Exercise sheet #5

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due on May 27 at the beginning of the lecture

Exercise A (10 points). Let X be a topological space and let Y be a subset of X.

1. Let F be a subset of Y. Let $\overline{F}$ denote the closure of F in X and $\overline{F}^Y$ denote the closure of F in space Y. Show that $\overline{F}^Y = \overline{F} \cap Y$.
2. Assume that Y is dense. Show that X is irreducible if and only if Y is irreducible in the induced topology.

Exercise B (10 points). Let U and V be quasi-affine sets over $\mathbb{C}$ and let $\alpha : \mathbb{C}[V] \to \mathbb{C}[U]$ be a morphism of $\mathbb{C}$-algebras.
Give an example where α is not the comorphism φ^* of a regular map $\varphi : U \to V$.

Exercise C (10 points). Let A be the subalgebra of $\mathbb{C}[t]$ of all polynomials $f(t)$ such that $f(1) = f(-1)$.

1. Show that A is generated by $t^2 - 1$ and $t^3 - t$.

Let X be an algebraic set such that $\mathbb{C}[X] \simeq A$.

2. Show that X is birationally equivalent to $\mathbb{A}^1_{\mathbb{C}}$.
3. Show that X is isomorphic to $V(Y^2 - X^3 - X^2) \subset \mathbb{C}^2$.

Exercise D (Ring of invariants and quotient variety, 20 points). Let A be a finitely generated $\mathbb{C}$-algebra and let G be a finite subgroup of the automorphism group of A.
Let A^G the the subalgebra of all $a \in A$ such that $g(a) = a$ for all $g \in G$. We aim at proving that A^G is finitely generated. For $a \in A$ and $1 \leq i \leq |G|$, let $\tau_i(a)$ be the coefficient of $T^{|G|-i}$ in the polynomial $\prod_{g \in G}(T - g(a))$.

1. a) Show that $\tau_i(a) \in A^G$ for all $a \in A$ and all $1 \leq i \leq |G|$.
   b) Show that if $a \in A^G$, then $\tau_1(a) = -|G|a$.
   c) Show that $\tau_1 : A \to A^G$ is an A^G-linear map.

Let $u_1, \dots, u_r$ be a set of generators of the algebra A. Let B be the subalgebra of A^G generated by the $\tau_i(u_j)$, for $1 \leq i \leq |G|$ and $1 \leq j \leq r$.

2. a) Show that A is finitely generated B-module.
   b) Show that A^G is a finite generated B-module. *(Hint: use questions 1b and 1c.)*
   c) Conclude that A^G is finitely generated $\mathbb{C}$-algebra.

3. Let X be an algebraic set. Let G be a finite subgroup of the automorphism group of X. Show that there exists an algebraic set Y and a regular map $\pi : X \to Y$ such that the following universal property holds:
   (i) $\pi \circ g = \pi$ for all $g \in G$;
   (ii) for all algebraic set Z and all regular map $f : X \to Z$ such that $f \circ g = f$ for all $g \in G$, there exists a unique map $h : Y \to Z$ such that $f = h \circ \pi$.

   Moreover, show that Y is unique up to isomorphism.