Exercise sheet #2

Prof. Peter Bürgisser, Dr. Pierre Lairez, Paul Breiding and Jesko Hüttenhain

April 27, 2016

Exercise 1. Let K be a field. Recall that a nonzero polynomial $f \in K[X]$ is called *separable* if $\gcd(f, f') = 1$ and that f is called *square-free* if it is not divided by some polynomial g^2, with $\deg g > 0$.

1. Show that if $f \in K[X] \setminus \{0\}$ is separable then f is square-free.
2. Show that if K is not perfect, then there exist a nonzero polynomial $f \in K[X]$ which is square-free but not separable.

Exercise 2. Show that every field extension of degree 2 is normal.

Exercise 3. Let $K \subseteq L$ be a Galois field extension with Galois group G. Let $\alpha \in L$. Show that $L = K(\alpha)$ if and only if the stabilisor of α for the action of G on L is trivial.

Exercise 4. Let F be a finite field with q elements, and let $f \in F[X]$ be a monic irreducible polynomial of degree $d > 0$.

1. Show that there is a field extension $F \leqslant L$ of degree d in which f has a root $\alpha \in L$.
2. Show that
$$f = \prod_{i=0}^{d-1} \left(X - \alpha^{q^i} \right).$$
3. Show that the extension $F \subseteq L$ is Galois.