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Multivariate Polynomials

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Exercise session 3

Exercise 1. Let $\mathbb{K}$ be a field and $f \in \mathbb{K}[X]$ be a univariate polynomial over $\mathbb{K}$ of degree d. Show that $L_{\text{ns}}(f) \leq 2\sqrt{d}$.

Exercise 2. Let $f_1, \dots, f_s \in \mathbb{K}[X_1, \dots, X_n]$. Show that

$$L(f_1, \dots, f_s) \geq \dim_{\mathbb{K}} \operatorname{span}_{\mathbb{K}} \{1, X_1, \dots, X_n, f_1, \dots, f_s\} - (n+1).$$

Exercise 3. For $e \in \{0,1\}^n$ we denote by p_e the $\left(\sum_{i=1}^n e_i 2^{i-1}\right)$-th prime number. Consider the multivariate polynomial

$$f_n := \sum_{e \in \{0,1\}^n} \sqrt{p_e} \cdot X_1^{e_1} \cdot \ldots \cdot X_n^{e_n} \in \mathbb{C}[X_1, \dots, X_n].$$

Prove that $L(f_n) \geq \Omega\left(\sqrt{2^n/n}\right)$.
Hint: Imitate the argument of the lecture.

Exercise 4. Suppose that $\mathbb{K} = \mathbb{C}$. Show that there exists $c > 0$ such that for all $d \geq 1$ there exist $e_1, \dots, e_d \in \{0,1\}$ such that $L\left(\sum_{i=1}^d e_i X^i\right) \geq c\sqrt{d/\log d}$.
Hint: Use that if $H \in \mathbb{Q}[Y_1, \dots, Y_d] \setminus \{0\}$ is multilinear, then there exists $y \in \{0,1\}^d$ with $H(y) \neq 0$.

Exercise 5. Let $n \geq 4$ and $f = \prod_{i=1}^n X_i$. Show that $L\left(\left\{ \frac{\partial^2 f}{\partial X_i \partial X_j} \,\middle|\, 1 \leq i,j \leq n \right\}\right) \geq \binom{n}{2}$. Conclude that

$$\text{"}\forall f \in \mathbb{K}[X_1, \dots, X_n]: \quad L\left(\left\{ \frac{\partial^2 f}{\partial X_i \partial X_j} \,\middle|\, 1 \leq i,j \leq n \right\}\right) = \mathcal{O}(L(f))\text{"}.$$

is not true.

Exercise 6. Show that the assumption

$$\forall f \in \mathbb{K}[X_1, \dots, X_n]: \quad L\left(\left\{ \frac{\partial^2 f}{\partial X_i \partial X_j} \,\middle|\, 1 \leq i,j \leq n \right\}\right) = \mathcal{O}(L(f) + n^2) \quad (\dagger)$$

implies that

$$\operatorname{MatMu}_n := L\left(\left\{ \sum_{k=1}^n a_{ik} \cdot b_{kj} \,\middle|\, 1 \leq i,j \leq n \right\}\right) = \mathcal{O}(n^2).$$

Remark: MatMu_n is the complexity of matrix multiplication. It is unknown whether $(\dagger)$ is true or not. The best known upper bound for MatMu_n as of today is $\mathcal{O}(n^{2.3728639})$, which was proven by Le Gall in 2014.