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# Exercises for Algebraic Geometry 1

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## Exercise sheet 9

**Exercise 1.** Let $X \subseteq \mathbb{A}^n$ be an affine variety with vanishing ideal $I := I(X) \subseteq k[x_1, \ldots, x_n]$. Consider the ideal $I_1 := I \cap k[x_1, \ldots, x_{n_1}]$, called the *first elimination ideal* of $I$, and the projection

$$\begin{aligned} \pi : \mathbb{A}^n &\longrightarrow \mathbb{A}^{n-1}, \\ (a_1, \ldots, a_n) &\longmapsto (a_1, \ldots, a_{n-1}). \end{aligned}$$

Prove that $I_1$ is the vanishing ideal of $\pi(X)$

**Exercise 2.** Let $X \subseteq \mathbb{A}^n$ be an affine variety. Suppose that $(0 : \ldots : 0 : 1) \in \mathbb{P}^n$ is not contained in the projective closure $\overline{X} \subseteq \mathbb{P}^n$ of $X$. Prove that the projection $X \to \mathbb{A}^{n-1}, (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_{n-1})$ is a closed map.

**Exercise 3.** The *Grassmannian* $\mathrm{Gr}(1, \mathbb{P}^n)$ is the set of lines in $\mathbb{P}^n$. It has a natural bijection with the set $\mathrm{Gr}(2, \mathbb{A}^{n+1})$ of two-dimensional linear subspaces of $\mathbb{A}^{n+1}$.
We define the set-theoretic map

$$\begin{aligned} \gamma : \mathrm{Gr}(1, \mathbb{P}^n) &\longrightarrow \mathbb{P}\left(\bigwedge^2 \mathbb{A}^{n+1}\right), \\ \mathrm{span}(a, b) &\longmapsto [a \wedge b]. \end{aligned}$$

In coordinates: For every line $L \subseteq \mathbb{P}^n$ choose two distinct point $a = (a_0 : \ldots : a_n)$ and $b = (b_0 : \ldots : b_n)$ on $L$ and define $\gamma(L)$ to be the point in $\mathbb{P}\left(\bigwedge^2 \mathbb{A}^{n+1}\right)$ whose homogeneous coordinates are the $\binom{n+1}{2}$ maximal minors of the matrix $\begin{pmatrix} a_0 & \cdots & a_n \\ b_0 & \cdots & b_n \end{pmatrix}$ in any fixed order.
Show that:

(1) The map $\gamma$ is well-defined and injective.
   It is called the *Plücker embedding*.

(2) $\mathrm{Gr}(1, \mathbb{P}^1)$ is a point, $\gamma(\mathrm{Gr}(1, \mathbb{P}^2)) \cong \mathbb{P}^2$, and $\gamma(\mathrm{Gr}(1, \mathbb{P}^3))$ is the zero locus of a quadratic equation in $\mathbb{P}(\bigwedge^2 \mathbb{A}^4) \cong \mathbb{P}^5$.

(3) The image of $\gamma$ is a projective variety that has a finite cover by affine spaces $\mathbb{A}^{2(n-1)}$. In particular, its dimension is $2(n-1)$.
   *(Hint: recall that by the Gaussian algorithm most matrices are equivalent to one of the form* $\begin{pmatrix} 1 & 0 & a'_2 & \cdots & a'_n \\ 0 & 1 & b'_2 & \cdots & b'_n \end{pmatrix}$*.)*

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We discuss these exercises on **14.12.2017**!