Exercise sheet 4 for Algebra 4: representation theory

TU Berlin, Prof. Bürgisser, SS 2017

18. Let G be a finite group which operates on a finite set S. Prove that the multiplicity of the trivial representation of G in $\mathbb{C}[S]$ equals the number of orbits of G in S.

19. Now suppose that the finite group G acts transitively on the set S. Then G acts on $S \times S$ via $g(s,t) := (gs, gt)$. One says that the action of G is *doubly transitive* if for all $s,t,u,v \in S$ with $s \neq u$, $t \neq v$ there exists $g \in G$ such that $gs = t$ und $gu = v$. Prove that the following three statements are equivalent:

(1) G acts doubly transitively on S,

(2) the action of G on $S \times S$ has exactly two orbits.

(3) the orthogonal complement of the trivial module $\mathbb{C}$ in $\mathbb{C}[S]$ is irreducible.

The next three exercises deal with the Fourier transform of a finite group G. You should check the meaning of the statements in the special case where G is a cyclic group!

The *Fourier transform* of a function $\phi\colon G \to \mathbb{C}$ assigns to an irreducible representation $D\colon G \to \mathrm{GL}(V)$ the endomorphism

$$\widehat{\phi}(D) \ := \ \sum_{g\in G} \phi(g)\, D(g) \in \mathrm{End}(V)\ .$$

The *convolution* of two functions $\phi, \psi\colon G \to \mathbb{C}$ is defined as the function $\phi * \psi\colon G \to \mathbb{C}$

$$(\phi * \psi)(g) := \sum_{h\in G} \phi(h)\, \psi(h^{-1}g) \qquad \text{for } g \in G.$$

Let $W_1, \dots, W_k$ be an isomorphism list of irreducible G-modules. We denote by $D_1, \dots, D_k$ and $\chi_1, \dots, \chi_k$ the corresponding representations and characters, respectively.

20. Prove that $\widehat{\phi}(D) \circ \widehat{\psi}(D) \ = \ \widehat{\phi * \psi}(D)$.

21. Prove the *Fourier inversion formula*: for $g \in G$ we have

$$\phi(g) \ = \ \frac{1}{|G|} \sum_{i=1}^{k} \dim(W_i)\, \mathrm{tr}\left(D_i(g^{-1}) \circ \widehat{\phi}(D_i) \right)\ .$$

(Hint: Use the character of the regular representation.)

22. Prove the *Plancherel formula*:

$$\sum_{g\in G} \phi(g^{-1})\psi(g) \ = \ \frac{1}{|G|} \sum_{i=1}^{k} \dim(W_i)\, \mathrm{tr}\left(\widehat{\phi}(D_i)\, \widehat{\psi}(D_i) \right)\ .$$

(Hint: Use the Fourier inversion formula.)