Zero Cycles and Factorizable Forms

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Contents

0 Introduction **3**

1 Multisymmetric Polynomials **7**

1.1 The Field of Multisymmetric Rational Functions 12

2 Quotients of Tensor Algebras and Schur Modules **17**

2.1 Schur Modules 20

3 Zero Cycles in Projective Spaces **23**

4 When Is a Form Factorizable? **26**

4.1 Polars of Forms 27

4.2 Apolar Covariant of Forms 28

4.3 Brill's Covariant and Factorizable Forms 29

5 Interpretation of Brill's covariant **34**

5.1 The Lie Algebra $\mathfrak{sl}(2, \mathbb{C})$ and its Generators 34

5.2 Projection onto the Space of Covariants 37

6 Zusammenfassung **44**

7 Bibliography **44**

0 Introduction

"Invariant theory is the great romantic story of mathematics. For one hundred and fifty years, from its beginnings with Boole to the time, around the middle of this century, when it branched off into several independent disciplines, mathematicians of all countries were brought together by their common faith in invariants: in England, Cayley, MacMahon, Sylvester and Salmon, and later, Alfred Young, Aitken, Littlewood and Turnbull. In Germany, Clebsch, Gordan, Grassmann, Sophus Lie, Study; in France, Hermite, Jordan and Laguerre; in Italy, Capelli, d'Ovidio, Brioschi, Trudi and Corrado Segre; in America, Glenn, Dickson, Carus (of the Carus Monographs), Eric Temple Bell and later Hermann Weyl. Seldom in history has an international community of scholars felt so united by a common scientific ideal for so long a stretch of time. In our century, Lie theory and algebraic geometry, differential algebra and algebraic combinatorics are offsprings of invariant theory. No other mathematical theory, with the exception of the theory of functions of a complex variable, has had as deep and lasting an influence on the development of mathematics."

— G.-C. Rota, *What is invariant theory, really?*

Consider a binary quadratic form where we take the coefficients to be complex numbers

$$\begin{aligned} Q(x,y) &= ax^2 + 2bxy + cy^2 \\ &= [x\ y]\begin{bmatrix} a & b \\ b & c \end{bmatrix}[x\ y]^T. \end{aligned}$$

The discriminant of Q is defined to be $\Delta := ac - b^2$. It is the determinant of the symmetric matrix associated to Q. Now if we do any (complex) invertible linear substitution of variables

$$\bar{x} = a_{11}x + a_{12}y, \quad \bar{y} = a_{21}x + a_{22}y, \quad a_{11}a_{22} - a_{12}a_{21} \neq 0,$$

it will map Q to a new binary quadratic form $\bar{Q}$ according to

$$\bar{Q}(\bar{x},\bar{y}) = \bar{Q}(a_{11}x + a_{12}y, a_{21}x + a_{22}y) = Q(x,y).$$

We can write this new form as follows

$$\bar{Q}(\bar{x},\bar{y}) = \bar{a}\bar{x}^2 + 2\bar{b}\bar{x}\bar{y} + \bar{c}\bar{y}^2.$$

Its discriminant is then $\bar{\Delta} = \bar{a}\bar{c} - \bar{b}^2$. It is a straightforward verification that

$$\Delta = ac - b^2 = (a_{11}a_{22} - a_{12}a_{21})^2(\bar{a}\bar{c} - \bar{b}^2) = (a_{11}a_{22} - a_{12}a_{21})^2\bar{\Delta}.$$

So up to a scalar factor the discriminant Δ remains invariant under invertible linear transformations. Under the action of invertible linear substitutions with determinant 1, it is invariant in the "true" sense of the word.

The discriminant Δ can be thought as a polynomial in three variables taking values in the coefficients of a binary quadratic form. It is an "intrinsic" property of binary quadratic forms, unaffected by change of variables. If Δ is nonzero, then Q can be written as a product of two linearly indepenedent factors. If it is 0, then the factors are dependent. The discriminant motivates the general definition of an invariant in the case of binary forms of any degree.

Let k be a nonnegative integer. A nonconstant polynomial $I(A_0, A_1, \ldots, A_d)$ in the variables $A_0, A_1, \ldots, A_d$, which takes values in the coefficients of binary forms of degree d, is said to be an ***invariant*** of index k if for all binary forms $Q(x,y)$ of degree d and for all invertible linear substitutions of variables, the following identity holds

$$I(A_0, A_1, \ldots, A_d) = (a_{11}a_{22} - a_{12}a_{21})^k I(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_d).$$

Let us now consider a binary cubic form

$$Q(x, y) = ax^3 + 3bx^2y + 3cxy^2 + dy^3.$$

It also has an invariant called the discriminant of the cubic Q, which is given by

$$\Delta := a^2d^2 - 6abcd + 4ac^3 - 3b^2c^2 + 4b^3d.$$

A tedious computation shows that it is an invariant of index 6. This discriminant is 0 if and only if Q has a factor of multiplicity greater than 1.

But invariants are not sufficient to capture all the intrinsic properties of binary forms. We therefore look for polynomials that not only depend on the coefficients of a binary form, but also on the independent variables x and y.

Consider the following Hessian of a binary form Q of any degree

$$H := \frac{\partial^2 Q}{\partial x^2}\frac{\partial^2 Q}{\partial y^2} - \frac{\partial^2 Q}{\partial x \partial y}.$$

In the case of the binary cubic, the Hessian comes out to be the following quadratic polynomial

$$\frac{1}{36}H = (bd - c^2)x^2 + (ad - bc)xy + (ac - b^2)y^2.$$

Again using a straightforward verification it can be shown that under any invertible linear substitution of variables, the Hessian $\bar{H}$ of the new binary cubic form $\bar{Q}$ is related to H via

$$H = (a_{11}a_{22} - a_{12}a_{21})^2\bar{H}.$$

Like the discriminant, the Hessian also tells us about the factorizability properties of binary forms, namely, the Hessian of a binary form of degree d vanishes, $H \equiv 0$, if and only if it is the d-th power of a linear form, that is, $Q(x, y) = (c'x + d'y)^d$. The Hessian is an example of something we call a covariant of index 2.

Let k be a nonnegative integer. A nonconstant polynomial $J(A_0, A_1, \ldots, A_d, x, y)$ in the variables $A_0, A_1, \ldots, A_d$, x and y, which takes values in the coefficients of binary forms of degree d and also depends on the independent variables x and y, is said to be a ***covariant*** of index k if for all binary forms $Q(x, y)$ of degree d and for all invertible linear substitutions of variables, the following identity holds

$$J(A_0, A_1, \ldots, A_d, x, y) = (a_{11}a_{22} - a_{12}a_{21})^k J(\bar{A}_0, \bar{A}_1, \ldots, \bar{A}_d, \bar{x}, \bar{y}).$$

Invariants are just covariants that do not explicity depend on x and y. It is the basic objective of classical invariant theory to determine explicitly as possible all covariants of binary forms.

In the case of a binary quadratic form it can be proven that the discriminant is the only *fundamental* invariant. Any other invariant is a power Δ^m of the discriminant. Also, any covariant of a binary quadratic form Q is given by $J = \Delta^m Q^n$. A binary cubic, on the other hand, has 4 fundamental covariants: the form Q itself, its Hessian H, the cubic discriminant Δ and the Jacobi covariant T which is given by

$$T := \frac{\partial Q}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial Q}{\partial y}\frac{\partial H}{\partial x}.$$

Any other covariant of the binary cubic is a polynomial in these 4 covariants.

