Exercise sheet #6

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due on June 2 at the beginning of the lecture

Exercise A (10 points). Let k be a field and $I \subseteq k[x_1, \ldots, x_n]$ be a homogeneous ideal. Show that the radical of I is a homogeneous ideal.

Exercise B (10 points). Let k be a field and I be an homogeneous ideal of $A = k[X_1, \ldots, X_n]$. Assume that for all homogeneous polynomials $a, b \in A$, if $ab \in I$ then $a \in I$ or $b \in I$. Show that I is a prime ideal.

Exercise C (Veronese embedding, 30 points). Let n and d be positive integers and $N = \binom{n+d}{d} - 1$. Let $\mathcal{S}$ be the set of all $A = (a_0, \ldots, a_n) \in \mathbb{Z}_{\geq}^{n+1}$ such that $a_0 + \cdots + a_n = d$. For $A = (a_0, \ldots, a_n) \in \mathcal{S}$ we write $x^A = x_0^{a_0} \cdots x_n^{a_n}$. Let $A_0, \ldots, A_N$ be an enumeration of $\mathcal{S}$. Recall that the *Veronese embedding* is the following map between projective spaces:

$$\varphi : [x_0 : \cdots : x_n] \in \mathbb{P}^n \mapsto [x^{A_0} : \cdots : x^{A_N}] \in \mathbb{P}^N.$$

1. Check that φ is a regular map between projective varieties.
2. Show that the image of φ is not included in any hyperplane of $\mathbb{P}^N$. *(Recall that a hyperplane of $\mathbb{P}^N$ is the zero locus of a homogeneous polynomial of degree 1.)*

Let $y_0, \ldots, y_N$ denote the projective coordinates on $\mathbb{P}^N$. For $B \in \mathcal{S}$, let Y_B denote y_i, where $i \in \{0, \ldots, N\}$ is such that $B = A_i$. Let $Z \subseteq \mathbb{P}^N$ be the closed set

$$Z = V(\{Y_A Y_B - Y_C Y_D \mid A, B, C, D \in \mathcal{S} \text{ and } A + B = C + D\}).$$

3. Show that $\varphi(\mathbb{P}^n) \subseteq Z$.
4. For $0 \leq i, j \leq n$, let $C_{i,j} \in \mathcal{S}$ be such that $x^{C_{i,j}} = x_i^{d-1} x_j$. Let U_i denote the open affine subset $\left\{ Y_{C_{i,i}} \neq 0 \right\}$ of $\mathbb{P}^N$. Show that $Z \subset U_0 \cup U_1 \cup \cdots \cup U_n$.
5. The integer $0 \leq i \leq n$ being fixed, show that $[Y_{C_{i,0}} : Y_{C_{i,1}} : \ldots : Y_{C_{i,n}}]$ defines a regular map $\psi_i : U_i \cap Z \to \mathbb{P}^n$ such that $\varphi(\psi_i(y)) = y$ for all $y \in U_i \cap Z$.
6. Show that $\psi_i|_{U_i \cap U_j \cap Z} = \psi_j|_{U_i \cap U_j \cap Z}$ for all $0 \leq i, j \leq n$.

7. Conclude that φ induces an isomorphism $\mathbb{P}^n \simeq Z$.

8. Let $F \in \mathbb{C}[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree d defining an hypersurface $V(F) \subset \mathbb{P}^n$. Show that there exists a hyperplane $H \subset \mathbb{P}^N$ such that $\varphi(V(F)) = H \cap Z$.