Exercise sheet 2 for
Representations of S_N and $\mathrm{GL}(V)$

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6. *Kronecker coefficients* $k(\lambda,\mu,\nu)\in\mathbb{N}$ are defined as multiplicities in the tensor product decomposition of irreducible representations of symmetric groups. For $\lambda,\mu\vdash N$ we define

$$\mathscr{S}_\lambda\otimes\mathscr{S}_\mu=\bigoplus_{\nu\vdash N}k(\lambda,\mu,\nu)\mathscr{S}_\nu.$$

We denote by M^{S_N} the space of invariants of an S_N-module M. Prove that:

(1) $k((N),\mu,\nu)=0$ if $\mu\neq\nu$ and $k((N),\mu,\mu)=1$.
(2) $k(\lambda,\mu,\nu)=\dim\left(\mathscr{S}_\lambda\otimes\mathscr{S}_\mu\otimes\mathscr{S}_\nu\right)^{S_N}$.
(3) $k(\lambda,\mu,\nu)$ is invariant under permutations of the partitions λ,μ,ν.
(4) $k(\lambda',\mu',\nu)=k(\lambda,\mu,\nu)$.

7. The isotypical decomposition of $V^{\otimes 3}$ as an S_3-module is of the form

$$V^{\otimes 3}=S^3(V)\oplus\Lambda^3(V)\oplus T,$$

where $S^3(V)$ corresponds to the trivial representation, $\Lambda^3(V)$ corresponds to the sign representation, and T corresponds to the 2-dimensional irreducible S_3-representation.

(1) Determine the projection of $V^{\otimes 3}$ onto T along the isotypical decomposition (use the result from Algebra 4).
(2) Prove that a tensor $w\in V^{\otimes 3}$ lies in T if and only if $w+(123)w+(132)w=0$.
(3) Prove that $S^3(V)\oplus\Lambda^3(V)$ consists exactly of the tensors $w\in V^{\otimes 3}$ that are invariant under cyclic permutations.

8.* Let K denote the $(2,2)$-isotypical component of $\mathbb{C}[S_4]$ and $L:=\{b\in\mathbb{C}[S_4]\mid bK=0\}$ its annihilator.

(1) What are the dimension of K and L?
(2) Let V be a vector space, N be the $(2,2)$-isotypical component of $V^{\otimes 4}$, and $w\in N$. Conclude that

$$\forall b\in L\quad bw=0.$$

The Riemannian curvature tensor R is known to lie in the $(2,2)$-isotypical component of $V^{\otimes 4}$. Hence it satisfies the symmetry relations $bR=0$ forall $b\in L$.

(3) Prove that for any pairwise distinct i,j,k,ℓ

$$\mathrm{id}-(ij)(k\ell),\quad \mathrm{id}-(ijk)\in L.$$

Thus any $w\in N$ satisfies (in coordinates)

$$w_{\alpha\beta\gamma\delta}=w_{\beta\alpha\delta\gamma},\quad w_{\alpha\beta\gamma\delta}+w_{\gamma\alpha\beta\delta}+w_{\beta\gamma\alpha\delta},$$

which are known symmetry relations of the Riemannian curvature tensor.

(4) Compute a basis of K and show that $x=\sum_\pi x_\pi\pi\in K$ iff

$$x_{\mathrm{id}}=x_{(12)(34)}=x_{(13)(24)}=x_{(14)(23}$$
$$x_{(123)}=x_{(ijk)}\text{ for all 3-cycles }(ijk)$$
$$x_{(ijk\ell)}=x_{(ik)}=x_{(j\ell)}\text{ for all 4-cycles }(ijk\ell)$$
$$x_{(12)}+x_{(13)}+x_{(14)}=0.$$

(5) You may come up with a list of generators of L, which will capture all the symmetry properties of tensors in the $(2,2)$-component of $V^{\otimes 4}$.

9. Let $\lambda \vdash_m N$ and $\alpha \vdash_m N$. The *Kostka number* $K_\lambda(\alpha)$ equals the number of m-tableaux of shape λ and content α. Prove that

$$\alpha \trianglelefteq \lambda \Longrightarrow K_\lambda(\alpha) > 0.$$

10. Execute the Robinson-Schenstedt algorithm on the input word $(4, 2, 3, 6, 5, 1, 7)$.

10. Execute the Robinson-Schenstedt-Knuth algorithm:

(1) forward on the input $\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 \\ 8 & 5 & 6 & 4 & 4 & 4 & 1 & 1 & 2 & 3 \end{bmatrix}$;

(2) backword on the input $T = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$ and $S = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 3 \\ \hline 4 & 4 & 4 \\ \cline{1-3} 5 & 6 \\ \cline{1-2} 8 \\ \cline{1-1} \end{array}$.