Hilbert proved in his famous basis theorem that any finite system of binary forms[1], admits a finite list of fundamental covariants, such that all covariants of this system can

[1] In fact Hilbert proved it for forms in any number of variables but we first need to define what we mean by an invariant or a covariant in the multivariate context. For an introduction to classical invariant theory we refer the reader to [**29**].

be written as polynomials in these fundamental covariants. Today most of us are only aware of the reformulation of Hilbert's basis theorem which states that a polynomial ring over a Noetherian ring is also Noetherian.

We now address the question of factorizability in the case of a quadratic form in any number of variables. As we will see, by using a suitable linear transformation we can reduce it to a form from which it is easy to deduce the factors. But before we do that, for the sake of clarity, let us define the basic terms.

A ***form*** is a homogenous polynomial of degree d in n variables where, depending on what we are interested in studying, we may take the coefficients to be real numbers, integers, elements of a finite field, and so on. We concern ourselves **only** with the complex case. A form of degree 1 is called a ***linear form*** and a form of degree 2 is called a ***quadratic form***. A form is said to be ***factorizable*** if it can be written as a product of linear forms.

Now for quadratic forms in general, we have

$$Q(x_1, \ldots, x_n) = \sum_{1\leq i,j\leq n} a_{ij}x_ix_j.$$

For $i \neq j$, the coefficient of $x_ix_j (= x_jx_i)$ is $a_{ij} + a_{ji}$. If choose the value of both a_{ij} and a_{ji} to be equal to $\dfrac{a_{ij} + a_{ji}}{2}$ (if we are considering forms over more general fields then it must have $2 \neq 0$ for this to work), then we can associate to Q a complex $n \times n$ symmetric matrix A, such that

$$Q(x_1, \ldots, x_n) = [x_1, \ldots, x_n]A[x_1, \ldots, x_n]^T.$$

Theorem 0.1. *Q is factorizable if and only if rank$(A) \leq 2$.*

Proof. We first show there is a (complex) invertible linear substitution $y = Mx$ such that one obtains the following canonical form

$$Q'(y_1, \ldots, y_n) = d_1y_1^2 + \cdots + d_ny_n^2,$$

with $Q(x_1, \ldots, x_n) = Q'(y_1, \ldots, y_n)$.

To prove this we use induction on n. For $n = 1$, it holds. Now assume it is true for quadratic forms in $n-1$ variables. We have

$$\begin{aligned} Q(x_1, \ldots, x_n) &= \sum_{1\leq i,j\leq n} a_{ij}x_ix_j \\ &= a_{11}x_1^2 + 2\sum_{2\leq j\leq n} a_{ij}x_1x_j + \sum_{2\leq i,j\leq n} a_{ij}x_ix_j. \end{aligned}$$

Assume one of the a_{ii}'s is not equal to 0, say a_{11}. Let $y_1 = x_1 + \sum_{2\leq j\leq n} \dfrac{a_{1j}}{a_{11}}x_j$. Then

$$y_1^2 = x_1^2 + (\sum_{2\leq j\leq n} \frac{a_{1j}}{a_{11}}x_j)^2 + 2\sum_{2\leq j\leq n} \frac{a_{1j}}{a_{11}}x_1x_j,$$

which implies

$$a_{11}y_1^2 = a_{11}x_1^2 + a_{11}(\sum_{2\leq j\leq n} \frac{a_{1j}}{a_{11}}x_j)^2 + 2\sum_{2\leq j\leq n} a_{1j}x_1x_j.$$

Substituting this in $Q(x_1, \ldots, x_n)$ we get

$$Q(x_1, \ldots, x_n) = a_{11}y_1^2 - a_{11}(\sum_{2\leq j\leq n} \frac{a_{1j}}{a_{11}}x_j)^2 + \sum_{2\leq i,j\leq n} a_{ij}x_ix_j.$$

The terms on the right of $a_{11}y_1^2$ are quadratic forms in $n-1$ variables, so we can use the induction hypothesis and we get the result in the case when one of the diagonal elements is not equal to 0. If all the diagonal elements are equal to 0, but one of $a_{ij},\ i \neq j$ is not, then since

$$x_i x_j = \frac{1}{4}\Big((x_i + x_j)^2 - (x_i - x_j)^2\Big),$$

using the linear substitution $y_i = \frac{1}{2}(x_i + x_j)$ and $y_j = \frac{1}{2}(x_i - x_j)$, we obtain a quadratic form which contains a diagonal element not equal to 0. So we return back to the first case.

Now suppose $\mathrm{rank}(A) = 1$, then there exists an invertible matrix M such that $Q'(y_1, \ldots, y_n) = d_1 y_1^2$, where $y = Mx$ and $d_1 \neq 0$. So $Q(x_1, \ldots, x_n) = d_1(Mx_1)^2$. If $\mathrm{rank}(A) = 2$, then then there exists an M such that $Q'(y_1, \ldots, y_n) = d_1 y_1^2 + d_2 y_2^2$, where $y = Mx$ and $d_1, d_2 \neq 0$. Now

$$\begin{aligned} Q'(y_1, \ldots, y_n) &= d_1 y_1^2 + d_2 y_2^2 \\ &= d_1(y_1^2 - \frac{-d_2}{d_1} y_2^2) \\ &= d_1(y_1^2 - k y_2^2), \quad \text{where } k^2 = \frac{-d_2}{d_1} \\ &= (d_1 y_1 - d_1 k y_2)(y_1 + k y_2). \end{aligned}$$

So $Q(x_1, \ldots, x_n) = (d_1 M x_1 - d_1 k M x_2)(M x_1 + k M x_2)$. So in both cases of $\mathrm{rank}(A) = 1$ or 2, we see that Q can be factored.

Conversely, suppose Q factors such that

$$Q(x_1 \ldots, x_n) = a(a_1 x_1 + \cdots + a_n x_n)^2, \quad a \neq 0.$$

Now one of the a_i's is not equal to 0, say a_1. Then by the invertible linear substitution $y_1 = a_1 x_1 + \cdots + a_n x_n$ and $y_i = x_i$ for $2 \leq i \leq n$, we get

$$Q'(y_1 \ldots, y_n) = a(y_1)^2,$$

so in this case the rank is equal to 1. Now suppose Q decomposes into linearly indepedent factors

$$Q(x_1 \ldots, x_n) = a(a_1 x_1 + \cdots + a_n x_n)(b_1 x_1 + \cdots + b_n x_n), \quad a \neq 0,$$

which is equivalent to saying the vectors $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are linearly independent. Permuting indices if necessary, we can assume (a_1, a_2) and (b_1, b_2) are linearly independent. Then by the invertible linear substitution

$$y = \begin{bmatrix} a_1 & a_2 & \cdots & \cdots & \cdots & a_n \\ b_1 & b_2 & \cdots & \cdots & \cdots & b_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} x,$$

we get the quadratic form

$$Q'(y_1 \ldots, y_n) = a y_1 y_2,$$

so the rank is equal to 2. □

Remark 0.1. The arguments in this proof hold for all fields with characteristic $2 \neq 0$ and where it possible to find certain square roots, otherwise we have to work in the quadratic extension.

What about forms of higher degrees? It is natural to ask: Under what conditions is a form of any degree in any number of variables factorizable? Brill provided an answer to this question using the methods of invariant theory [**4**]. He produced a set of covariants whose simultaneous vanishing is a necessary and sufficient condition for factorizability of a form. Gordan [**14**] identified in Brill's set of covariant a particular one, henceforth named Brill's covariant, which already provides the necessary and sufficient condition for factorizability. A summary of Gordan's presentation of Brill's covariant can be found in [**3**]. In this thesis we will give the modern representation theoretic interpretation of Brill's covariant which is due to [**13**].

