Exercise sheet #1

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due on April 21 at the beginning of the lecture

Each of the five questions is worth 10 points.

Let $\mathbb{K}$ be a field of characteristic different from 2. Let $C(\mathbb{K}) \stackrel{\text{def}}{=} \left\{(x,y) \in \mathbb{K}^2 \;\middle|\; x^2+y^2=1\right\}$. For t in $\mathbb{K}$, let L_t be the affine line $\{(u-1,tu) \mid u \in \mathbb{K}\}$.

1. a) Show that L_t and $C(\mathbb{K})$ meet in at most two points and that if $t^2 \neq -1$ then L_t meets $C(\mathbb{K})$ in exactly two points.

   b) For $t \in \mathbb{K}$ such that $t^2 \neq -1$, let $\varphi(t)$ be the unique point in $L_t \cap C(\mathbb{K})$ which is not $(-1,0)$. Assuming that $t^2 \neq -1$ for all $t \in \mathbb{K}$, show that φ defines a bijection $\mathbb{K} \to C(\mathbb{K}) \setminus \{(-1,0)\}$.

2. *(Generating all Pythagorean triples)* Prove that the following map is well defined and induces a bijection:

$$\left\{(k,u,v) \in \mathbb{Z}^3_> \;\middle|\; \gcd(u,v)=1,\ u>v,\ u+v \text{ is odd}\right\} \to \left\{(p,q,r) \in \mathbb{Z}^3_> \;\middle|\; p^2+q^2=r^2\right\}$$
$$(k,u,v) \mapsto \left(k(u^2-v^2), 2kuv, k(u^2+v^2)\right).$$

3. a) Using the coordinates $(u,v)=(x+iy,x-iy)$, show that $C(\mathbb{C})$ is homeomorphic to $\mathbb{C}^\times$.

   b) Show that $C(\mathbb{R})$ is not homeomorphic to any subset of $\mathbb{R}$.

For $n \in \mathbb{N}$, let $\mathbb{P}^n(\mathbb{K})$ denote the set of $\mathbb{K}$-linear subspaces of dimension 1 of $\mathbb{K}^{n+1}$. This is the *projective space.* For $\mathbf{x}=(x_0,\dots,x_n) \in \mathbb{K}^{n+1}$ a non-zero vector, let $[x_0 : \cdots : x_n]$ denotes $\mathbb{K}\mathbf{x} \in \mathbb{P}^n(\mathbb{K})$, the line spanned by $\mathbf{x}$. Note that $[x_0 : \cdots : x_n] = [\lambda x_0 : \cdots : \lambda x_n]$ for all $\lambda \neq 0$. This defines the *homogeneous coordinates* on $\mathbb{P}^n(\mathbb{K})$. There are two basic inclusions:

$$i_n : (x_1,\dots,x_n) \in \mathbb{K}^n \mapsto [1 : x_1 : \cdots : x_n] \in \mathbb{P}^n$$

$$\text{and } j_n : [x_0 : \cdots : x_{n-1}] \in \mathbb{P}^{n-1}(\mathbb{K}) \mapsto [0 : x_0 : \cdots : x_{n-1}] \in \mathbb{P}^n(\mathbb{K}).$$

4. a) Show that for all $n \geq 1$, $\mathbb{P}^n(\mathbb{K})$ is the disjoint union of $i_n(\mathbb{K}^n)$ and $j_n(\mathbb{P}^{n-1}(\mathbb{K}))$.

   b) Let $\overline{C}(\mathbb{K}) \stackrel{\text{def}}{=} \left\{[x:y:z] \in \mathbb{P}^2(\mathbb{K}) \;\middle|\; x^2+y^2=z^2\right\}$. Why is this definition well posed?

   c) Show that the map $[u:v] \in \mathbb{P}^1(\mathbb{K}) \mapsto [u^2-v^2 : 2uv : u^2+v^2] \in \mathbb{P}^2(\mathbb{K})$ induces a bijection between $\mathbb{P}^1(\mathbb{K})$ and $\overline{C}(\mathbb{K})$

5. *(Veronese embedding)* Let $n, d \in \mathbb{Z}_>$ and let $S_{n,d}$ be the set of all monomials of degree d in the variables $x_0, \dots, x_n$.
   a) Show that $S_{n,d}$ has $\binom{n+d}{d}$ elements.
   b) Let $m = \binom{n+d}{d} - 1$ and let $s_0, \dots, s_m$ be an enumeration of the elements of $S_{n,d}$. Show that the map $v_{n,d} : [x_0 : \cdots : x_n] \in \mathbb{P}^n(\mathbb{K}) \mapsto [s_0 : \cdots : s_m] \in \mathbb{P}^m(\mathbb{K})$ is well defined and injective. (The notation identifies variables and values of the variables, don't be picky.)
   c) How do $\overline{C}(\mathbb{C})$ and $v_{1,2}(\mathbb{P}^1(\mathbb{C}))$ compare?