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Exercises for Algebraic Geometry 1

Winter term 2017/2018

Exercise sheet 3

Exercise 1. Let $X \subset \mathbb{A}^n$ be an algebraic set and let $S \subset X$ be any subset. Show:

$$\overline{S} = Z(I(S)).$$

Exercise 2. Let $f : \mathbb{A}^n \to \mathbb{A}^m$ be a polynomial map (i.e., $f(P) = (f_1(P), \dots, f_m(P))$ with $f_i \in k[x_1, \dots, x_n]$). Are the following statements true or false?

(1) If $X \subset \mathbb{A}^n$ is an algebraic set, then the image $f(X) \subset \mathbb{A}^m$ is an algebraic set.

(2) If $Y \subset \mathbb{A}^m$ is an algebraic set, then the inverse image $f^{-1}(Y) \subset \mathbb{A}^n$ is an algebraic set.

(3) If $X \subset \mathbb{A}^n$ is an algebraic set, then the graph $\Gamma := \{(x, f(x)) \mid x \in X\} \subset \mathbb{A}^{n+m}$ is an algebraic set.

Exercise 3. In $\mathbb{A}^4$ with coordinates x, y, z, t, let X be the union of the two planes

$$Z(x, y) \qquad \text{and} \qquad Z(z, x - t).$$

(1) Find generators of the ideal $I := I(X) \subset k[x, y, z, t]$.

(2) For any $a \in k$, let $I_a \subset k[x, y, z]$ be the ideal obtained by substituting $t = a$ in I, and let $X_a = Z(I_a) \subset \mathbb{A}^3$.
Show that the family of algebraic sets X_a with $a \in k^\times$ describes two skew lines in $\mathbb{A}^3$ approaching each other, until they finally intersect for $a = 0$.

(3) Show that the ideals I_a are radical for $a \neq 0$, but that I_0 is not. Which elements are in $\sqrt{I_0} \setminus I_0$? Do they have a geometric meaning?

Exercise 4. Let $\pi : Y \to X$ be a map of topological spaces. For every open subset $U \subset X$ we let

$$\mathcal{S}(U) := \{\sigma : U \to \pi^{-1}(U) \mid \sigma \text{ continuous}, \pi \circ \sigma = \mathrm{id}_U\}.$$

Show that this defines a sheaf (called the *sheaf of sections of* π).

The following exercise is for those who are more interested in sheaf theory:
From every pre-sheaf $\mathcal{F}$ one can make a sheaf, called the *sheafification* of $\mathcal{F}$. It is the sheaf "best approximating" the pre-sheaf $\mathcal{F}$. The following exercise gives a step-by-step construction of the sheafification.

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Exercise 5. Let $\mathcal{F}$ be a pre-sheaf on a topological space X. For every open subset $U \subset X$ and every $P \in U$ there is a map $\mathcal{F}(U) \to \mathcal{F}_P$, sending a section $\varphi \in \mathcal{F}(U)$ to the equivalence class of (U, φ). We denote this class by $\varphi_P \in \mathcal{F}_P$.

(1) Show: If $\mathcal{F}$ is a sheaf, we have for every open $U \subset X$ and every section $\varphi \in \mathcal{F}(U)$ that

$$\varphi = 0 \quad \Leftrightarrow \quad \forall P \in U : \varphi_P = 0.$$

(2) We define $\overline{\mathcal{F}} := \bigcup_{P \in X}(\{P\} \times \mathcal{F}_P)$. Show that all the sets of the form

$$\mathcal{V}(U, \varphi) := \{(P, \varphi_P) \mid P \in U\}, \tag{0.1}$$

where $U \subset X$ is open and $\varphi \in \mathcal{F}(U)$, are a base for the open sets of a topology on $\overline{\mathcal{F}}$, i.e., show that

- the union of all sets in (0.1) is $\overline{\mathcal{F}}$, and
- the intersection of two sets in (0.1) can be written as the union of sets in (0.1).

(Note that the open sets of the generated topology on $\overline{\mathcal{F}}$ are the unions of the sets in (0.1).)

(3) Show that the natural map $\pi : \overline{\mathcal{F}} \to X$ is continuous (using the topology defined in (2)).

(4) Show that, for an open $U \subset X$ and $\varphi \in \mathcal{F}(U)$, the map

$$\begin{aligned} \sigma : U &\longrightarrow \overline{\mathcal{F}}, \\ P &\longmapsto (P, \varphi_P) \end{aligned}$$

is a *continuous section of π over U*, i.e., show that

- σ is continuous, and
- $\pi \circ \sigma$ is the identity on U.

(5) Conversely, show that any continuous map $\sigma : U \to \overline{\mathcal{F}}$ with $\pi \circ \sigma = \mathrm{id}_U$ arises as in (4), if $\mathcal{F}$ is a sheaf.

We have defined a map $\mathcal{F}(U) \to \mathcal{S}(U)$ (see Exercise 4) for every open subset $U \subset X$. This is a bijection if $\mathcal{F}$ is a sheaf (injectivity follows from (1), surjectivity from (5)).
Hence, the sheaf $\mathcal{S}$ of sections of $\pi : \overline{\mathcal{F}} \to X$ is isomorphic to $\mathcal{F}$, if $\mathcal{F}$ is a sheaf.
If $\mathcal{F}$ is not a sheaf, the sheaf $\mathcal{S}$ of sections of $\pi : \overline{\mathcal{F}} \to X$ is still a sheaf, called the **sheafification** of $\mathcal{F}$. It is the unique sheaf $\mathcal{S}$ with a morphism of presheaves $\mathcal{F} \to \mathcal{S}$ such that for all $P \in X$ the induced map $\mathcal{F}_P \to \mathcal{S}_P$ is an isomorphism.