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# Exercises for Algebraic Geometry 1

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**Sheet 13**

**Exercise 1.** For a subset $S \subseteq \mathbb{P}^n$, we denote by $\mathcal{C}(S) \subseteq \mathbb{A}^{n+1}$ the affine cone over $S$. In particular, for $p \in \mathbb{P}^n$, the cone $\mathcal{C}(p)$ is the affine line spanned by $p$.

(a) Show for $p \in \mathbb{P}^n$ that

$$T_p\mathbb{P}^n \cong \mathbb{A}^{n+1}/\mathcal{C}(p).$$

(b) More generally, show for a projective variety $X \subseteq \mathbb{P}^n$ and $p \in X$ that

$$T_pX \cong T_x\mathcal{C}(X)/\mathcal{C}(p), \quad \text{where } x \in \mathcal{C}(p) \setminus \{0\}.$$

**Exercise 2.** For a commutative ring $R$, we define $\operatorname{Spec} R$ to be the set of all prime ideals of $R$. Find all the elements of $\operatorname{Spec} R$ when $R$ is

(a) $\mathbb{Z}/(6)$,

(b) $\mathbb{C}[x]/(x^2)$.

**Exercise 3.** Let $R$ be a commutative ring. Each $f \in R$ defines a function of $\operatorname{Spec} R$: for a prime ideal $\mathfrak{p} \subseteq R$, we denote by $\kappa(\mathfrak{p})$ the quotient field of the domain $R/\mathfrak{p}$, and we define $f(\mathfrak{p}) \in \kappa(\mathfrak{p})$ to be the image of $f$ via the canonical maps

$$R \longrightarrow R/\mathfrak{p} \longrightarrow \kappa(\mathfrak{p}).$$

What is the value of the function $15 \in \mathbb{Z}$ at the point $\mathfrak{p} \in \operatorname{Spec} \mathbb{Z}$ when $\mathfrak{p}$ is

(a) $(7)$,

(b) $(5)$?

**Exercise 4.**

(a) Consider $p \in \mathbb{C}[x]$ and $\alpha \in \mathbb{C}$. Show that there is a natural identification of $\kappa((x-\alpha))$ with $\mathbb{C}$ such that value of the function $p$ at the point $(x-\alpha) \in \operatorname{Spec} \mathbb{C}[x]$ is the number $p(\alpha)$.

(b) More generally, if $R$ is the coordinate ring of an affine variety $X$ over an algebraically closed field $K$ and $\mathfrak{p}$ is the maximal ideal corresponding to a point $x \in X$, then there is a natural identification of $\kappa(\mathfrak{p})$ with $K$ such that the value of a function $f \in R$ at $\mathfrak{p} \in \operatorname{Spec} R$ is the value $f(x)$.

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We discuss these exercises on **01.02.2018**!