From an algebraic geometric point of view, the projectivization of the space of all factorizable forms of degree d in $n+1$ variables is a variety, namely, the Chow variety of 0-cycles of degree d in a complex projective space of dimension n. We denote it by $\mathrm{Chow}(d,n)$. Studying the conditions under which a form is factorizable is equivalent to looking for the equations defining $\mathrm{Chow}(d,n)$ set-theoretically.

In Section 1, we introduce multisymmetric polynomials over fields with characteristic 0, which we will later require in Section 3. They offer a generalization of the usual symmetric polynomials and can be thought as polynomials in vector variables which remain unchanged under permutation of these variables. We will prove a generalization of the fundamental theorem of symmetric polynomials. We then study the field of fractions of the ring of multisymmetric polynomials which turns out to be a purely transcendental extension of its ground field. In the complex case, we can make an even more precise statement about the transcendence base.

In Section 2, we introduce Schur modules which we will later require in interpreting the apolar covariant as they help decompose representations of linear groups. But we need to set up the basic machinery first. So at the beginning we quickly go over the notion of a tensor algebra and other algebras, like symmetric and exterior algebra, derived from it.

In Section 3, we first introduce the Grassmannian, which helps motivate the notion of an algebraic cycle and that of a Chow variety. The Chow variety basically allows us to view the set of algebraic cycles as a variety by embedding it into a projective space, but we will not concern ourselves with this general notion. We are only interested in the specific case $\mathrm{Chow}(d,n)$. By using what we proved in Section 1, we will show that $\mathrm{Chow}(d,n)$ is isomorphic to the variety $\mathrm{Sym}^d(\mathbb{P}^n)$ and that it is a rational variety.

In Section 4 we finally give the explicit description of Brill's covariant, but first we need to introduce polars of a form and the apolar covariant, which are essential tools in constructing Brill's covariant.

In Section 5, we interpret the formula for the apolar covariant in the language of modern representation theory. With the help of the Schur module decomposition stated in Section 4 and examining the action of the generators of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ on the space of bihomogenous forms, we will arrive at the formula for the apolar covariant via a projection map.

Note: Throughout this thesis we denote a field with characteristic 0 by $\mathbb{K}$, though some results may apply to more general fields. We denote the symmetric group on d letters by S_d. By a variety, we mean a quasi-projective variety unless specified. And just to avoid confusion, by a positive integer, we mean an integer greater than 0.

1 Multisymmetric Polynomials

A polynomial $f \in \mathbb{K}[x_1, \ldots, x_d]$ is called a ***symmetric polynomial*** if it remains unchanged under permutation of the variables. That is,

$$f(x_1, \ldots, x_d) = f(x_{\sigma(1)}, \ldots, x_{\sigma(d)}) \text{ for all } \sigma \in S_d.$$

For example, if the variables are x_1 and x_2, then $x_1 + x_2$ and $x_1^2 x_2^2$ are symmetric polynomials.

Let

$$X := \begin{bmatrix} x_{11} & x_{12} & x_{13} & \dots & x_{1n} \\ x_{21} & x_{22} & x_{23} & \dots & x_{2n} \\ \multicolumn{5}{c}{\dotfill} \\ x_{d1} & x_{d2} & x_{d3} & \dots & x_{dn} \end{bmatrix}$$

be a matrix of indeterminates. Let S_d act on it by permuting the rows. A polynomial $f \in \mathbb{K}[x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{dn}]$ is called a ***multisymmetric polynomial*** if it remains unchanged under this action of S_d. That is,

$$f(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{dn}) = f(x_{\sigma(1)1}, \dots, x_{\sigma(1)n}, x_{\sigma(2)1}, \dots, x_{\sigma(d)n}) \text{ for all } \sigma \in S_d.$$

For example, if $n = 2$ and $d = 2$, then $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$. Let $f = x_{11}x_{22} + x_{12}x_{21}$. For $\sigma = (12) \in S_2$, $x_{\sigma(1)1}x_{\sigma(2)2} + x_{\sigma(1)2}x_{\sigma(2)1} = x_{21}x_{12} + x_{22}x_{11}$, so f is multisymmetric.

Multisymmetric polynomials are basically polynomials in vector variables that remain unchanged under permutation of these variables. If $n = 1$, we get back the symmetric polynomials in scalar variables. So this provides a generalisation.

Note: From now on, we will write symmetric instead of multisymmetric.

For a monomial in $\mathbb{K}[x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{dn}]$, we can construct its ***exponent matrix*** by replacing each x_{ij} in the matrix X by its power in the monomial. For example, if we have the monomial $x_{11}^2 x_{22} = x_{11}^2 x_{12}^0 x_{21}^0 x_{22}^1$, then its exponent matrix is given by $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. Note that $x_{11}^2 x_{22}$ is not symmetric. We can make it symmetric by addding to it monomials corresponding to all possible permutations of its exponent matrix. Now the monomial corresponding to $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ is $x_{12}x_{21}^2$. So we get $x_{11}^2 x_{22} + x_{12}x_{21}^2$, which is a symmetric polynomial. We call symmetric polynomials of this type ***monomial symmetric polynomials***.

For each $\omega \in \mathbb{Z}_{\geq 0}^{d \times n}$, we have a corresponding monomial $\mathbf{x}^\omega := \prod_{\substack{1 \leq i \leq d \\ 1 \leq j \leq n}} x_{ij}^{\omega_{ij}}$. We can also write it as $\mathbf{x}_1^{\omega(1)} \cdots \mathbf{x}_d^{\omega(d)}$, where $\omega(1), \dots, \omega(d) \in \mathbb{Z}_{\geq 0}^n$ are rows of ω and $\mathbf{x}_i^{\omega(i)} := x_{i1}^{w_{i1}} \cdots x_{in}^{w_{in}}$.

For $\lambda \in \mathbb{Z}_{\geq 0}^{d \times n}$, let $m_\lambda := \sum_{\omega \in S_d \cdot \lambda} \mathbf{x}^\omega$ be the sum of all possible monomials corresponding to permutation of the rows of λ under the action of S_d. Now m_λ is a monomial symmetric polynomial and $\{m_\lambda\}$, where λ runs over some system of representatives of the orbit set $\mathbb{Z}_{\geq 0}^{d \times n}/S_d$, forms a basis of the space of symmetric polynomials over $\mathbb{K}$. There is no natural choice of basis. For example, if $n = 1$, we have $\lambda = [\lambda(1), \dots, \lambda(d)]^T$, and we could choose λ such that $\lambda(1) \geq \dots \geq \lambda(d)$ or $\lambda(1) \leq \dots \leq \lambda(d)$.

We now consider two types of monomial symmetric polynomials: the elementary symmetric polynomials and the power sum symmetric polynomials.

