Exercise sheet #11

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Exercise 1. Finish up the sheet #10.

In the following we discuss scalar extensions and restrictions of modules.
Let A and B be a commutative rings with a ring morphism $f : A \to B$. Note that B is an A-module for the multiplication $a \cdot b := f(a)b$.
For any B-module N, we define an A-module N_A with the group structure of N and the following multiplication: $a \cdot x := f(a)x$, with $a \in A$ and $x \in N$. This is a *restriction of the scalars.*
For any A-module M, we define $B \otimes_A M$ as the B-module generated by the symbols $b \otimes m$, for $b \in B$ and $m \in M$ and subject to the relations:

- $\forall b \in B, m, m' \in M, b \otimes (m + m') = b \otimes m + b \otimes m'$;
- $\forall b, b' \in B, m \in M, (b + b') \otimes m = b \otimes m + b' \otimes m$;
- $\forall b, b' \in B, m \in M, b(b' \otimes m) = (bb') \otimes m$;
- $\forall a \in A, b \in B, m \in M, b \otimes am = (f(a)b) \otimes m$.

This is an *extension of the scalars.*
The map $\otimes : (b, m) \in B \times M \mapsto b \otimes m \in B \otimes M$ is, by construction, a bilinear map.

Exercise 2. Let M and N be A-modules and $f : B \times M \to N$ be an A-bilinear map. Show that there exists a unique A-linear map $h : B \otimes M \to N$ such that $f(b, m) = h(b \otimes m)$, for any $b \in B$ and $m \in M$.

Exercise 3. Let M be an A-module and N be a B-module. Show that $\mathrm{Hom}_A(M, N_A) \simeq \mathrm{Hom}_B(B \otimes_A M, N)$.

Exercise 4. We consider the usual inclusion $\mathbb{R} \to \mathbb{C}$.

1. Let M be an $\mathbb{R}$-vector space of dimension p. Show that $\mathbb{C} \otimes_{\mathbb{R}} M$ is a $\mathbb{C}$-vector space of dimension p.
2. Let N be a $\mathbb{C}$-vector space of dimension q. Show that $N_{\mathbb{R}}$ is an $\mathbb{R}$-vector space of dimension $2q$.