Invariant Theory

4h lecture at TU Berlin, SS 2018, Prof. P. Bürgisser

The course focused on the basics of linear algebraic groups and geometric invariant theory. A basic source was the book "Geometrische Invariantentheorie" by Hanspeter Kraft, Vieweg 1984. We worked over algebraically closed fields K of characteristic zero.

Chap. 1. Introduction

1.1 Finite group actions (equivalence problem, Hilbert's Finiteness Theorem

1.2 Conjugation of matrices (generating system of invariants, orbits and orbit closures, closed orbits)

Chap. 2. Linear algebraic groups

1.1 Basics (constructible subgroups are closed, decomposition into components)

1.2 Examples (GL_n, K^+, K^*, T_n, U_n, SL_n connected)

1.3 Regular operations (Orbits, irreducibility and dimension for operations and morphisms, rational modules and representations)

1.4 Examples ($K[X]_d \simeq S^d(n)$, orbits of $S^2(n)$, O_n, SO_n connected, orbits of $\Lambda^2(n)$, Sp_n connected)

1.5 Rational modules (Schur, centers, $V \oplus W$, V^*, $V \otimes W$, $S^d(V)$, $\Lambda^d(V)$)

1.6 Linearisation ($\mathcal{O}(Z)$ locally rational, every G-variety is closed G-subvariety of a rational module, every linear groups is closed subgroup of some GL_n)

Chap. 3. Linear reduktive groups

3.1 The Lie algebra of a linear group (functoriality, gl_n, sl_n, o_n, properties with respect to morphisms, connected subgroups, rational modules, normal subgroups, center)

3.2 Tori (representation theory, character group, maximal tori of T_n conjugated)

3.3 Unipotent groups (connectedness, no characters, existence of fixed points, maximal unipotent subgroups of GL_n are conjugated, orbits are closed)

3.4 Solvable groups ($[G,G]$ closed, Lie-Kolchin, Borel subgroups of GL_n are conjugated, solvalble Lie algebras)

3.5 Semisimple groups (radical, SL_n, SO_n, O_n Sp_n semisimple, semisinple Lie algebras, Weyl's Theorem, the classical groups are reductive)

3.5 Representation theory of GL_n (action of B_n on weight vectors, highest weights and highest weight vectors, classification, construction of Weyl modules, examples)

Chap. 4. Invariants

4.1 Isotypical decomposition (semisimple locally rational modules, isotypical decomposition, $W_\omega \otimes \mathrm{Hom}_G(W_\omega, V) \simeq V_{(\omega)}$, representation theory of $G \times H$, Burnside's Theorem, isotypical decomposition of $\mathcal{O}(G)$, $\mathrm{mult}_\omega(\overline{Gz}) \leq \dim \omega$)

4.2 Hilbert's Finiteness Theorem (categorical quotients, properties, $\mathcal{O}(Z)_{(\omega)}$ finitely generated $\mathcal{O}(Z)^G$-module, all $\mathrm{mult}_\omega(Z) < \infty$ iff $Z//G$ is finite)

4.3 Examples (K^n as S_n-module, reprise: conjugation of matrices, definition and characterization of the null cone of a rational module*)

4.4 Applications to group theory (product of quotients are quotients, factor groups by linear reductive groups, G linear reductive iff $\mathcal{O}(G)$ semisimple iff $\varphi(V^G) = W^G$ for all $\varphi\colon V \to W$. G linear reductive iff $N \trianglelefteq G$ linear reductive and G/N linear reductive. G reductive iff $R(G)$ is a torus. Characterisations of diagonalizable groups. Characterization of the semisimple among the linear reductive groups.)

Chap. 5. Hilbert-Mumford criterion

5.1 $\mathbb{C}$-topology (w.o.p.: $\overline{M}^{\mathbb{C}} = \overline{M}$ if M is constructible)

5.2 One parameter subgroups (pairing of $X(T), Y(T)$, $\lim_{\varepsilon \to 0} \lambda(\varepsilon) z = y$)

5.3 Intermezzo on convex cones (if $C \cap (-C) \subseteq W$, then $C \setminus W$ lies one side of a hyperplane containing W)

5.4 Hilbert criterion for tori

5.5 Hilbert criterion for GL_n (only over $\mathbb{C}$, Cartan decomposition, characterizations of null cones)

5.6 Examples of null cones (binary forms, simultaneous conjugation, simultaneous left-right action)