The ***elementary symmetric polynomials*** are those m_λ, where λ has at most one nonzero entry in each row and this entry is 1 if it exists. These polynomials are uniquely determined by the column sums of λ. So instead of m_λ, we write $e_{k_1, \dots, k_n}$, where k_j is the sum of the entries in the j-th column of λ. Note that $0 \leq k_1 + \dots + k_n \leq d$, and if b_j represents the standard basis vector $(0, \dots, 1, \dots, 0)$, where 1 is in the j-th position, then λ has k_1 rows equal to b_1, k_2 rows equal to $b_2, \dots,$ and $d - (k_1 + \dots + k_n)$ rows equal to 0. In the scalar case $n = 1$, we get back the usual elementary symmetric polynomials. If $n = 2$ and $d = 2$, we have $e_{0,0} = 1$, $e_{0,1} = x_{12} + x_{22}$, $e_{1,0} = x_{11} + x_{21}$, $e_{1,1} = x_{11}x_{22} + x_{12}x_{21}$, $e_{2,0} = x_{11}x_{21}$ and $e_{0,2} = x_{12}x_{22}$.

Consider the polynomial $\prod_{1 \le i \le d} (1 + x_i t)$, where we have introduced a new variable t. This polynomial remains unchanged under permutation of $x_1, \dots, x_n$. We have

$$\prod_{1 \le i \le d} (1 + x_i t) = 1 + \sigma_1 t^1 + \cdots + \sigma_d t^d,$$

where σ_i is a polynomial in $x_1, \dots, x_n$. Since the left-hand side remains unchanged under permutation, the coefficients of powers of t also remain unchanged, so they are symmetric polynomials. σ_k, the coefficient of t^k, must contain the term $x_1 x_2 \cdots x_k$ and therefore also permutations of it. Since the left-hand side only has 1 as its constant coefficient, the same is true for σ_k. Therefore σ_k is in fact the usual elementary symmetric polynomial e_k.

We generalize this by considering

$$\prod_{1 \le i \le d} (1 + x_{i1} t_1 + x_{i2} t_2 + \cdots + x_{in} t_n).$$

This polynomial remains unchanged under the permutation $x_{\sigma(i)1}, \dots, x_{\sigma(i)n}$, for all $\sigma \in S_d$. The coefficents $\sigma_{k_1,\dots,k_n}$ of $t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}$ are then symmetric polynomials by the same argument as above. $\sigma_{k_1,\dots,k_n}$ must contain the monomial term whose exponent matrix has k_1 rows equal to b_1, k_2 rows equal to $b_2, \dots,$ and $d - (k_1 + \cdots + k_n)$ equal to zero. Since it is symmetric, it must contain all possible permutations of it. Also, it has only 1 as its constant coefficient by the same argument as above. Therefore it is in fact the elementary symmetric polynomial $e_{k_1,\dots,k_n}$. So we have

$$\prod_{1 \le i \le d} (1 + x_{i1} t_1 + x_{i2} t_2 + \cdots + x_{in} t_n) = 1 + \sum_{k_1,\dots,k_n} e_{k_1,\dots,k_n} t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n}, \tag{1.1}$$

where the sum is over all multi-indices $(k_1, \dots, k_n)$ with $1 \le k_1 + \cdots + k_n \le d$. For example, if $n = 2$ and $d = 2$, we have

$$\begin{aligned}(1 + x_{11}t_1 + x_{12}t_2)(1 + x_{21}t_1 + x_{22}t_2) &= 1 + (x_{11} + x_{21})t_1 + (x_{12} + x_{22})t_2 \\ &\quad + (x_{21}x_{12} + x_{11}x_{22})t_1 t_2 + x_{11}x_{21}t_1^2 + x_{12}x_{22}t_2^2 \\ &= 1 + e_{1,0}t_1 + e_{0,1}t_2 + e_{1,1}t_1 t_2 + e_{2,0}t_1^2 + e_{0,2}t_2^2.\end{aligned}$$

We want to show that any symmetric polynomial can be written as a polynomial in the elementary symmetric polynomials, but before that we need to introduce power sum symmetric polynomials.

The ***power sum symmetric polynomials*** are those monomial symmetric polynomials m_λ, where λ has at most one nonzero row. For each $\rho \in \mathbb{Z}^n_{\ge 0}$, we denote by p_ρ those m_λ, where λ is a matrix with one row equal to ρ and the rest equal to 0. So we have

$$p_\rho := \mathbf{x}_1^\rho + \cdots + \mathbf{x}_d^\rho.$$

If $n = 1$, this reduces to ***power sums*** $p_r = x_1^r + \cdots + x_d^r$, $r \ge 0$. For each matrix $\lambda \in \mathbb{Z}^{d \times n}_{\ge 0}$, we set $p_\lambda := p_{\lambda(1)} \cdots p_{\lambda(d)}$, where $\lambda(1), \dots, \lambda(d)$ are the rows of λ.

Proposition 1.1. *$\{p_\lambda\}$, where λ runs over some system of representatives of the orbit set $\mathbb{Z}^{d \times n}_{\ge 0}/S_d$, forms a basis of the space of symmmetric polynomials over $\mathbb{K}$.*

Proof. We choose λ such that the nonzero rows are at the top. We number the entries from left to right and top to bottom. On such λ's we define an ordering such that for

$\lambda_1,\lambda_2 \in \mathbb{Z}_{\geq 0}^{d\times n}$, $\lambda_1 > \lambda_2$ if and only if in $\lambda_1 - \lambda_2$, the rightmost nonzero entry is positive. This is called the inverse lexicographic ordering. For $\lambda \in \mathbb{Z}_{\geq 0}^{d\times n}$, let $l(\lambda)$ denote the number of nonzero rows in λ. If $l(\lambda_1) > l(\lambda_2)$ then $\lambda_1 > \lambda_2$.

Let $d = 2$, then

$$\begin{aligned} p_\lambda &= (\mathbf{x}_1^{\lambda(1)} + \mathbf{x}_2^{\lambda(1)})(\mathbf{x}_1^{\lambda(2)} + \mathbf{x}_2^{\lambda(2)}) = \mathbf{x}_1^{\lambda(1)+\lambda(2)} + \mathbf{x}_1^{\lambda(1)}\mathbf{x}_2^{\lambda(2)} + \mathbf{x}_2^{\lambda(1)}\mathbf{x}_1^{\lambda(2)} + \mathbf{x}_2^{\lambda(1)+\lambda(2)} \\ &= m_\lambda + \mathbf{x}_1^{\lambda(1)+\lambda(2)} + \mathbf{x}_2^{\lambda(1)+\lambda(2)}. \end{aligned}$$

So p_λ is equal to m_λ plus linear combination of monomial symmetric polynomials whose exponent matrices have less nonzero rows than λ, so they are less than λ with respect to the inverse lexicographic ordering. For $\lambda \in \mathbb{Z}_{\geq 0}^{d\times n}$, by generalizing this argument we get

$$\begin{aligned} p_\lambda = p_{\lambda(1)} \cdots p_{\lambda(d)} &= (\mathbf{x}_1^{\lambda(1)} + \cdots + \mathbf{x}_d^{\lambda(1)}) \cdots (\mathbf{x}_1^{\lambda(d)} + \cdots + \mathbf{x}_d^{\lambda(d)}) \\ &= c_\lambda m_\lambda + (\text{linear combinations of } m_\mu \text{ with } l(\mu) < l(\lambda)), \end{aligned}$$

where $c_\lambda > 0$. Let $\{p_\mu\}$ and $\{m_\mu\}$be sets indexed by $\mu \leq \lambda$. Both of these are finite sets. By the reasoning above we have

$$\begin{bmatrix} * & m_\mu & * & m_\lambda \end{bmatrix} \begin{bmatrix} * & * & * & * & * \\ & c_\mu & * & * & * \\ & & * & * & * \\ & 0 & & * & * \\ & & & & c_\lambda \end{bmatrix} = \begin{bmatrix} * & p_\mu & * & p_\lambda \end{bmatrix},$$

where on the left we have an upper triangular matrix with nonzero diagonal entries. It is therefore invertible, and acts as the transition matrix from $\{m_\mu\}$ to $\{p_\mu\}$. □

