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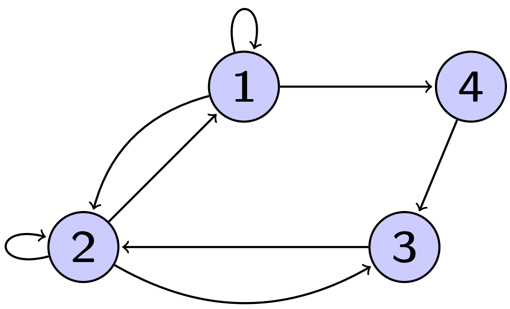
Multivariate Polynomials

Sommersemester 2015

Exercise session 4

Exercise 1. Consider the following weighted, directed graph G with vertices $\{1,2,3,4\}$. All weights of the edges displayed are equal to 1.



(1) Find all cycles in G. Does G have a cycle cover?

(2) Find all closed walk sequences in G.

Consider the closed walk sequence $\mathcal{W} = ((12),(23))$. Find its partner $\mathcal{W}' := I(\mathcal{W})$ according to the lecture and verify that $w(\mathcal{W}) = w(\mathcal{W}')$ and $\operatorname{sgn}(\mathcal{W}) = -\operatorname{sgn}(\mathcal{W})$.

Exercise 2. Let $G = (V,E)$ be an undirected bipartite graph with $A \in \{0,1\}^{\frac{|V|}{2} \times \frac{|V|}{2}}$ being its adjacency matrix. Show that $\operatorname{per}(A)$ equals the number of perfect matchings in G .

Exercise 3. Let $G = (V,E)$ be a directed, simple graph with $A \in \{0,1\}^{|V| \times |V|}$ being its adjacency matrix. Show that $\operatorname{per}(A)$ equals the number of cycle covers of G.

Exercise 4. Let R be some commutative ring, $n,m \geq 1$, $A \in R^{n\times n}$ and $v \in R^n$. Show that there exists an arithmetic circuit Φ computing

$$v,\ Av,\ A^2v,\ \ldots,\ A^{m-1}v$$

with size $\mathcal{O}(n^3 \log m)$ and depth $\mathcal{O}(\log(m)\log(n))$.
Hint: Without restriction let $m = 2^l$. Compute A^{2^i}, $i = 1,\ldots,l-1$. Then consider the matrices $M_k := \begin{bmatrix} v, & Av, & A^2v, & \ldots & A^{2^k-1}v \end{bmatrix}$, $1 \leq k \leq l$.

Exercise 5. Let R be a commutative ring and $n \geq 1$. Use the ideas from the lecture to construct an arithmetic circuit Φ with size $n^{\mathcal{O}(1)}$ and depth $\mathcal{O}(\log^2 n)$, that computes the characteristic polynomial $\det(XI_n - A)$ of a matrix $A \in R^{n\times n}$.