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# Exercises for Invariant Theory

Summer term 2018

## Exercises

**Exercise 1.** In the lecture we have seen that every rational $\mathbb{C}^\times$-modules splits into one-dimensional submodules. Does the same statement hold for *continuous* representations?
More concretely, let $V$ be a $\mathbb{C}^\times$-module with a continuous representation $D : \mathbb{C}^\times \to \mathrm{GL}(V)$. Is $V$ the sum of one-dimensional submodules?

**Exercise 2.** Show the following assertions:

(1) All continuous group homomorphisms $\mathbb{R} \to \mathbb{R}$ are of the form $x \mapsto ax$ for some $a \in \mathbb{R}$.

(2) All continuous group homomorphisms $\mathbb{R} \to S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ are of the form $x \mapsto e^{iax}$ for some $a \in \mathbb{R}$.

(3) All continuous group homomorphisms $S^1 \to S^1$ are of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.

(4) All continuous group homomorphisms $S^1 \to \mathbb{C}^\times$ are of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.

**Exercise 3.** Let $X \in \mathrm{L}(\mathrm{SO}(3))$, $X \neq 0$. Show that there is $w \in \ker X$ such that $Xv = w \times v$ holds for every $v \in \mathbb{R}^3$, where $\times$ denotes the cross product on $\mathbb{R}^3$.
Interpret $w \times v$ as an infinitesimal rotation of $v$ around the $w$-axis.

**Exercise 4.** We consider $K^{n\times n}$ as a $\mathrm{GL}_n$ module via conjugation:

$$\begin{aligned} \mathrm{GL}_n \times K^{n\times n} &\longrightarrow K^{n\times n}, \\ (g, A) &\longmapsto gAg^{-1}. \end{aligned}$$

Moreover, we consider the isotypical decomposition of $K^{n\times n}$ as a $T_n$-module.

(1) For $n = 2$, compute the $U_2$-invariant subspace of each isotypical component.

(2) Verify that each $\mathrm{GL}_2$-module generated by such a $U_2$-invariant subspace is an irreducible submodule of $K^{2\times 2}$, and that all irreducible $\mathrm{GL}_2$-submodules of $K^{2\times 2}$ are obtained like this.

    Hence, using the $U_2$-invariant subspaces of the isotypical components, we managed to compute the decomposition of $K^{2\times 2}$ into irreducible $\mathrm{GL}_2$-submodules.

(3) Can you generalize this result to arbitrary $n$?

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We discuss these exercises on **Friday**, **2018-06-01**!