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Exercises for Algebraic Geometry 1

Winter term 2017/2018

Exercise sheet 8

Exercise 1. Let L_1 and L_2 be two lines disjoint lines in $\mathbb{P}^3$, and let $P \in \mathbb{P}^3 \setminus (L_1 \cup L_2)$ be a point. Show that there is a unique line $L \subseteq \mathbb{P}^3$ meeting L_1, L_2 and P (i.e., such that $P \in L$ and $L \cap L_i \neq \emptyset$ for $i = 1, 2$).

Exercise 2. Consider the map

$$\begin{aligned} f : \mathbb{P}^1 &\longrightarrow \mathbb{P}^3, \\ (s : t) &\longmapsto (x : y : z : w) := (s^3 : s^2 t : s t^2 : t^3). \end{aligned}$$

(1) Show that f is an isomorphism onto the *twisted cubic curve* $C := Z(I)$, where

$$I := \langle y^2 - xz, \quad z^2 - yw, \quad yz - xw \rangle.$$

(2) Let $P = (0 : 0 : 1 : 0) \in \mathbb{P}^3$, and let H be the hyperplane defined by $z = 0$. Consider the projection $\varphi : C \to H \cong \mathbb{P}^2$ from P to H, i.e., the map associating to a point Q of C the intersection point of the unique line through P and Q with H.

Show that φ is a morphism, and determine the equation of the curve $\varphi(C)$ in $H \cong \mathbb{P}^2$.

(3) Is $\varphi : C \to \varphi(C)$ an isomorphism onto its image?

Exercise 3. Let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Define I^h to be the ideal generated by $\{f^h \mid f \in I\} \subseteq k[x_0, \ldots, x_n]$, where

$$f^h(x_0, \ldots, x_n) := x_0^{\deg(f)} \cdot f\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right)$$

denotes the homogenization of f with respect to x_0. Show that:

(1) I^h is a homogeneous ideal.

(2) $Z_{\mathbb{P}}(I^h) \subseteq \mathbb{P}^n$ is the closure of $Z_{\mathbb{A}}(I) \subseteq \mathbb{A}^n$ in $\mathbb{P}^n$.

We call $Z_{\mathbb{P}}(I^h)$ the *projective closure* of $Z_{\mathbb{A}}(I)$.

(3) Let $I = \langle f_1, \ldots, f_k \rangle$. Show by an example that $I^h \neq \langle f_1^h, \ldots, f_k^h \rangle$ in general.

(Hint: You may consider the twisted cubic curve.)

We discuss these exercises on **07.12.2017**!