Exercise sheet #8

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For any finite field extension $K \subseteq L$ and any $a \in L$, we define the *multiplication map* $m_a : x \in L \mapsto ax \in L$ which we regard as a linear endomorphism of the K-linear space L.
For $a \in L$, we define the *norm of a over K* with $\mathrm{N}_{L/K}(a) = \det(m_a)$, and the *trace of a over K* with $\mathrm{Tr}_{L/K}(a) = \mathrm{Tr}(m_a)$.

Exercise 1. Let $\zeta \in \mathbb{C}$ be a primitive nth root of unity, $n > 1$, and let K be the cyclotomic extension $\mathbb{Q}(\zeta)$.

1. Show that $\mathrm{N}_{K/\mathbb{Q}}(1-\zeta) = \Phi_n(1)$.
   Hint: Consider the basis $1, \zeta, \ldots, \zeta^{[K:\mathbb{Q}]-1}$ *of K over* $\mathbb{Q}$ *and figure out what the eigenvalues of* m_ζ *are.*

2. Show that $\mathrm{N}_{K/\mathbb{Q}}(1-\zeta)$ equals 1 if n is not the power of a prime number and equals p if n is a power of the prime number p.

Exercise 2. Let $K \subseteq L$ be a finite separable extension.

1. Let $\sigma_1, \ldots, \sigma_n$ be the distinct embeddings $L \to \overline{K}$. Show that for any $\alpha \in L$,

$$\mathrm{Tr}_{L/K}(\alpha) = \sum_{i=1}^{r} \sigma_i(\alpha).$$

2. Show that $\mathrm{Tr}_{L/K} : L \to K$ is not identically zero.

Assume that L is a cyclic Galois extension of K, whose Galois group is generated by some σ.

3. For $\beta \in L$, show that $\mathrm{Tr}_{L/K}(\beta) = 0$ if and only if $\beta = \alpha - \sigma\alpha$ for some $\alpha \in L$.

Exercise 3. Let K be a field of characteristic $p > 0$ and let $K \subset L$ be a Galois extension of degree p.

1. Show that L/K is a cyclic extension.

Let σ be a generator of the Galois group.

2. Show that there is an $\alpha \in L$ such that $\sigma(\alpha) = \alpha + 1$.
3. Show that $L = K(\alpha)$.
4. Show that L is the splitting field of a polynomial $X^p - X - a$, for some $a \in K$.

Conversely, let $a \in K$ and let $f(X) = X^p - X - a$.

5. Show that the splitting field of f over K is either K or an extension of degree p.
   Hint: Show that $f(X) = \prod_{i=1}^{p}(X - \alpha - i)$, *where* α *is a root of* f.