Exercise sheet 4 for
Representations of S_N and $\mathrm{GL}(V)$

TU Berlin, Prof. Bürgisser, WS 2017/18

14. Give a direct proof of the formula

$$h_\mu = \sum_{\lambda \vdash_m N} K_\lambda(\mu) s_\lambda, \quad \mu \vdash_m N$$

using the RSK-correspondence.

15. Use the RSK-correspondence to prove Cauchy's identity:

$$\sum_{\lambda \vdash m} s_\lambda(x_1, \ldots, x_m)\, s_\lambda(y_1, \ldots, y_m) = \prod_{i=1}^{m} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j}.$$

16. Let $U \simeq V \simeq \mathbb{C}^m$. View $S^N(U \otimes V)$ as a $\mathrm{GL}(U) \times \mathrm{GL}(V)$-module via the natural action

$$(g, h)(u_1 \otimes v_1) \cdots (u_N \otimes v_N) = (g(u_1) \otimes h(v_1)) \cdots (g(u_N) \otimes h(v_N))$$

(dots denoting symmetric products). Using the Cauchy formula (previous exercise) prove that

$$S^N(U \otimes V) = \bigoplus_{\lambda \vdash_m N} \mathscr{S}_\lambda(U) \otimes \mathscr{S}_\lambda(V),$$

where the action on the right-hand side is

$$(g, h)\Big(\bigoplus_{\lambda} x_\lambda \otimes y_\lambda\Big) = \bigoplus_{\lambda} g(x_\lambda) \otimes h(y_\lambda)$$

for $x_\lambda \in \mathscr{S}(U)$ and $y_\lambda \in \mathscr{S}(V)$.

17. Let $\lambda = (\lambda_1, \lambda_2)$ and denote by f_λ the number of standard tableaux of shape λ. Derive from the hook formula that

$$f_\lambda = \frac{\lambda_1 - \lambda_2 + 1}{\lambda_1 + 1} \binom{\lambda_1 + \lambda_2}{\lambda_1}.$$

Also, give a direct proof of this by finding a recursion formula for f_λ. In the special case $m = \lambda_1 = \lambda_2$ we obtain the *Catalan numbers* $f_\lambda = \frac{1}{m+1}\binom{2m}{m}$.

18. (Applications of the Murnaghan-Nakayama rule)
Suppose $\pi \in S_N$ has ρ_1 fixed points and ρ_2 transpositions.

(1) Let $\lambda = (N-1, 1)$. Show that

$$\chi_\lambda(\pi) = \rho_1 - 1.$$

(2) Let $\lambda = (N-2, 1, 1)$. Show that

$$\chi_\lambda(\pi) = \frac{1}{2}(\rho_1 - 1)(\rho_2 - 1) - \rho_2.$$

19. Express $h_3 \cdot h_3 \cdot h_2$ as a sum of Schur polynomials using Pieri's formulas.