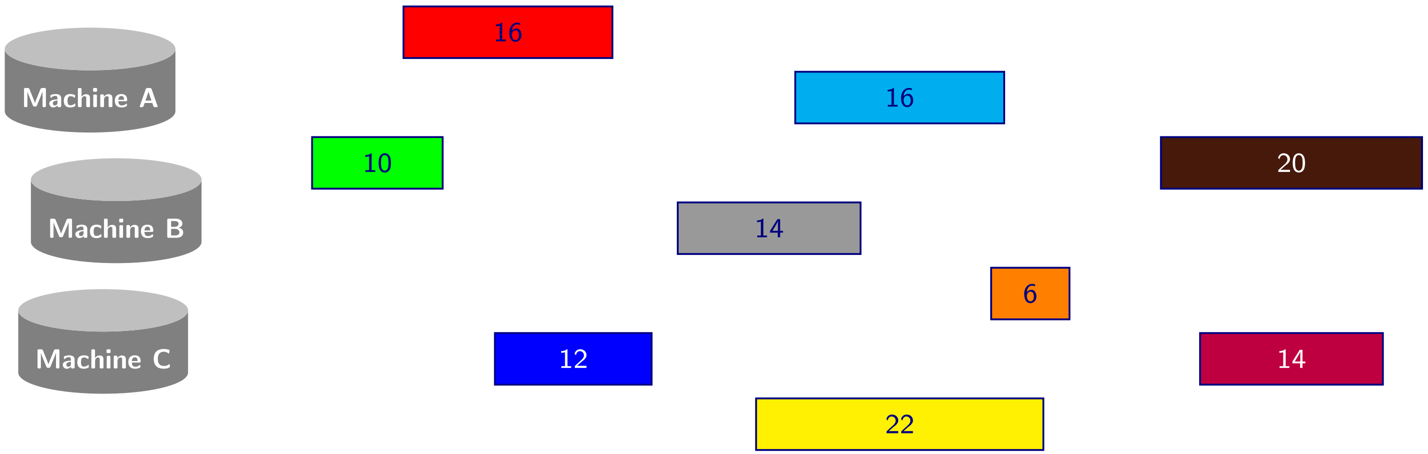
- Summary:

	Objective	
	$\sum C_j$	$\max C_j$
single machine	$\mathcal{NP}$-hard	polynomial (greedy algorithm)
$\geq$ 2 machines	$\mathcal{NP}$-hard	$\mathcal{NP}$-hard
unlimited machines	...?	polynomial (critical path method)

- Jobs can be carried out on one of 3 identical machines



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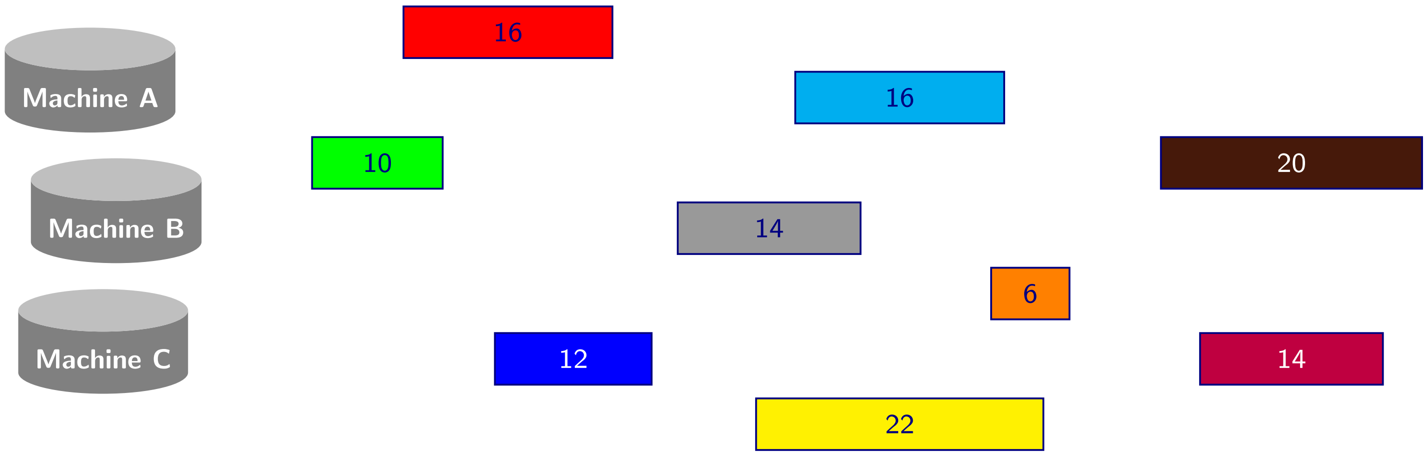

