Exercise sheet #2

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due on April 28 at the beginning of the lecture

Let k be a field.

Exercise 1 (Noether normalization lemma in practice, 10 points)**.** Recall that the Noether normalization lemma states that if A is a finitely generated commutative k-algebra, then there exist $d \in \mathbb{N}$ and $x_1, \ldots, x_d \in A$ algebraically independent elements over k such that A is an integral extension *(ganze Ringerweiterung)* of $k[x_1, \ldots, x_d]$. Find explicitly such elements for the following algebras:

1. $k[x, y, z]/(xy)$
2. $k[x, x^{-1}]$

Exercise 2 (Finite extensions of fields are fields, 10 points)**.** Let B be a domain *(Integritätsring)* which is an algebraic extension of a field A. Show that B is also a field.

Exercise 3 (Zariski's topology in a degenerate situation, 10 points)**.** Assume that k is a finite field. Show that any subset of k^n is closed for Zariski's topology.

Exercise 4 (Elimination and implicitization, 20 points)**.** Let $A = k[t, x, y]$ and $B = k[x, y]$. We consider the lexicographic order on the monomials of A, with $t > x > y$. That is, we say that $t^n x^i y^j < t^{n'} x^{i'} y^{j'}$ if $n < n'$ or ($n = n'$ and $i < i'$) or ($n = n'$ and $i = i'$ and $j < j'$). Recall that $\mathrm{lm}(f)$ denotes the leading monomial of a non zero element f of A. For example $\mathrm{lm}(2t^2 + 4ty^5 - x^6) = t^2$. Recall also that a Gröbner basis of an ideal $I \subseteq A$ (resp. $I \subseteq B$) is a finite subset G of I such that $\langle \mathrm{lm}(I) \rangle = \langle \mathrm{lm}(G) \rangle$.

1. Let $f \in A$. Show that $\mathrm{lm}(f) < t$ if and only if $f \in B$.
2. Let $I \subseteq A$ be an ideal. Show that if G is a Gröbner basis of I for the lexicographic monomial order, then $G \cap B$ is a Gröbner basis of the ideal $I \cap B$.
3. Let $a(t)$ and $b(t)$ be two rational functions in $k(t)$.
   a) Show that there exists a non-zero polynomial $f \in k[x, y]$ such that $f(a(t), b(t)) = 0$.
   b) Assuming an algorithm to compute Gröbner bases, devise an algorithm to compute such a polynomial f, given $a(t)$ and $b(t)$.