Exercise sheet 1 for Algebra 4: representation theory

TU Berlin, Prof. Bürgisser, SS 2017

1. Show that the S_n-module $\{x \in \mathbb{C}^n \mid \sum_i x_i = 0\}$ is simple if $n \geq 3$.

2. (a) Suppose that the finite group G acts on the finite set X. We denote by X/G the set of G-orbits of X. We say that $x \in X$ is a fixed point of $g \in G$ if $gx = x$. Prove the following result of Burnside:

$$\frac{1}{|G|} \sum_{g \in G} \text{number of fixed points of } g = |X/G|.$$

Hint: double counting.

(b) As an application, determine the average number of fixed points of a random permutation $\pi \in S_n$ w.r.t. the uniform distribution.

3. Suppose that V is a G-module. The dual space $V^* := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ becomes a G-module via

$$(g\ell)(v) := \ell(g^{-1}v),$$

for $g \in G$, $\ell \in V^*$, $v \in V$. One calls V^* the *dual module* of V. Express the character of V^* in terms of the character of V.

4. Let D_n denote the *dieder group* of order n. In the lecture, we provide a concrete list of all possible D_n-modules, up to isomorphy, in the case where n is odd. Provide such a list in the case where n is even. Also, describe the character table of D_n in that case.

5. In the lecture, we provide a concrete list of all possible S_4-modules,

(1) Provide a concrete list of all possible A_4-modules, up to isomorphy.
(2) Describe the structure of the Wedderburn isomorphism for $\mathbb{C}[A_4]$.
(3) Give the charakter table of A_4 an.