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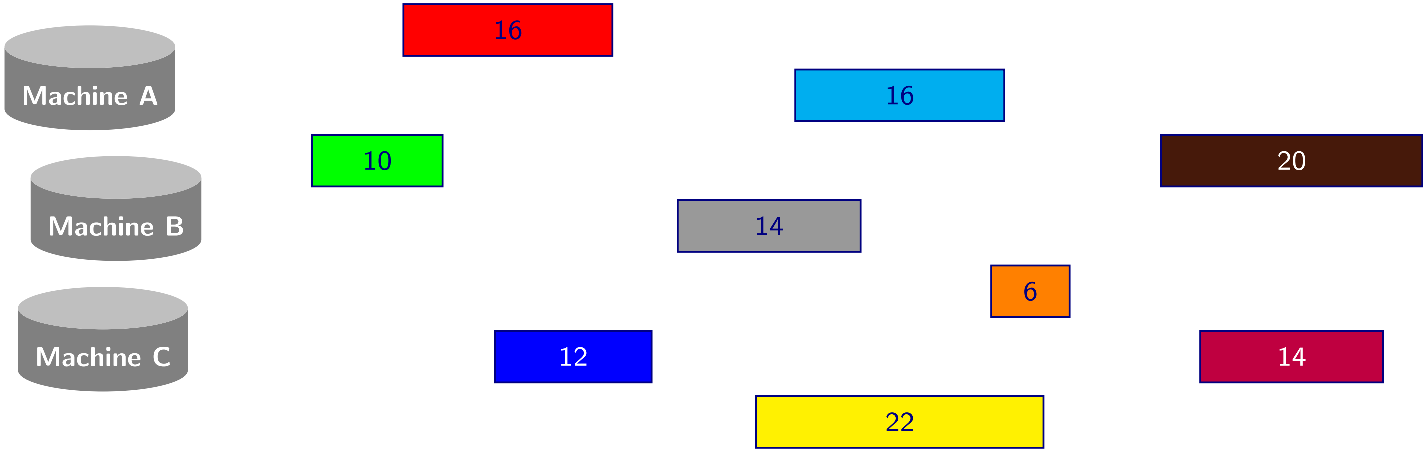

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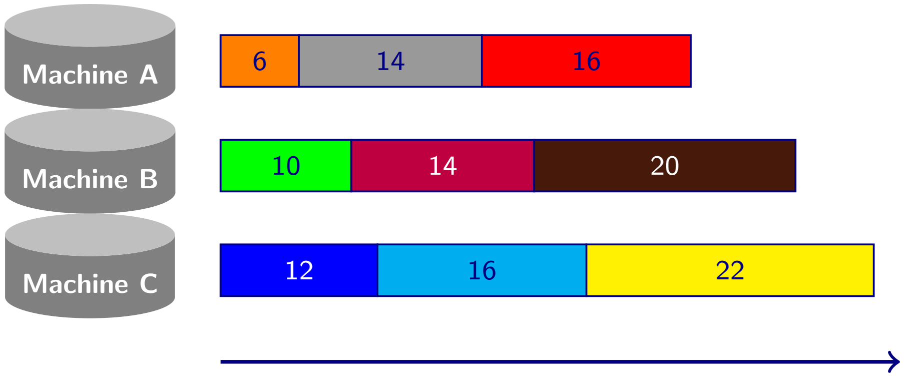

