Exercise sheet #9

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due on June 23 at the beginning of the lecture

Exercise A (20 points). Let $\varphi : (x,y) \in \mathbb{A}^2 \mapsto (x, xy^2, x^2y) \in \mathbb{A}^3$. Let (U,V,W) denote the coordinates on $\mathbb{A}^3$.

1. Show that $\overline{\varphi(\mathbb{A}^2)}$ is the zero set of the polynomial $U^3V - W^2$.
2. Describe $\varphi(\mathbb{A}^2)$ precisely (with equations and inequations).

Exercise B (Extension theorem, 30 points). Let $X, Y_1, \dots, Y_n$ denote the coordinates on $\mathbb{A}^1 \times \mathbb{A}^n = \mathbb{A}^{n+1}$. Let π be the projection $(x, y_1, \dots, y_n) \mapsto (y_1, \dots, y_n)$. Let $Z \subseteq \mathbb{A}^{n+1}$ be a closed subset. This exercise aim at giving a sufficient condition for a point of $\overline{\pi(Z)}$ to be in $\pi(Z)$.

Let $A = K[Y_1, \dots, Y_n]$ and $B = A[X]$, where K is an algebraically closed field. Let $I \subseteq B$ an ideal such that $Z = V(I) \subset \mathbb{A}^{n+1}$. Let $J = I \cap A$. It is an ideal of A called *the first elemination ideal of I*.

1. Show that $\overline{\pi(Z)} = V(J)$.

Let $\overline{Z}$ be the closure of Z in $\mathbb{P}^1 \times \mathbb{A}^n$.

2. Show that $\overline{\pi(Z)} = \pi(\overline{Z})$.

For $f = \sum_{k=0}^d a_k X^k$, with $a_d \in A \setminus \{0\}$, the leading coefficient of f, denoted $\operatorname{lc}(f)$, is defined to be a_d. The homogenization of f, denoted $\overline{f}$, is defined to be $\sum_{k=0}^d a_k X_0^{d-k} X_1^k \in A[X_0, X_1]$.

Let S be the ideal of A generated by the $\operatorname{lc}(f)$, for $f \in I \setminus \{0\}$. Finally, let $\overline{I}$ be the ideal of $A[X_0, X_1]$ generated by the $\overline{f}$, for $f \in I$.

3. Show that $\overline{Z} \setminus Z = \{\infty\} \times V(S)$. *(Hint: remember that $\overline{Z} = V(\overline{I})$.)*
4. Show that $\overline{\pi(Z)} \setminus V(S) \subseteq \pi(Z)$.