Exercise sheet #10

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due on June 30 at the beginning of the lecture

Let K be an algebraically closed field.

Exercise A (10 points)**.** Let $Z = V(X^3 - Y^2) \in K^2$. Show that the map $f : t \in \mathbb{C} \mapsto (t^2, t^3) \in Z$ is finite.

Exercise B (20 points)**.**

1. Let $f \in K[X_1, ..., X_n]$ be a non constant polynomial. Show that the inclusion map $\mathbb{A}^n \setminus V(f) \to \mathbb{A}^n$ is not a finite map.

Let X and Y be irreducible affine varieties and $\varphi : X \to Y$ a dominant regular map. It induces an injective ring morphism $\varphi^* : K[Y] \to K[X]$ and so a field morphism $\varphi^* : K(Y) \to K(X)$. The *degree* of φ is the dimension of $K(X)$ as a $\varphi^* K(Y)$-vector space.

2. Let $f \in K[Y]$ and $U = Y \setminus V(f)$. Show that $K[\varphi^{-1}(U)] = K[X][1/\varphi^*(f)]$.

3. Assume that φ has finite degree. Show that there exists a non empty open subset $U \subseteq Y$ such that the map $\varphi^{-1}(U) \to U$ induced by φ is finite.

Exercise C (20 points)**.** Let X be an affine variety and let $\varphi : X \to \mathbb{A}^n$ be a finite map. Let D be the degree of φ, which is finite since φ is a finite map. Let $A = K[\mathbb{A}^n]$ and $B = K[X]$.

1. Let $f \in K[X]$. Show that there exists a monic polynomial $p \in A[T]$ of degree at most D such that $p(f) = 0$. *(Hint: A is a unique factorization domain and you can use Gauß' lemma.)*

2. Let $S \subseteq X$ be a finite subset of X. Show that there exists a function $a \in K[X]$ such that $\#a(S) = \#S$.

3. Let $y \in Y$. Show that $\varphi^{-1}(y)$ contains at most D points.