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Exercises for Algebraic Geometry 1

Winter term 2017/2018

Christmas sheet

Exercise 1. Let $\mathrm{Gr}(1, \mathbb{P}^n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Show that:

(1) The set $\{(P, L) \mid P \in L\} \subseteq \mathbb{P}^n \times \mathrm{Gr}(1, \mathbb{P}^n)$ is closed.

(2) If $\Sigma \subseteq \mathrm{Gr}(1, \mathbb{P}^n)$ is any closed subset, then the union of all lines $L \subseteq \mathbb{P}^n$ such that $L \in \Sigma$ is closed in $\mathbb{P}^n$.

(3) If $X, Y \subseteq \mathbb{P}^n$ are disjoint projective varieties, then the union of all lines in $\mathbb{P}^n$ intersecting X and Y is a closed subset of $\mathbb{P}^n$. It is called the *join* of X and Y.

Exercise 2. (1) Let $X, Y \subseteq \mathbb{P}^n$ be projective varieties. Show that $X \cap Y$ is not empty if $\dim X + \dim Y \geq n$.

(2) Give an example of a projective variety Z and closed subsets $X, Y \subseteq Z$ with $\dim X + \dim Y \geq \dim Z$ and $X \cap Y = \emptyset$.

Exercise 3. Let V be the k-vector space of homogeneous quadratic polynomials in three variables x_0, x_1, x_2, and let $\mathbb{P}(V) \cong \mathbb{P}^5$ be its projectivization. A one-dimensional closed subset C of $\mathbb{P}^2$ is called a *conic* if its vanishing ideal is generated by a quadratic polynomial. A conic is called *irreducible* if its defining polynomial is irreducible.

(1) Show that the set of those polynomials in V which are squares of linear polynomials is the cone over a projective variety X_1 in $\mathbb{P}(V)$.

This shows that the space of conics in $\mathbb{P}^2$ can be identified with the open subset $\mathbb{P}(V) \setminus X_1$ of $\mathbb{P}(V)$. What geometric objects can be associated to points in X_1?

(2) Show that the set of reducible polynomials in V is the cone over a projective variety X_2 in $\mathbb{P}(V)$.

This shows that the space of irreducible conics in $\mathbb{P}^2$ can be identified with the open subset $\mathbb{P}(V) \setminus X_2$ of $\mathbb{P}(V)$. What geometric objects can be associated to the points in $X_2 \setminus X_1$?

(3) Compute the vanishing ideals of X_1 and X_2, for example with `Macaulay2`.

(Hint: You can define a ring homomorphism with the command `map` *and then compute its kernel using the command* `kernel`*.)*

(4) Let $P \in \mathbb{P}^2$ be a point. Show that there is a hyperplane $H \subseteq \mathbb{P}(V)$ such that the conics passing through P are exactly those in $(\mathbb{P}(V) \setminus X_1) \cap H$.

What do points in $X_1 \cap H$ correspond to?

We discuss these exercises on **11.01.2018**!

(5) Prove that there is a unique conic through any five given points in $\mathbb{P}^2$, as long as no three of them lie on a line. Moreover, show that this conic is irreducible.

What happens if three of the given points do lie on a line?

Exercise 4. The goal of this exercise is to show that for four general lines $L_1, \ldots, L_4 \subseteq \mathbb{P}^3$, there are exactly two lines in $\mathbb{P}^3$ intersecting all the L_i.
We consider the Plücker embedding $\gamma : \mathrm{Gr}(1, \mathbb{P}^3) \to \mathbb{P}^5$ and write $\gamma(\mathrm{span}(a, b)) = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, where $p_{ij} := a_i b_j - a_j b_i$.

(1) For a line $L = Z(c_0x_0 + \ldots + c_3x_3,\ d_0x_0 + \ldots + d_3x_3) \subseteq \mathbb{P}^3$ $(c_i, d_j \in k)$, we define the *dual Plücker coordinates* $p_{ij}^{(d)} := c_i d_j - c_j d_i$ for $0 \le i < j \le 3$. Show that:

$$\gamma(L) = \left(p_{23}^{(d)} : -p_{13}^{(d)} : p_{12}^{(d)} : p_{03}^{(d)} : -p_{02}^{(d)} : p_{01}^{(d)} \right).$$

(2) Let $L_1, L_2 \in \mathrm{Gr}(1, \mathbb{P}^3)$ and write $\gamma(L_1) = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$ and $\gamma(L_2) = (q_{01} : q_{02} : q_{03} : q_{12} : q_{13} : q_{23})$. Show that:

$$L_1 \cap L_2 \neq \emptyset \quad \Leftrightarrow \quad p_{01}q_{23} - p_{02}q_{13} + p_{03}q_{12} + p_{12}q_{03} - p_{13}q_{02} + p_{23}q_{01} = 0.$$

(Hint: Write $L_1 = \mathrm{span}(a, b)$ and L_2 as a zero locus as in (1). Then express $L_1 \cap L_2 \neq \emptyset$ using a resultant.)

(3) For $L \in \mathrm{Gr}(1, \mathbb{P}^3)$ with $\gamma(L) = (p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, we set

$$\begin{aligned} F_L := {} & y_{01}p_{23} - y_{02}p_{13} + y_{03}p_{12} + y_{12}p_{03} - y_{13}p_{02} + y_{23}p_{01} \\ & \in \mathbb{P}(k[y_{01}, y_{02}, y_{03}, y_{12}, y_{13}, y_{23}]_1). \end{aligned}$$

Moreover, we denote $\mathcal{G} := \mathrm{Gr}(1, \mathbb{P}^3) \times \mathrm{Gr}(1, \mathbb{P}^3) \times \mathrm{Gr}(1, \mathbb{P}^3) \times \mathrm{Gr}(1, \mathbb{P}^3)$.

Show: There is a closed subset $\Sigma_1 \subsetneq \mathcal{G}$ such that

$$Z\left(F_{L_1}, \ldots, F_{L_4}\right) \cong \mathbb{P}^1 \quad \Leftrightarrow \quad (L_1, \ldots, L_4) \notin \Sigma_1$$

holds for all $(L_1, \ldots, L_4) \in \mathcal{G}$.

(4) Show: There is a closed subset $\Sigma_2 \subsetneq \mathcal{G}$ such that

$$Z\left(F_{L_1}, \ldots, F_{L_4}\right) \subseteq Z(y_{01}y_{23} - y_{02}y_{13} + y_{03}y_{12}) \quad \Leftrightarrow \quad (L_1, \ldots, L_4) \in \Sigma_2$$

holds for all $(L_1, \ldots, L_4) \in \mathcal{G} \setminus \Sigma_1$.

(5) Show: There is a closed subset $\Sigma_3 \subsetneq \mathcal{G}$ such that

$$Z\left(F_{L_1}, \ldots, F_{L_4}\right) \cap Z(y_{01}y_{23} - y_{02}y_{13} + y_{03}y_{12}) \cong \mathbb{P}^0 \quad \Leftrightarrow \quad (L_1, \ldots, L_4) \in \Sigma_3$$

holds for all $(L_1, \ldots, L_4) \in \mathcal{G} \setminus (\Sigma_1 \cup \Sigma_2)$.

(6) Conclude that for all $(L_1, \ldots, L_4) \in \mathcal{G} \setminus (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3)$ there are exactly two lines in $\mathbb{P}^3$ intersecting all the L_i.