Prof. Dr. Peter Bürgisser
Kathlén Kohn

Technische Universität Berlin
Fakultät II, Institut für Mathematik
Sekretariat MA 3-2

Exercises for Algebraic Geometry 1

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Sheet 12

Exercise 1. Let X and Y be two varieties. Show that the following assertions are equivalent:

(a) X and Y are birationally equivalent.

(b) There are non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ which are isomorphic.

(c) The functions fields $K(X)$ and $K(Y)$ are isomorphic.

Exercise 2. Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $p \in X$. Consider the projection $\pi_p : X \dashrightarrow \mathbb{P}^{n-1}$ away from the point p, which is defined on all of $X \setminus \{p\}$. The *blow up* $\mathrm{Bl}_p(X) \subseteq X \times \mathbb{P}^{n-1}$ of X at p is defined to be the Zariski closure of the graph of π_p.

(a) Show that $\mathrm{Bl}_p(X)$ is irreducible and has the same dimension as X.

For two distinct points $x, y \in \mathbb{P}^n$, we denote by $L(x, y)$ the line spanned by x and y.

(b) Let $H \subseteq \mathbb{P}^n$ be a hyperplane which does not contain p. Show that

$$\mathrm{Bl}_p(\mathbb{P}^n) \cong \{(x, y) \in \mathbb{P}^n \times H \mid x \in L(p, y)\}.$$

(c) Consider the two projections

$$\begin{array}{ccccc} & & \mathrm{Bl}_p(X) & & \\ & \swarrow \pi_1 & & \pi_2 \searrow & \\ X & & & & \mathbb{P}^{n-1} \end{array}$$

and show in the case $X = \mathbb{P}^n$ that each fiber of π_2 is a line, whereas the fiber of $x \in \mathbb{P}^n$ under π_1 is a point if $x \neq p$ and a $\mathbb{P}^{n-1}$ if $x = p$.

We define the *strict transform* of a subvariety $Y \subseteq X$ as $\tilde{Y} := \overline{\pi_1^{-1}(Y \setminus \{p\})} \subseteq \mathrm{Bl}_p(X)$.

(d) Show that $\tilde{Y} = \mathrm{Bl}_p(Y)$.

(e) Let $L_1, L_2 \subseteq \mathbb{P}^n$ be two distinct lines that intersect at p. Show that the strict transforms $\tilde{L}_1, \tilde{L}_2 \subseteq \mathrm{Bl}_p(\mathbb{P}^n)$ are disjoint lines.

Exercise 3. Show that the *Fermat cubic surface* $S := Z(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{P}^3$ contains exactly 27 lines. Compute these explicitly and show the following:

(a) Each of the 27 lines intersects exactly 10 of the other lines.

(b) For each pair (L_1, L_2) of disjoint lines on S there are exactly five lines on S that meet both L_1 and L_2.