Exercise sheet #3

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due on May 5 at the beginning of the lecture

Exercise A (Zariski closure, 10 points)**.**

1. Let $M \subseteq \mathbb{C}^n$ and let $\overline{M}$ be the closure of M in the Zariski topology (also called the *Zariski closure*). Show that $V(I(M)) = \overline{M}$.
2. In $\mathbb{C}^2$, describe by a finite number of polynomial equations the Zariski closure of the following spaces: (a) $\mathbb{C}^2 \setminus \{(0,0)\}$; (b) $\mathbb{Z}^2$; (c) $\left\{(n,n^2) \in \mathbb{C}^2 \;\middle|\; n \in \mathbb{N}\right\}$; (d) $\left\{(z,e^z) \in \mathbb{C}^2 \;\middle|\; z \in \mathbb{C}\right\}$.

Exercise B (Density and irreducibility, 20 points)**.** Let $X \subseteq \mathbb{C}^n$ be a non-empty algebraic subset. Zariski's topology on X is the induced topology by Zariski's topology on $\mathbb{C}^n$. That is to say, the open subsets of X are the $U \cap X$, where U is a Zariski open subset of $\mathbb{C}^n$.

1. Show that X is irreducible if and only if every non-empty Zariski open subset of X is Zariski dense.
2. Using a density argument in the Zariski topology, show that $\chi_{AB} = \chi_{BA}$ for all matrices $A, B \in \mathbb{C}^{n\times n}$, where χ_M denotes the characteristic polynomial of M. *(Hint: Fix A, observe that the property is easy when B is invertible, conclude.)*

Exercise C (Solving equations via eigenvalues and eigenvectors, 20 points)**.** Let I an ideal of $A = \mathbb{C}[x_1,\dots,x_n]$. Let B be the quotient ring A/I. We assume that B is a finite dimensional $\mathbb{C}$-algebra. We aim at giving an algorithm to compute the elements of $V(I) \in \mathbb{C}^n$, assuming that we know how to compute eigenvectors and eigenvalues of matrices. For $f \in A$, the equivalence class of f modulo I is denoted $[f]$, this is an element of B. Let also m_f be the $\mathbb{C}$-linear map $[g] \in B \mapsto [fg] \in B$.

1. Show that $f \in A \mapsto m_f \in \mathrm{End}_{\mathbb{C}}(B)$ is an ring homomorphism whose kernel is I.

Let $f \in A$.

2. a) Let χ be the characteristic polynomial of m_f. Show that $\chi(f) \in I$.

b) Show that if $p \in V(I)$, then $f(p)$ is an eigenvalue of m_f.

3. a) Let $g \in A$. Show that if g does not vanish on any point of $V(I)$ then $[g]$ has a multiplicative inverse in B. *(Hint: Nullstellensatz)*

   b) Show that if $\lambda \in \mathbb{C}$ is an eigenvalue of m_f then there exists $p \in V(I)$ such that $\lambda = f(p)$.

4. Suggest a method to compute all the elements of $V(I)$.