Exercise sheet #7

Prof. Peter Bürgisser, Dr. Pierre Lairez, Paul Breiding and Jesko Hüttenhain

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Exercise 1. Let $f \in \mathbb{Q}[X]$ an irreducible cubic polynomial with three real roots. We aim at proving that no root of f can be expressed using only real radicals.
Let α_1, α_2 and $\alpha_3 \in \mathbb{R}$ be the roots of f. Let Δ be the discriminant of f, namely

$$\Delta = (\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2.$$

1. Show that $\Delta > 0$.

Assume that we have a tower of field extensions $\mathbb{Q}(\sqrt{\Delta}) = L_0 \subset L_1 \subset \cdots \subset L_n \subset \mathbb{R}$ such that α_1 is in L_n but not in L_{n-1} and such that for any $0 < i \leqslant n$, there is some positive real number $u_i \in L_{i-1}$ and some integer $s_i > 0$ such that L_i is generated over L_{i-1} by the positive real s_i-th root of u_i.

2. Show that, without loss of generality, we may assume that every s_i is prime.

Let K be the splitting field of f over L_{n-1}.

3. Show that $\mathrm{Gal}(K/L_{n-1}) \simeq \mathbb{Z}/(3)$.
4. Show that $K = L_n$.
5. Show that L_n cannot be a real extension.

Exercise 2. Let K be a field. A *preorder* on K is a subset $P \subset K$ such that:
(i) $\forall x, y \in P, x + y \in P$ and $xy \in P$; (ii) $P \cap (-P) = \{0\}$; (iii) $\forall x \in K, x^2 \in P$. Moreover, a preorder P is an *order* if $P \cup (-P) = K$.

1. If P is a preorder and $a \notin P$, show that $P - aP$ is a preorder.
2. Show that for any preorder P, there exist an order that contains P.

   Hint: Zorn's Lemma.
3. Show that there exists a preorder in K if and only if -1 cannot be written as a sum of squares $\sum_{i=1}^r a_i^2$, with $a_1, \ldots, a_r \in K$.