In the scalar case of two variables, we have $(1 + x_1t)(1 + x_2t) = 1 + e_1t + e_2t^2$. We take the log on both sides. On the left we have

$$\log(1 + x_1t)(1 + x_2t) = \log(1 + x_1t) + \log(1 + x_2t),$$

where by the Taylor expansion of log we get

$$\begin{aligned} \log(1 + x_1t) + \log(1 + x_2t) &= \sum_{m=1}^{\infty} (-1)^{m+1} \frac{(x_1^m + x_2^m)t^m}{m} \\ &= \sum_{m=1}^{\infty} (-1)^{m+1} \frac{p_m(x_1, x_2)t^m}{m}. \end{aligned}$$

On the right we get a series in t whose coefficients are polynomials in e_1 and e_2. Comparing the coefficients of powers of t on both sides, we see that the power sums $p_m(x_1, x_2)$ can be written as polynomials in the elementary symmetric polynomials e_1 and e_2. By generalizing this argument we get the following

Proposition 1.2. *The power sum symmetric polynomials can be written as polynomials in the elementary symmetric polynomials over $\mathbb{K}$.*

Proof. Let $y_i = x_{i1}t_1 + \cdots + x_{in}t_n$, $1 \leq i \leq d$. Then by Taylor expansion we have

$$\log(1 + x_{i1}t_1 + \cdots + x_{in}t_n) = \log(1 + y_i) = y_i - \frac{{y_i}^2}{2} + \cdots,$$

where

$$y_i^m = (x_{i1}t_1 + \cdots + x_{in}t_n)^m = \sum_{k_1+\cdots+k_n=m} \binom{m}{k_1, k_2, \ldots, k_n} (x_{i1}t_1)^{k_1} \cdots (x_{in}t_n)^{k_n}$$
$$= \sum_{k_1+\cdots+k_n=m} \frac{m!}{k_1!\, k_2! \cdots k_n!} x_{i1}^{k_1} \cdots x_{in}^{k_n} t_1^{k_1} \cdots t_n^{k_n}.$$

Now

$$\log(1+y_i) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{y_i^m}{m}$$
$$= \sum_{m=1}^{\infty} [\sum_{k_1+\cdots+k_n=m} \frac{m!(-1)^{m+1}}{k_1!\, k_2! \cdots k_n! m} x_{i1}^{k_1} \cdots x_{in}^{k_n} t_1^{k_1} \cdots t_n^{k_n}]$$
$$= \sum_{0 \neq \rho = (k_1, \ldots, k_n) \in \mathbb{Z}^n_{\geq 0}} \frac{|\rho|!(-1)^{|\rho|+1}}{k_1!\, k_2! \cdots k_n! |\rho|} x_{i1}^{k_1} \cdots x_{in}^{k_n} t_1^{k_1} \cdots t_n^{k_n}$$
$$= \sum_{0 \neq \rho \in \mathbb{Z}^n_{\geq 0}} \frac{|\rho|!(-1)^{|\rho|+1}}{k_1!\, k_2! \cdots k_n! |\rho|} \mathbf{x}_i^{\rho} t_1^{k_1} \cdots t_n^{k_n}.$$

Substituting this in $\log \prod_{1 \leq i \leq d} (1+y_i)$ we get

$$\log \prod_{1 \leq i \leq d} (1 + x_{i1}t_1 + x_{i2}t_2 + \cdots + x_{in}t_n) = \sum_{i=1}^{d} \log(1 + x_{i1}t_1 + x_{i2}t_2 + \cdots + x_{in}t_n)$$
$$= \sum_{0 \neq \rho \in \mathbb{Z}^n_{\geq 0}} \frac{|\rho|!(-1)^{|\rho|+1}}{k_1!\, k_2! \cdots k_n! |\rho|} (\mathbf{x}_1^{\rho} + \cdots + \mathbf{x}_d^{\rho}) t_1^{k_1} \cdots t_n^{k_n}$$
$$= \sum_{0 \neq \rho \in \mathbb{Z}^n_{\geq 0}} \frac{|\rho|!(-1)^{|\rho|+1}}{k_1!\, k_2! \cdots k_n! |\rho|} p_\rho t_1^{k_1} \cdots t_n^{k_n}.$$

So the p_ρ are obtained up to rational factors as coefficients in the taylor expansion of $\log \prod_{1 \leq i \leq d} (1 + x_{i1}t_1 + x_{i2}t_2 + \cdots + x_{in}t_n)$ which is equal to $\log(1 + \sum_{k_1, \ldots k_n} e_{k_1, \ldots, k_n} t_1^{k_1} t_2^{k_2} \cdots t_n^{k_n})$ by (1.1). That is, the p_ρ are obtained as polynomials in $e_{k_1, \ldots, k_n}$. □

Remark 1.1. Note that the field $\mathbb{K}$ must have charateristic 0 in order to contain the rational factors.

As a consequence of both of the previous propositions we get the following generalisation of the fundamental theorem of symmetric polynomials

Theorem 1.1. *Any symmetric polynomial (in vector variables) can be expressed as a polynomial in the elementary symmetric polynomials over* $\mathbb{K}$.

Remark 1.2. This expression is not necessarily unique. We will go into more details shortly.

If $n = 1$ in the last equation, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} p_k(x_1, \ldots, x_d) t^k = \log[1 + \sum_{i=1}^{d} e_i(x_1, \ldots, x_d) t^i].$$

Now

$$(e_1t^1+\cdots+e_dt^d)^m = \sum_{k_1+\cdots+k_d=m} \frac{m!}{k_1!\,k_2!\cdots k_d!} e_1^{k_1}\cdots e_d^{k_d} t^{k_1}t^{2k_2}\cdots t^{dk_d},$$

therefore

$$\begin{aligned}\log[1+(e_1t^1+\cdots+e_dt^d)] &= \sum_{m=1}^{\infty}\Big[\sum_{k_1+\cdots+k_d=m} \frac{m!(-1)^{m+1}}{k_1!\,k_2!\cdots k_d!m} e_1^{k_1}\cdots e_d^{k_d} t^{k_1}t^{2k_2}\cdots t^{dk_d}\Big] \\ &= \sum_{0\neq m=(k_1,\ldots,k_d)\in\mathbb{Z}^n_{\geq 0}} \frac{|m|!(-1)^{|m|+1}}{k_1!\,k_2!\cdots k_d!|m|} e_1^{k_1}\cdots e_d^{k_d} t^{k_1}t^{2k_2}\cdots t^{dk_d}.\end{aligned}$$

Comparing the coefficients of t^k, we get

$$\frac{(-1)^{k+1}}{k} p_k(x_1,\ldots,x_d)t^k = \sum_{k_1+2k_2+\cdots+dk_d=k} \frac{(k_1+\cdots+k_d)!(-1)^{(k_1+\cdots+k_d)+1}}{k_1!\,k_2!\cdots k_d!(k_1+\cdots+k_d)} e_1^{k_1}\cdots e_d^{k_d} t^k,$$

which is equivalent to

$$p_k(x_1,\ldots,x_d) = (-1)^k k \sum_{k_1+2k_2+\cdots+dk_d=k} \frac{(k_1+\cdots+k_d-1)!(-1)^{k_1+\cdots+k_d}}{k_1!\,k_2!\cdots k_d!} e_1^{k_1}\cdots e_d^{k_d}. \quad (1.2)$$

This is known as the ***Newton-Girard formula***.