Scheduling on 3 identical machines

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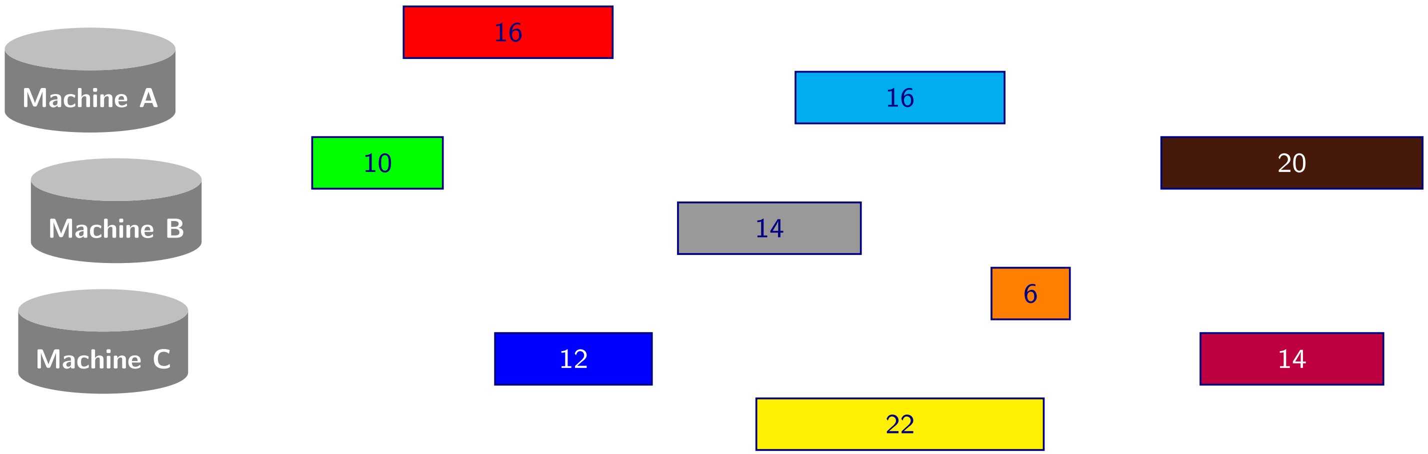

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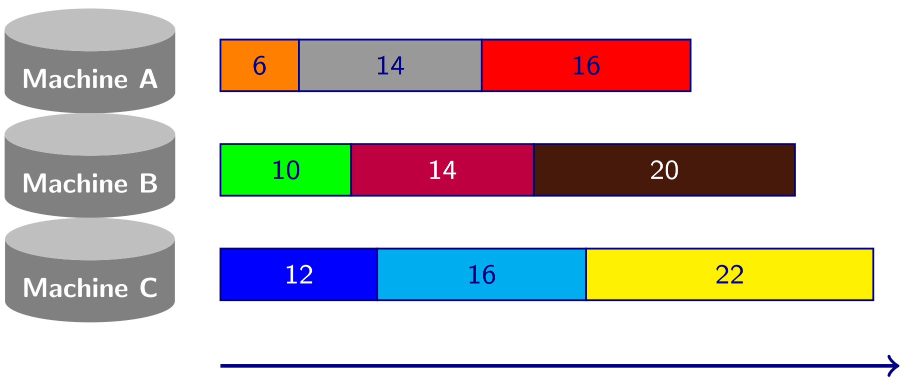

