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Exercises for Algebraic Geometry 1

Winter term 2017/2018

Exercise sheet 2

Exercise 1. Let $X \subset \mathbb{A}^3$ be the algebraic set given by the equations

$$x^2 - yz = xz - x = 0.$$

Find the irreducible components of X. What are their vanishing ideals?

Exercise 2.

(1) Show that the common zero locus $Z(f, g)$ of two polynomials $f, g \in k[x, y]$ without common factor is finite.
(Hint: You can use the following result about resultants:
The resultant of two polynomials with coefficients in an integral domain is zero if and only if they have a common divisor of positive degree.)

(2) Let $X \subset \mathbb{A}^2$ be an irreducible algebraic set. Show that either

- $X = Z(0)$, i.e., X is the whole space $\mathbb{A}^2$, or
- $X = Z(f)$ for some irreducible polynomial $f \in k[x, y]$, or
- $X = Z(x - a, y - b)$ for some $a, b \in k$, i.e., X is a single point.

(3) Deduce that $\dim(\mathbb{A}^2) = 2$.

Exercise 3. In the lecture we have seen that, for ideals $I, J \subset \mathbb{C}[x_1, \ldots, x_n]$, we have

$$I \subset J \;\Rightarrow\; Z(J) \subset Z(I),$$
$$Z(I) \cap Z(J) = Z(I + J),$$
$$Z(I) \cup Z(J) = Z(IJ),$$
$$I(Z(I)) = \sqrt{I}.$$

(1) Show that $\sqrt{I}$ as defined in the lecture is indeed an ideal.

(2) Show that: $Z(I) \cup Z(J) = Z(I \cap J)$

(3) Show that $I \cap J$ is radical if both I and J are radical.

(4) Find an example of two radical ideals I, J such that IJ is not radical.

(5) Find an example of two radical ideals I, J such that $I + J$ is not radical.

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Exercise 4. In the lecture we have seen that, for algebraic sets $X, Y \subset \mathbb{A}^n$, we have

$$X \subset Y \;\Rightarrow\; I(Y) \subset I(X),$$
$$Z(I(X)) = X.$$

Show that:

(1) $I(X \cup Y) = I(X) \cap I(Y)$

(2) $I(X \cap Y) = \sqrt{I(X) + I(Y)}$