Remark 1.3. In the above equation e_1^k occurs in p_k with nonzero coefficient 1. In a grading such that $\deg(e_i)=i$, p_k is a homogenous polynomial of degree k.

For example, if d and $k=2$, we have

$$p_2(x_1,x_2) = (-1)^2\cdot 2 \sum_{k_1+2k_2=2} \frac{(k_1+k_2-1)!(-1)^{(k_1+k_2)}}{k_1!k_2!} e_1^{k_1}e_2^{k_2}.$$

$k_1+2k_2=2$ implies $k_1=2$ and $k_2=0$ or $k_1=0$ and $k_2=1$, so we have

$$2\cdot\frac{(2+0-1)!(-1)^{(2+0)}}{2!0!}e_1^2e_2^0 + 2\cdot\frac{(0+1-1)!(-1)^{(0+1)}}{0!1!}e_1^0e_2^1,$$

and we get $p_2(x_1,x_2) = x_1^2+x_2^2 = (x_1+x_2)^2-2x_1x_2$.

1.1 The Field of Multisymmetric Rational Functions

Fix d. Instead of the polynomial ring $\mathbb{K}[\mathbf{X}]_n := \mathbb{K}[x_{11},\ldots,x_{1n},x_{21},\ldots,x_{dn}]$, we now consider its field of fractions $\mathbb{K}(\mathbf{X})_n := \mathbb{K}(x_{11},\ldots,x_{1n},x_{21},\ldots,x_{dn})$. Let S_d act on it in the same way as described in the beginning of Section 1. The set of elments in $\mathbb{K}(\mathbf{X})_n$ invariant under this action is again a field. We denote this invariant subfield by $\mathbb{K}(\mathbf{X})_n^{S_d}$. By Theorem 1.1, this invariant subfield is generated by elementary symmetric polynomials. If $n=1$, they are algebraically independent over $\mathbb{K}$, so $\mathbb{K}(\mathbf{X})_1^{S_d}$ is a purely transcendental extension of $\mathbb{K}$. For $d,n>1$, the elementary symmetric polynomials cannot be algebraically independent over $\mathbb{K}$, since there are more than dn of them. Nevertheless [**23**]

Theorem 1.2. *The field of symmetric rational functions $\mathbb{K}(\boldsymbol{X})_n^{S_d}$ (in vector variables) is a purely transcendental extension of $\mathbb{K}$.*

Proof. We use induction on n. If $n = 1$, we already know it's true. Now let $n > 1$. From the field $\mathbb{K}(x_{11}, \ldots, x_{1n-1})$, select any d elements $a_1^{(1)}, \ldots, a_1^{(d)}$ linearly independent over $\mathbb{K}$ and denote by $a_i^{(1)}, \ldots, a_i^{(d)}$ their images in $\mathbb{K}(x_{i1}, \ldots, x_{in-1})$ under the action of S_d. There are unique $t_1, \ldots, t_d \in \mathbb{K}(\mathbf{X})_n$ that satisfy the following system of equations

$$\begin{aligned} a_1^{(1)} t_1 + \cdots + a_1^{(d)} t_d &= x_{1n} \\ a_2^{(1)} t_1 + \cdots + a_2^{(d)} t_d &= x_{2n} \\ \vdots \qquad & \quad \vdots \\ a_d^{(1)} t_1 + \cdots + a_d^{(d)} t_d &= x_{dn}. \end{aligned}$$

The t_i's are invariant under S_d, since by construction the above system of equations is invariant under S_d and this implies $\mathbb{K}(\mathbf{X})_{n-1}^{S_d}(t_1, \ldots, t_d) \subseteq \mathbb{K}(\mathbf{X})_n^{S_d}$. We want to show that

$$\mathbb{K}(\mathbf{X})_n^{S_d} = \mathbb{K}(\mathbf{X})_{n-1}^{S_d}(t_1, \ldots, t_d). \tag{$*$}$$

To show $(*)$, it is enough to show that $\mathbb{K}(\mathbf{X})_n$ has degree $d!$ over both fields in $(*)$. By Galois theory we know that $[\mathbb{K}(\mathbf{X})_n : \mathbb{K}(\mathbf{X})_n^{S_d}] = |S_d| = d!$. The above system of equations show that $\mathbb{K}(\mathbf{X})_n = \mathbb{K}(\mathbf{X})_{n-1}(t_1, \ldots, t_d)$. So we have

$$\begin{aligned} [\mathbb{K}(\mathbf{X})_n : \mathbb{K}(\mathbf{X})_{n-1}^{S_d}(t_1, \ldots, t_d)] &= [\mathbb{K}(\mathbf{X})_{n-1}(t_1, \ldots, t_d) : \mathbb{K}(\mathbf{X})_{n-1}^{S_d}(t_1, \ldots, t_d)] \\ &= [\mathbb{K}(\mathbf{X})_{n-1} : \mathbb{K}(\mathbf{X})_{n-1}^{S_d}] = |S_d| = d!. \end{aligned}$$

Now by inductive hypothesis, $\mathbb{K}(\mathbf{X})_{n-1}^{S_d}$ is a purely transcendental extension of $\mathbb{K}$, so $(*)$ implies the same for $\mathbb{K}(\mathbf{X})_n^{S_d}$. □

We can make an even more precise statement if $\mathbb{K} = \mathbb{C}$ but it requires some work.

Let $\{x_1, \ldots, x_d\}$ be an unordered tuple of d points in $\mathbb{C}$. Suppose we know the values of the elementary symmetric polynomials $e_k(x_1, \ldots, x_d)$, $1 \leq k \leq d$ beforehand. We have

$$\begin{aligned} \prod_{1 \leq i \leq d} (1 + x_i t) &= 1 + e_1(x_1, \ldots x_d) t^1 + \cdots + e_d(x_1, \ldots, x_d) t^d \\ &= (t - y_1) \cdots (t - y_d), \end{aligned}$$

for some $y_1, \ldots, y_d \in \mathbb{C}$ by the fundamental theorem of algebra. Now using the right-hand side which we already know, we can determine the left-hand side. It also possible to determine the value of unordered tuple of d points $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ in $\mathbb{C}^n$, if we know the values of the elementary symmetric polynomials (in vector variables) beforehand, but only under some restrictions.

Lemma 1.1. An unordered tuple $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ of d points $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbb{C}^n$ with pairwise distinct first coordinates x_{i1}, can be uniquely determined if we know the values of the elementary symmetric polynomials

(i) $e_{k,0,\ldots,0}$, where $1 \leq k \leq d$.

(ii) $e_{k,0,\ldots,0,1,0,\ldots,0}$, where $0 \leq k \leq d-1$ and 1 is any postion from 2 to n.

Proof. The elementary symmetric polynomial $e_{k,0,\ldots,0}(\mathbf{x}_1, \ldots, \mathbf{x}_d)$ is the usual elementary symmetric polynomial $e_k(x_{11}, \ldots, x_{d1})$ in the first coordinates of the $\mathbf{x}_i$'s. As we have already discussed, using these we can determine the tuple $\{x_{11}, \ldots, x_{d1}\}$.

