Exercise sheet 3 for
Representations of S_N and $\mathrm{GL}(V)$

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11. Decompose the permutation module $\mathscr{M}_{(3,2,1)}$ into a direct sum of Specht modules.

12. The branching rule expresses how the Schur-Weyl module $\mathscr{W}_\lambda$ splits when the action is restricted from GL_m to the subgroup GL_{m-1}. Namely, for $\lambda \vdash_m N$:

$$\mathscr{W}_\lambda = \bigoplus_\mu \mathscr{W}_\mu,$$

where the sum is over all $\mu \vdash_{m-1}$ satisfying the interleaving property

$$\lambda_1 \geq \mu_1 \geq \ldots \geq \lambda_{m-1} \geq \mu_{m-1} \geq \lambda_m.$$

Deduce from the branching rule that $\dim \mathscr{W}_\lambda$ equals the number of m-tableaux of shape λ.

13. We call a polynomial $p \in \mathbb{C}[X_1, \ldots, X_m]$ *skew symmetric* if $\pi p = \mathrm{sgn}(\pi) p$ for all $\pi \in S_m$. For $\lambda \vdash_m N$ we consider the skew symmetric polynomial

$$f_\lambda := \det[X_i^{\lambda_j+m-j}]_{1\leq i,j\leq m}.$$

Prove that $(f_\lambda)_{\lambda \vdash_m N}$ is a basis of the vector space of skew symmetric homogeneous polynomials of degree $N + \binom{m}{2}$ in m variables.