Exercise sheet 2 for Algebra 4: representation theory

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6. Let G and H be finite groups, $U_1, \ldots, U_k$ be an isomorphism list of simple G-modules, and $V_1, \ldots, V_\ell$ be an isomorphism list of simple H-modules. We define a $G \times H$-module structure on $U_i \otimes V_j$ by

$$(g, h)(u \otimes v) := (gu) \otimes (hv)$$

for $g \in G$, $h \in H$, $u \in U_i$, $v \in V_j$. Prove that $(U_i \otimes V_j)_{1 \le i \le k, 1 \le j \le \ell}$ is an isomorphism list of simple $G \times H$-modules.

7. Let k be a field. A k-*division algebra* is a k-algebra A such that $A \neq 0$ and every nonzero $a \in A$ is a unit, that is, there exists $b \in A$ with $ab = ba = 1$.

(1) Let M be a simple A-module. Prove that $\mathrm{End}_A(M)$ is a k-division algebra.
(2) Determine $\mathrm{End}_A(M)$ for $A = k^{n \times n}$ and $M = k^n$.

8. Let k be a field.

(1) Prove that the k-algebra $k[X]/(X^n)$ is not semi-simple if $n \ge 2$.
(2) More generally, let $f \in k[X]$ be a nonzero polynomial. Determine when $k[X]/(f)$ is semi-simple.

9. Let A and B be finite dimensional k-algebras, where k is a field.

(1) Show that the tensor product $A \otimes B$ of k-vector spaces carries the structure of a k-algebra, whose multiplication is characterized by

$$(a \otimes b)(a' \otimes b') = (aa') \otimes (bb'),$$

for $a, a' \in A$ and $b, b' \in B$.
(2) Show that the tensor product of algebras is characterized by the following universal property: For every k-algebra C and k-algebra morphisms $\alpha\colon A \to C$ and $\beta\colon B \to C$, there is a unique k-algebra morphism $\gamma\colon A \otimes B \to C$ such that

$$\gamma(a \otimes b) = \alpha(a)\beta(b)$$

for all $a \in A$, $b \in B$.
(3) Prove that $\mathrm{End}(U) \otimes \mathrm{End}(V) \simeq \mathrm{End}(U \otimes V)$ as k-algebras, where U and V are finite dimensional k-vector spaces.