We now want to determine $\{x_{1j}, \dots, x_{dj}\}$, $2 \le j \le n$. For that purpose, we consider the polynomial

$$f_j(t) \coloneqq \sum_{i=1}^{d} x_{ij} \prod_{l \neq i} (t - x_{l1}), \quad 2 \le j \le n.$$

Now

$$f_j(t) = x_{1j} \prod_{l \neq 1} (t - x_{l1}) + \cdots + x_{dj} \prod_{l \neq d} (t - x_{l1}),$$

where

$$\begin{aligned} \prod_{l \neq i} (t - x_{l1}) &= t^{d-1} - e_1(x_{11}, \dots, \widehat{x_{i1}}, \dots, x_{d1}) t^{d-2} \\ &\quad + \cdots + (-1)^{d-1} e_{d-1}(x_{11}, \dots, \widehat{x_{i1}}, \dots, x_{d1}) \\ &= \sum_{k=0}^{d-1} e_k(x_{11}, \dots, \widehat{x_{i1}}, \dots, x_{d1}) (-1)^k t^{(d-1)-k}. \end{aligned}$$

The hat $\widehat{\ }$ represents omission. Multiplying by x_{ij} we get

$$x_{ij} \prod_{l \neq i} (t - x_{l1}) = \sum_{k=0}^{d-1} x_{ij} e_k(x_{11}, \dots, \widehat{x_{i1}}, \dots, x_{d1}) (-1)^k t^{(d-1)-k}.$$

Substituting this in $f_j(t)$ we get

$$\begin{aligned} f_j(t) &= \sum_{k=0}^{d-1} [\sum_{i=1}^{d} x_{ij} e_k(x_{11}, \dots, \widehat{x_{i1}}, \dots, x_{d,1})] (-1)^k t^{(d-1)-k} \\ f_j(t) &= \sum_{k=0}^{d-1} [x_{1j} e_k(\widehat{x_{11}}, \dots, x_{d1}) \\ &\quad + \cdots + x_{dj} e_k(x_{11}, \dots, \widehat{x_{d1}})] (-1)^k t^{(d-1)-k}. \end{aligned}$$

So up to sign the coefficients of powers of t in $f_j(t)$ are the polynomials

$$x_{1j} e_k(\widehat{x_{11}}, \dots, x_{d1}) + \cdots + x_{dj} e_k(x_{11}, \dots, \widehat{x_{d1}}),\ k = 0, \dots, d-1.$$

These are symmetric polynomials and in fact the elementary symmetric polynomials $e_{k,0,\dots 0,1,0,\dots,0}$, where the 1 is in the j-th position.

To illustrate, let's look at an example in the case $d, n = 3$ and $j = 2$,

$$\begin{aligned} f_2(t) &= \sum_{i=1}^{3} x_{i2} \prod_{l \neq i} (t - x_{l1}) \\ &= x_{12} \prod_{l \neq 1} (t - x_{l1}) + x_{2,2} \prod_{l \neq 2} (t - x_{l1}) + x_{32} \prod_{l \neq 3} (t - x_{l1}) \\ &= x_{12}[t^2 - (x_{21} + x_{31})t + x_{21}x_{31}] + x_{22}[t^2 - (x_{11} + x_{31})t + x_{11}x_{31}] \\ &\quad + x_{32}[t^2 - (x_{11} + x_{21})t + x_{11}x_{21}] \\ &= (x_{12} + x_{22} + x_{32})t^2 - [x_{12}(x_{21} + x_{31}) + x_{22}(x_{11} + x_{31}) + x_{32}(x_{11} + x_{21})]t \\ &\quad + x_{12}(x_{21}x_{31}) + x_{22}(x_{11}x_{31}) + x_{32}(x_{11}x_{21}) \\ &= (x_{12} + x_{22} + x_{32})t^2 - [x_{12}x_{21} + x_{12}x_{31} + x_{22}x_{11} + x_{22}x_{31} + x_{32}x_{11} + x_{32}x_{21}]t \\ &\quad + x_{12}x_{21}x_{31} + x_{22}x_{11}x_{31} + x_{32}x_{11}x_{21}. \end{aligned}$$

The exponent matrix of $x_{22}x_{11}$ is given by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and the rest of the coefficients of t are monomials correpsonding to all possible permuations of this. So the coefficient of t is in fact the elementary symmetric polynomial $e_{1,1,0}$.

The polynomial $f_j(t)$ satisfies

$$\begin{aligned} f_j(x_{k1}) &= \sum_{i=1}^{d} x_{ij} \prod_{l \neq i} (x_{k1} - x_{l1}), \quad 2 \leq j \leq n \\ &= x_{kj} \prod_{l \neq k} (x_{i1} - x_{l1}). \end{aligned}$$

The first coordinates $\{x_{11}, \ldots, x_{d1}\}$ are distinct, so once we know f_j and the collection $\{x_{11}, \ldots, x_{d1}\}$, both of which are determined by elementary symmetric polynomials stated in the lemma, we can determine $\{x_{1j}, \ldots, x_{dj}\}, \ 2 \leq j \leq n$. □

In what is about to follow we use the notion of a complex (analytic) space, which we will not introduce here as it is beyond the scope of this thesis. We only use certain facts about complex spaces and holomorphisms (morphisms between complex spaces) to establish that a certain map is biholomorphic. For an explicit description of a complex space, we refer the reader to [**15**][**18**].

Lemma 1.2. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial. The set of points in $\mathbb{C}^n$ at which f is nonzero is an open path-connected set in $\mathbb{C}^n$.

Proof. We can assume f is not constant. A polynomial gives a continuous function from $\mathbb{C}^n$ to $\mathbb{C}$, so since $\mathbb{C}\backslash\{0\}$ is open, $f^{-1}(\mathbb{C}\backslash\{0\})$ is open. Let $z_1, z_2 \in f^{-1}(\mathbb{C}\backslash\{0\})$. Consider the set $L = \{tz_1 + (1-t)z_2 \mid t \in \mathbb{C}\}$, which contains both z_1 and z_2. $f(tz_1 + (1-t)z_2) \in \mathbb{C}[t]$ is a polynomial in one variable, so it has finitely many zeros by the fundamental theorem of algebra. So $L \cap f^{-1}(\mathbb{C}\backslash\{0\})$ is L with finitely many points removed. Now L the image of $\mathbb{C}$ under the map $t \mapsto t(z_1) + (1-t)z_2$, so after removal of finite number of points it remains path-connected, as it is then the image of $\mathbb{C}$ with finitely many points removed. □

We require the following three results about complex spaces and morphisms between them.

Lemma 1.3. If $f : X \to Y$ is a holomorphic surjection with finite fibres, then $\dim X = \dim Y$.

Lemma 1.4. Let $f : X \to Z$ be an injective holomorphic mapping from a pure dimensional reduced complex space X into a manifold Z of the same dimension. Then f is open and $f : X \to f(X)$ is biholomorphic.

The following lemma due to H. Cartan [**5**] is stated in [**24**].

Lemma 1.5. If X is a reduced complex space and $G \in Aut(X)$ is a finite subgroup of the automorphism group of X (biholomorphisms from X to itself), then the quotient space X/G is a reduced complex space, and the projection map $X \to X/G$ is holomorphic.

Let S_d act on $(\mathbb{C}^n)^d$ in the same way as described in the beginning of Section 1. A subset $U \subseteq (\mathbb{C}^n)^d$ is S_d-invariant if it remains invariant under the action of S_d. A holomoprhic function $f : U \to \mathbb{C}$ on an S_d-invariant subset $U \subseteq (\mathbb{C}^n)^d$ is called ***symmetric*** if

$$f(x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{dn}) = f(x_{\sigma(1)1}, \ldots, x_{\sigma(1)n}, x_{\sigma(2)1}, \ldots, x_{\sigma(d)n}) \quad \text{for all } \sigma \in S_d.$$

Proposition 1.3. *Let $f : (\mathbb{C}^n)^d \to \mathbb{C}$ be a symmetric holomorphic function. Then f can be expressed as a holomorphic function of the elementary symmetric functions given by the elementary symmetric polynomials*

(i) $e_{k,0,\ldots,0}$, *where* $1 \leq k \leq d$.

