# Exercise sheet 1 for
# Representations of $S_N$ and $\mathrm{GL}(V)$

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**1.** Let $\lambda, \mu \vdash N$ be partitions. We say that $\lambda \lhd_1 \mu$ iff

$$\exists\ u < v \quad (\mu_1 \dots, \mu_N) = (\lambda_1, \dots, \lambda_{u-1}, \lambda_u + 1, \lambda_{u+1}, \dots, \lambda_{v-1}, \lambda_v - 1, \lambda_{v+1}, \dots, \lambda_N).$$

Prove that

$$\lambda \unlhd \mu \Longleftrightarrow \exists\ \kappa_0, \dots, \kappa_r \vdash N : \kappa_0 = \lambda, \kappa_r = \mu,\ \forall \rho < r\ \kappa_\rho \lhd_1 \kappa_{\rho+1}.$$

**2.**$^*$ Let $\lambda, \mu \vdash_m N$. The Gale-Ryser Theorem states that the equivalence of the following two conditions:

(1) There exists $[\alpha_{ij}] \in \{0,1\}^{m \times m}$ such that $\sum_k \alpha_{ik} = \lambda_i$ and $\sum_k \alpha_{kj} = \mu_j$ for all $i, j$.
(2) $\mu \unlhd \lambda'$.

In the lecture we only showed that (1) implies (2). The reverse implication can be elegantly derived from Fulkerson's max-flow/min cut theorem. Try to derive this, if you know this theorem.

**3.** Let $V$ be an $S_N$-module. Show that

(1) The character of $V$ takes only real values.
(2) $V^* \simeq V$.

**4.** Prove from the definition of Specht modules $\mathscr{S}_\lambda$ that $\mathscr{S}_\lambda \otimes \mathbb{C}_{\mathrm{sgn}} \simeq \mathscr{S}_{\lambda'}$ for $\lambda \vdash N$.

**5.** Show that the $(2,1)$-isotypical component of the group algebra $\mathbb{C}[S_3]$ is given by

$$K = \Big\{ \sum_{\pi \in S_3} a_\pi \pi \ \Big|\ a_{\mathrm{id}} + a_{(123)} + a_{(132)} = 0, \quad a_{(12)} + a_{(13)} + a_{(23)} = 0 \Big\}.$$