Exercise sheet #10

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Let A be a commutative ring.

Exercise 1. Recall that a free A-module is a module over A which is isomophic to A^I for some index set I (not necessarily finite).
Let P be an A-module. Show that the following properties are equivalent:

(i) For any A-modules M and M', any morphism $f : P \to M$ and any surjective morphism $g : M' \to M$, there exist a morphism $h : P \to M'$ such that $g \circ h = f$.

(ii) Every exact sequence of A-modules $0 \to M' \xrightarrow{f} M \xrightarrow{g} P \to 0$ splits. (That is, if f is injective, g is surjective and $\ker(g) = \mathrm{im}(f)$, then there exists a morphism $h : P \to M$ such that $g \circ h = \mathrm{id}_P$.)

(iii) There exists an A-module M such that $M \oplus P$ is free.

A module P that satisfies the properties above is called *projective*.

Exercise 2. Let I be an A-module. Show that the following properties are equivalent:

(i) For any module M, any submodule $M' \subseteq M$, and any morphism $f : M' \to I$, there exist a morphism $h : M \to I$ such that $h_{|M'} = f$. (We say that f *extends to* M.)

(ii) Every exact sequence $0 \to I \to M' \to M \to 0$ splits.

A module I that satisfies the properties above is called *injective*.

Exercise 3. Let D be a $\mathbb{Z}$-module such that for any nonzero integer n, the multiplication map $x \in I \mapsto nx \in D$ is surjective. Such a module is called *divisible*.

1. Let M be a module, $M' \subseteq M$ a submodule, x an element of M, and $f : M' \to D$ be a morphism. Show that f extends to the submodule generated by M' and x.
2. Show that D is injective.

Exercise 4. Let M be a A-module and P be a $\mathbb{Z}$-module.

1. Show that for any A-module N, the Abelian group $\mathrm{Hom}_{\mathbb{Z}}(N, P)$ is given a structure of A-module with the multiplication

$$a \cdot f \stackrel{\text{def}}{=} (x \in N \mapsto f(ax) \in P) .$$

2. Show that the A-module $\mathrm{Hom}_A(M, \mathrm{Hom}_{\mathbb{Z}}(A, P))$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(M, P)$.

3. If P is a divisible $\mathbb{Z}$-module, show that $\mathrm{Hom}_{\mathbb{Z}}(A, P)$ is an injective A-module.

Exercise 5. Let M be an A-module and let T be the Abelian group $\mathbb{Q}/\mathbb{Z}$. For any A-module N, let N^* denote the A-module $\mathrm{Hom}_{\mathbb{Z}}(N, T)$.

1. Show that T is divisible.

2. Show that the map $x \in M \mapsto (f \mapsto f(x)) \in M^{**}$ is an injective morphism of A-modules.

3. Show that every $\mathbb{Z}$-module is a submodule of an injective module.

   Hint: Find an inclusion in T^I, for some set I.

4. Show that every A-module is a submodule of an injective module.

   Hint: Use the previous exercise.