Exercise sheet #8

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due on June 16 at the beginning of the lecture

Exercise A (20 points)**.** Let $n, m \geq 0$ be integers. The notation $X_{p\smile q}$ means $X_p, \dots, X_q$. An element $f \in \mathbb{C}[X_{0\smile n}, Y_{1\smile m}]$ is *partly homogeneous* if there exists a $p \geq 0$ such that for $f(\lambda X_0, \dots, \lambda X_n, Y_{1\smile m}) = \lambda^p f$ for all $\lambda \in \mathbb{C}$. If S is a set of partly homogeneous polynomials, then $V_{\mathbb{P}}(S)$ denotes $\{([x], y) \in \mathbb{P}^n \times \mathbb{A}^m \mid \forall F \in S,\ F(x, y) = 0\}$.

1. Show that $Z \subseteq \mathbb{P}^n \times \mathbb{A}^m$ is closed if and only if there exists partly homogeneous polynomials $F_1, \dots, F_r$ such that $Z = V_{\mathbb{P}}(F_1, \dots, F_r)$.

For $f \in \mathbb{C}[X_{1\smile n}, Y_{1\smile m}]$, let f^h denote the *partial homogenization* of f, defined by

$$f^h = X_0^d f\left(\frac{X_1}{X_0}, \dots, \frac{X_n}{X_0}, Y_{1\smile m}\right), \quad \text{where } d = \deg_{X_{1\smile n}} f.$$

For $F \in \mathbb{C}[X_{0\smile n}, Y_{1\smile m}]$, the *dehomogenization* of F, denoted F^a is defined by $F^a = F(1, X_{1\smile n}, Y_{1\smile m})$.

2. For $F \in \mathbb{C}[X_{0\smile n}, Y_{1\smile m}]$ partly homogeneous, show that there exists a d such that $F = X_0^d \cdot (F^a)^h$.

Let $I \subseteq \mathbb{C}[X_{1\smile n}, Y_{1\smile m}]$ be a radical ideal and $Z = V(I) \subset \mathbb{A}^n \times \mathbb{A}^m$. Let $\overline{Z}$ be the closure of Z in $\mathbb{P}^n \times \mathbb{A}^m$. Let I^h be the set of all f^h, with $f \in I$.

3. Show that $Z \subset V_{\mathbb{P}}(I^h)$.
4. Show that $\overline{Z} = V_{\mathbb{P}}(I^h)$. *(Hint: consider partly homogeneous polynomials $F_1, \dots, F_r$ such that $\overline{Z} = V_{\mathbb{P}}(F_1, \dots, F_r)$.)*

Exercise B (Composition law on a cubic curve, 30 points)**.**

1. Show that the set of all $[a : b : c : d] \in \mathbb{P}^3$ such that the homogeneous polynomial $aX^3 + bX^2Y + cXY^2 + dY^3 \in \mathbb{C}[X, Y]$ has three distinct zeros in $\mathbb{P}^1$ is open in $\mathbb{P}^3$. *(Hint: consider the discriminant.)*

Let $f \in \mathbb{C}[X_0, X_1, X_2]$ be a homogeneous polynomial of degree 3. Let $Z = V(f) \subset \mathbb{P}^2$. For a and $b \in \mathbb{P}^2$ such that $a \neq b$, let $L_{a,b}$ be the unique projective line passing through a and b. Let $U = \{(a, b) \in Z \times Z \mid a \neq b \text{ and } \#(L_{a,b} \cap Z) = 3\}$.

2. Show that U is open in $Z \times Z$. *(Hint: consider affine charts.)*

3. For $(a,b) \in U$, let $f(a,b)$ be the point of X such that $L_{a,b} \cap Z = \{a, b, f(a,b)\}$. Show that $f : U \to Z$ defines a regular map.