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Multivariate Polynomials

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Exercise session 1

Throughout this exercise $\mathbb{K}$ denotes a field.

Exercise 1 (Expressions to polynomials). As in the lecture we represent expressions with trees. Determine $E(\cdot), D(\cdot)$ and $\text{val}(\cdot)$ for the trees below.

(1)

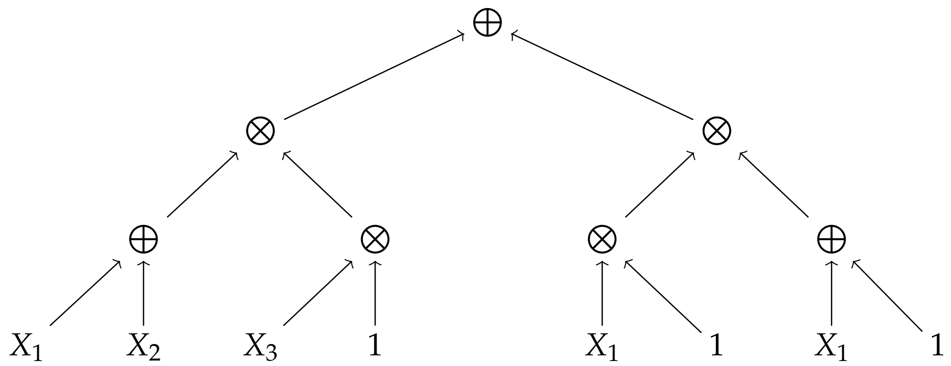

Figure 1: Tree representation for φ_1.

(2)

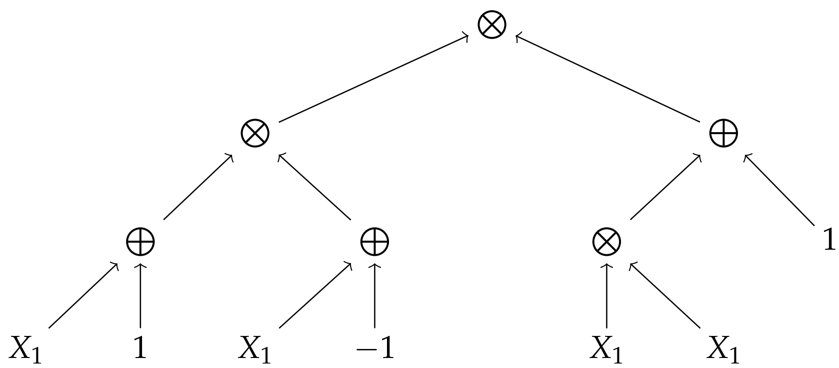

Figure 2: Tree representation for φ_2.

Can you give an expression φ_3, such that $\text{val}(\varphi_2) = \text{val}(\varphi_3)$ and $E(\varphi_3) < E(\varphi_2)$?

Exercise 2 (Polynomials to expressions). Let $n \geq 1$. Find expressions φ_1, φ_2 and φ_3, such that

(1) $\operatorname{var}(\varphi_1) = \prod_{i=1}^n X_i$.

(2) $\operatorname{var}(\varphi_2) = \sum_{i=1}^n X_i$.

(3) $\operatorname{var}(\varphi_3) = X_1^2 - X_2^2$.

As in exercise 1 you may represent the expression with trees.

Exercise 3. Let $n \geq 1$ and $f \in \mathbb{K}[X_1, \dots, X_n]$. Show that

(1) $E(f) \geq \deg f - 1$.

(2) $D(f) \geq \log_2(\deg f)$.

Exercise 4. Let $n \geq 1$ and $f = \prod\limits_{i=1}^{n} X_i \in \mathbb{K}[X_1, \dots, X_n]$. Show that

(1) $E(f) = n - 1$.

(2) $D(f) = \lceil \log_2(n) \rceil$.

Hint: Use exercise 3.

Exercise 5. Let $f \in \mathbb{K}[X]$. Use Horner's method to show that $E(f) \leq 2\deg(f)$.

Exercise 6. Let $n, d \geq 1$. Show that

$$(X_1 + \ldots + X_n)^d = \sum_{1 \leq i_1, \ldots, i_d \leq n} X_{i_1} \cdot \ldots \cdot X_{i_d} = \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| = d}} \binom{d}{\alpha} \cdot X^\alpha,$$

where $\binom{d}{\alpha} := \frac{d!}{\alpha_1! \cdot \ldots \cdot \alpha_n!}$, $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $X^\alpha := X_1^{\alpha_1} \cdot \ldots \cdot X_n^{\alpha_n}$.

Exercise 7 (Waring's problem for polynomials). Let $n, d \geq 1$. Show that there exists $N = N(n, d)$, such that for any $f \in \mathbb{C}[X_1, \dots, X_n]_d$ there exist linear forms $l_1, \dots, l_N$ with

$$f = \sum_{i=1}^{N} l_i^d.$$

Hint: A possible solution involves exercise 6.

Exercise 8. Let $m \geq 1$, $n \geq 3$. Recall the definition of the *sum-product polynomial*

$$S_m^n := \sum_{i=1}^{m} \prod_{j=1}^{n} x_{ij} \in \mathbb{K}[X_{11}, \dots, X_{mn}]_n.$$

Show that S_m^n is characterized by symmetry.