$$
\begin{aligned}
\sum_{j=1}^{n} C_j &= 6 + 20 + 36 \\
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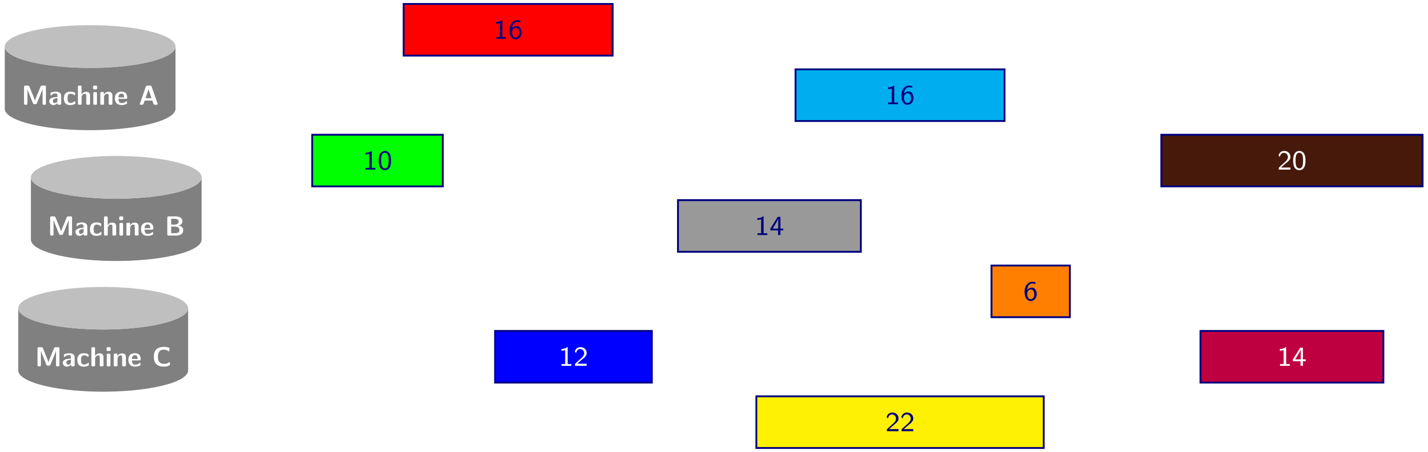

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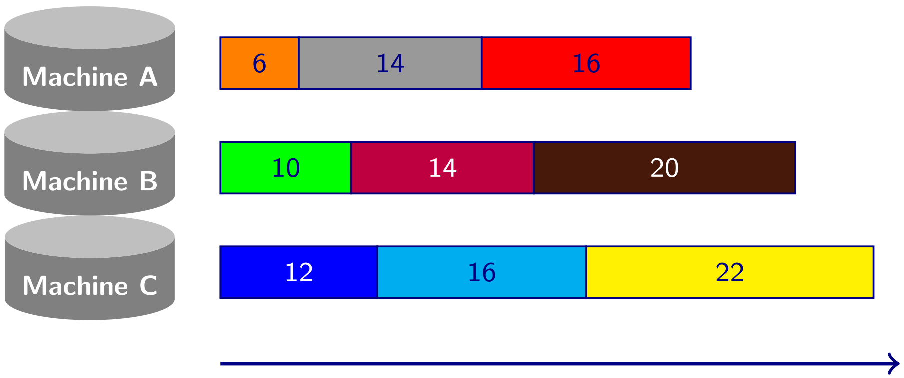

$$\sum_{j=1}^{n} C_j = 6 + 20 + 36 = 10 + 24 + 44 = 12 + 28 + 50 = 230$$

➡ Optimal: schedule by non-decreasing processing times, on earliest available machine

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    - ➡ Makespan minimization for job shop scheduling can also be solved using networks

- Models, Data and Algorithms
- Linear Optimization
- Mathematical Background: Polyhedra, Simplex-Algorithm
- Sensitivity Analysis; (Mixed) Integer Programming
- MIP Modelling
- MIP Modelling: More Examples; Branch & Bound
- Cutting Planes; Combinatorial Optimization: Examples, Graphs, Algorithms
- TSP-Heuristics
- Network Flows
- Shortest Path Problem
- Complexity Theory
- Nonlinear Optimization
- Scheduling
- Lot Sizing & Intro to Multiobjective Optimization (Feb 01)
- Summary (Feb 08)
- Oral exam (Feb 15)