Prof. Dr. Peter Bürgisser
Kathlén Kohn

Technische Universität Berlin
Fakultät II, Institut für Mathematik
Sekretariat MA 3-2

Exercises for Algebraic Geometry 1

Winter term 2017/2018

Exercise sheet 4

Exercise 1. Let X be an affine variety, $Y \subseteq X$ closed, and $J \subseteq A(X)$ an ideal. We define the vanishing ideal $I_X(Y) := \{f \in A(X) \mid \forall p \in Y : f(p) = 0\}$ and the zero set $Z_X(J) := \{p \in X \mid \forall f \in J : f(p) = 0\}$. Show that $Z_X(I_X(Y)) = Y$ and that $I_X(Z_X(J)) = \sqrt{J}$.

Macaulay2

Exercise 2 (Exercise 1 on Sheet 2). Let $I = \langle x^2 - yz, xz - x \rangle$ and $X := Z(I) \subseteq \mathbb{A}^3$. Use `Macaulay2` to verify that I is radical and to compute the prime ideals of the irreducible components of X.

Exercise 3 (Exercise 3 on Sheet 3). In $\mathbb{C}^4$ with coordinates x, y, z, t, let X be the union of the two planes

$$Z(x, y) \qquad \text{and} \qquad Z(z, x - t).$$

(1) Find the vanishing ideal $I := I(X) \subset \mathbb{C}[x, y, z, t]$ with `Macaulay2`.

(2) For any $a \in \mathbb{C}$, let $I_a \subset \mathbb{C}[x, y, z]$ be the ideal obtained by substituting $t = a$ in I, and let $X_a = Z(I_a) \subset \mathbb{A}^3$.
Compute with `Macaulay2` the prime ideals of the irreducible components of X_1 and X_0, and see that X_1 is two skew lines, whereas X_0 is two lines intersecting at the origin.

(3) Verify with `Macaulay2` that I_1 is radical but that I_0 is not. Compute the radical ideal of I_0.

(4) Why is it enough to consider $a = 1$ to deduce that all ideals I_a for $a \neq 0$ are radical?

Exercise 4 (Exercise 1 on Sheet 1). Consider the following curve in $\mathbb{C}^3$:

$$C := \{(t^3, t^4, t^5) \mid t \in \mathbb{C}\} = \{(x, y, z) \in \mathbb{C}^3 \mid x^3 = yz,\ y^2 = xz,\ z^2 = x^2y\}.$$

Verify with `Macaulay2` that one needs indeed three equations to define C.
Helpful command: mingens

Technische Universität Berlin
Prof. Dr. Peter Bürgisser — Fakultät II, Institut für Mathematik
Kathlén Kohn — Sekretariat MA 3-2

Exercise 5 (Exercise 2 on Sheet 1). Consider the set

$$X := \left\{ \left(\begin{smallmatrix} m_{00} & m_{01} & m_{02} \\ m_{10} & m_{11} & m_{12} \end{smallmatrix}\right) \in \mathbb{C}^{2\times 3} \mid m_{00}m_{11} = m_{10}m_{01}, m_{00}m_{12} = m_{10}m_{02}, m_{01}m_{12} = m_{11}m_{02} \right\}$$

of all 2×3-matrices of rank at most 1. Verify with `Macaulay2` that X has dimension four and that one needs indeed three equations to define X.

Exercise 6 (Related to Exercise 3 on Sheet 1). Consider the cubic surface $S \subseteq \mathbb{R}^3$ defined by

$$\begin{aligned} f = {} & 81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + xy^2 + xz^2 + y^2z + yz^2) \\ & + 54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1. \end{aligned}$$

Verify with `Macaulay2` that there are 27 **real** lines on S and compute them explicitly! How many of these lines are defined over $\mathbb{Q}$?