Exercise sheet #4

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due on **May 19** at the beginning of the lecture

Exercise A (10 points).

1. Show that the automorphisms of the affine line $\mathbb{C}$ are all of the form $z \mapsto az + b$, with $a, b \in \mathbb{C}$, $a \neq 0$.
2. Let H be the set of all maps $\mathbb{C}^2 \to \mathbb{C}^2$ of the form $(x, y) \mapsto (ax, by + P(x))$, where a and b are non-zero complex numbers and P a univariate polynomial. Show that H is a subgroup of the automorphism group of $\mathbb{C}^2$.

Exercise B (10 points). Let $\varphi : A \to B$ a morphism between commutative rings.

1. Let $\mathfrak{p}$ be a prime ideal of B. Show that $\varphi^{-1}(\mathfrak{p})$ is a prime ideal of A.
2. Find an example of a map φ where $\varphi^{-1}(\mathfrak{p})$ is not a maximal ideal of A for some maximal ideal $\mathfrak{p}$ of B.
3. Let $\mathfrak{p}$ be a prime ideal of A. Show that if φ is surjective and $\ker \varphi \subseteq \mathfrak{p}$ then $\varphi(\mathfrak{p})$ is a prime ideal of B.
4. Find an example of a surjective map φ where $\varphi(\mathfrak{p})$ is not a prime ideal of B for some prime ideal $\mathfrak{p}$ of A.

Exercise C (10 points). Let $Z \subset \mathbb{C}^n$ be an algebraic set. Show that Z is *quasi-compact* for Zariski's topology, this means that if $(U_i)_{i \in I}$ is a family of Zariski open subsets of Z such that $Z = \cup_{i \in I} U_i$, then there exists a finite subset $J \subseteq I$ such that $Z = \cup_{i \in J} U_i$.

Exercise D (10 points). Let A be a domain *(Integritätsring)* and $I \subset A$ be a prime ideal. Let K be the field of fractions of A. Let A_I be the subring of K generated by A and all $1/p$ for $p \in A \setminus I$. Let IA_I be the ideal of A_I generated by I. Show that the field of fractions of A/I is isomorphic to A_I / IA_I.

Exercise E (10 points). In $\mathbb{C}^3$, what are the irreducible components of $V(x^2 + y^2 + z^2, x^2 - y^2 - z^2 - 1)$?

Exercise F (10 points). Let $f : X \to Y$ be a regular map between algebraic sets. Let $\Gamma_f = \{(x, f(x)) \in X \times Y \mid x \in X\}$. Show that Γ_f is a Zariski closed in $X \times Y$ and isomorphic to X.

The following exercises need content from the lectures coming next week.

Exercise G (10 points). Show that the algebraic set $V(xy - 1) \subset \mathbb{C}^2$ is birationally equivalent to $\mathbb{C}$ but not isomorphic to $\mathbb{C}$.

Exercise H (10 points). Let $Z = V(x^2 + y^2 - 1) \subset \mathbb{C}^2$. Determine the domain of definition of the rational map $Z \to \mathbb{C}$, $(x, y) \mapsto (1 - y)/x$.

Exercise I (20 points). Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be an irreducible polynomial of degree 2, with $n \geq 2$. Show that $V(f)$ is birationally equivalent to $\mathbb{C}^{n-1}$. *(Hint: remember the exercise sheet #1, question 1.)*