Exercise sheet 3 for Algebra 4: representation theory

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10. Let G and H be finite groups and k be a field. Prove that the group algebra $k[G\times H]$ is isomorphic to the tensor product $k[G]\otimes k[H]$ of the group algebras of G and H. (Compare Exercise 9.)

In the following, V denotes a finite dimensional k-vector space and V^* stands for its dual space. We assume $\mathrm{char} k = 0$.

11. Let $N \geq 1$. Prove that there is vector space isomorphism

$$\varphi\colon (V^*)^{\otimes N} \to (V^{\otimes N})^*$$

such that for $\ell_1, \ldots, \ell_N \in V^*$, $\varphi(\ell_1 \otimes \cdots \otimes \ell_N)$ maps $v_1 \otimes \cdots \otimes v_N$ to $\ell_1(v_1)\cdots\ell_N(v_N)$.

12. The isotypical decomposition of $V^{\otimes 3}$ as an S_3-module is of the form

$$V^{\otimes 3} = S^3(V) \oplus \Lambda^3(V) \oplus T,$$

where $S^3(V)$ corresponds to the trivial representation, $\Lambda^3(V)$ corresponds to the sign representation, and T corresponds to the 2-dimensional irreducible S_3-representation.

(1) Determine the projection of $V^{\otimes 3}$ onto T along the isotypical decomposition (use the result from the lecture).
(2) Prove that a tensor $w \in V^{\otimes 3}$ lies in T if and only if $w + (123)w + (132)w = 0$.
(3) Prove that $S^3(V)\oplus\Lambda^3(V)$ consists exactly of the tensors $w \in V^{\otimes 3}$ that are invariant under cyclic permutations.

Remark. From Schur-Weyl duality, it will become clear later in the lecture that T splits into two isomorphic irreducible $\mathrm{GL}(V)$-modules.

13. Consider the setting of Exercise 13 in the case $V = k^2$. Let e_1, e_2 denote the standard basis of V. Recall that the $e_i \otimes e_j \otimes e_k$, for $1 \leq i, j, k \leq 2$, form a basis of the 8-dimensional $V^{\otimes 3}$

(1) Show that $\Lambda^3(V) = 0$, so that we have the decomposition
$$V^{\otimes 3} = S^3(V) \oplus T.$$
(2) What are the dimensions of $S^3(V)$ and T?
(3) Provide an explicit basis of T.