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Exercises for Algebraic Geometry 1

Winter term 2017/2018

Exercise sheet 5

A ring is called *reduced* if it has no non-zero nilpotent elements. A k-algebra is called *reduced* if it is reduced as a ring.

Exercise 1. Let k be an algebraically closed field.

(1) Let I be a proper ideal of $k[x_1, \dots, x_n]$. Show that:

$$\sqrt{I} = \bigcap_{M \in \mathcal{M}} M, \qquad \text{where } \mathcal{M} := \{M \subseteq k[x_1, \dots, x_n] \mid M \text{ maximal ideal}, I \subseteq M\}.$$

(2) Let A be a reduced, finitely generated k-algebra. Show that:

$$\bigcap_{M \subseteq A \text{ maximal ideal}} M = \{0\}.$$

(Recall that a k-algebra is finitely generated if and only if it is isomorphic to a quotient ring $k[x_1, \dots, x_n]/I$ for some $n \in \mathbb{N}$ and ideal $I \subseteq [x_1, \dots, x_n]$.)

These statements are also true over fields which are not algebraically closed. For our purposes, it is enough to consider algebraically closed fields.

Let R be a commutative ring. A sequence

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \xrightarrow{\varphi_3} \dots \xrightarrow{\varphi_{n-1}} M_n$$

of R-modules $M_1, \dots, M_n$ and module homomorphisms $\varphi_1, \dots, \varphi_{n-1}$ is called *exact* if the image of each homomorphism is equal to the kernel of the next, i.e.,

$$\forall k = 1, \dots, n-2 : \quad \operatorname{im}(\varphi_k) = \ker(\varphi_{k+1}).$$

Exercise 2. Let M_1, M_2 and M_3 be R-modules. Show that

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} M_3 \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow \operatorname{Hom}(M_3, N) \xrightarrow{\varphi_2^*} \operatorname{Hom}(M_2, N) \xrightarrow{\varphi_1^*} \operatorname{Hom}(M_1, N)$$

is exact for **every** R-module N, where $\varphi_i^*(\varphi) := \varphi \circ \varphi_i$ for $i \in \{1, 2\}$.

These exercises are meant as preparation for our Übung on tensor products, on **16.11.2017**!