Exercise sheet #11

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due on July 7 at the beginning of the lecture

Let K be an algebraically closed field.

Exercise A (10 points). Show that an irreducible closed subset of $\mathbb{P}^n$ of codimension 1 is the zero locus of a homogeneous polynomial.

Exercise B (10 points). Let X be an irreducible quasi-projective variety. Show that the dimension of X is the maximum integer n such that there exists a chain $\varnothing \neq Y_0 \subset Y_1 \subset \cdots \subset Y_n = X$ (strict inclusions) of irreducible closed sets.

Exercise C (10 points). Let $X \subseteq \mathbb{P}^N$ be a projective variety. Show that $N - \dim X - 1$ is the maximal dimension of a linear projective subspace $L \subseteq \mathbb{P}^N$ such that $L \cap X = \varnothing$.

Exercise D (20 points). Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be irreducible affine varieties. We know that the product $X \times Y$ is also irreducible. Recall that the canonical projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ induce injections $\pi_1^* : K(X) \to K(X \times Y)$ and $\pi_2^* : K(Y) \to K(X \times Y)$.

1. Show that the field $K(X \times Y)$ is generated by $\pi_1^* K(X)$ and $\pi_2^* K(Y)$.
2. Show that $\dim(X \times Y) = \dim X + \dim Y$.
3. Show that if X and Y are quasi-projective varieties then $\dim(X \times Y) = \dim X + \dim Y$.