(ii) $e_{k,0,\ldots,0,1,0,\ldots,0}$, *where* $0 \leq k \leq d-1$ *and* 1 *is any postion from* 2 *to* n.

Proof. Let $U = (\mathbf{x}_1, \ldots, \mathbf{x}_d) \in (\mathbb{C}^n)^d$ be the set consisting of points with pairwise distinct first coordinates. Consider the map

$$\begin{aligned} \phi : U &\to \phi(U) \\ (\mathbf{x}_1, \ldots, \mathbf{x}_d) &\mapsto (e_{1,0,\ldots,0}(\mathbf{x}), \ldots, e_{d,0,\ldots,0}(\mathbf{x}), e_{k,0,\ldots,0,1,0,\ldots,0}(\mathbf{x})), \end{aligned}$$

where on the right we have the elementary symmetric functions stated in the proposition. By Lemma 1.1, this induces a bijective map $\hat{\phi} : U/S_d \to \phi(U)$. But this is a set-theoretic bijection. We want to show that this map is biholomorphic.

U is the complement of the set of zeroes of the polynomial $\prod_{1\leq i<j\leq n} (x_{i1}-x_{j1})$. By Lemma 1.2, U is therefore an open and connected set. So it is a connected complex manifold. All complex manifolds are reduced complex spaces. So U is a connected reduced complex space. By Lemma 1.5, since S_d is a finite group, U/S_d is a reduced complex space and since the projection map $U \to U/S_d$ is holomorphic, it is also connected. A complex space that is both reduced and irreducible is of pure dimension. Since U/S_d is connected, it is irreducible, therefore U/S_d is a pure-dimensional reduced complex space. Since the projection map $U \to U/S_d$ is a holomorphic surjection with finite fibres, $\dim(U) = \dim(U/S_d)$ by Lemma 1.3, and therefore $\dim(U/S_d) = \dim(U) = \dim(\mathbb{C}^{dn})$. The map $\hat{\phi}$ satisfies the conditions of Lemma 1.4, so it is biholomorphic.

Now $f|_U : U \to \mathbb{C}$ is a symmetric holomorphic function. Using the map ϕ, we can express $f|_U$ as a holomorphic function in the elementary symmetric functions stated in the proposition. Since U is an open and connected set, this must also be globally true by the identity theorem. ☐

With the help of this proposition, we can now finally prove a more precise version of Theorem 1.2 in the case $\mathbb{K} = \mathbb{C}$.

Theorem 1.3. [2] *The field of symmetric rational functions $\mathbb{C}(\boldsymbol{X})_n^{S_d}$ (in vector variables) is isomorphic to the field of rational functions in dn variables over $\mathbb{C}$. The algebraically independent generators of this field are given by the elementary symmetric polynomials*

(i) $e_{k,0,\ldots,0}$, *where* $1 \leq k \leq d$.

(ii) $e_{k,0,\ldots,0,1,0,\ldots,0}$, *where* $0 \leq k \leq d-1$ *and* 1 *is any postion from* 2 *to* n.

Proof. A holomorphic function on $\mathbb{C}^m, m \geq 1$, can be expressed as a power series in m variables. By Propostion 1.3, it follows that a symmetric polynomial function, say f, can be expressed as a power series series in the elementary symmetric functions stated above. Since f is a polynomial, this power series must eventually terminate. So a symmetric polynomial function can be expressed as a polynomial function in the above elementary symmetric functions. From an algebraic point of view, we have that

$$\mathbb{C}[\mathbf{X}]_n^{S_d} = \mathbb{C}[e_{1,0,\ldots,0}(\mathbf{x}), \ldots, e_{d,0,\ldots,0}(\mathbf{x}), e_{k,0,\ldots,0,1,0,\ldots,0}(\mathbf{x})].$$

[2]The proof of this theorem given in [**13**] is incomplete since a bijective map between smooth varieties need not be an isomorphism.

By Galois theory, $\mathbb{C}(\mathbf{X})_n^{S_d}$ has transcendence degree dn over $\mathbb{C}$, and this forces the elementary symmetric polynomials stated above to be a transcendence basis. □

2 Quotients of Tensor Algebras and Schur Modules

Our goal in this section is to state certain results about representations of linear groups. For that purpose, we first quickly introduce tensor algebras, other algebras derived from it, and Schur modules. For a detailed discussion, we refer the reader to [**9**].

Let R be a commutative ring with 1. Let M be an R-module where the left and right actions of R on M are the same. For an integer $k \geq 1$, we define

$$T^k(M) \coloneqq M \otimes_R M \otimes_R \cdots \otimes_R M \quad (k\text{-times}),$$

and set $T^0(M) \coloneqq R$. The elements of $T^k(M)$ are called ***k-tensors***. We define

$$T(M) \coloneqq \bigoplus_{k=0}^{\infty} T^k(M) = R \oplus T^1(M) \oplus T^2(M) \oplus \cdots .$$

Every element of $T(M)$ is a finite linear combination of k-tensors and $M = T^1(M)$ is an R-submodule of $T(M)$. We call an element in $T(M)$ of the form $m_1 \otimes \cdots \otimes m_n$ a **simple tensor**. We obtain a product on $T(M)$ by first defining it on simple tensors

$$(m_1 \otimes \cdots \otimes m_i)(n_1 \otimes \cdots \otimes n_j) = m_1 \otimes \cdots \otimes m_i \otimes n_1 \otimes \cdots \otimes n_j$$

and then extending it using distributive laws. This makes $T(M)$ into an R-algebra containing M and with respect to this product $T^i(M)T^j(M) \subseteq T^{i+j}(M)$. The R-algebra $T(M)$ is called the ***tensor algebra*** of M.

A ring S is called a ***graded ring*** if it is the direct sum of additive subgroups: $S = \bigoplus_{k=0}^{\infty} S_k = S_0 \oplus S_1 \oplus S_2 \oplus \cdots$ such that $S_iS_j \subseteq S_{i+j}$ for $i, j \geq 0$. An ideal I of the graded ring S is called a ***graded ideal*** if $I = \bigoplus_{k=0}^{\infty}(I \cap S_k)$. A ring homomorphism $\varphi : S \to T$ between two graded rings is called a ***homomorphism of graded rings*** if $\varphi(S_k) \subseteq T_k$ for all $k \geq 0$, that is, it respects the grading structure. If I is a graded ideal of a graded ring S then the quotient S/I is also a graded ring and it is isomorphic (as a graded ring) to $\bigoplus_{k=0}^{\infty}(S_k/I_k)$.

The ***symmetric algebra*** $S(M)$ of the R-module M is the quotient of the tensor algebra $T(M)$ by the ideal generated by all elements of the form $m \otimes n - n \otimes m$, $m, n \in M$. That is, $S(M) = T(M)/ < m \otimes n - n \otimes m >$. We denote this ideal by $C(M)$. The tensor algebra $T(M)$ is generated by $R = T^0(M)$ and $M = T^1(M)$, and these generators commute in the quotient ring $S(M)$, so it is a commutative ring. An element of the ideal $C(M)$ can only be a finite linear combination of k-tensors with $k \geq 2$, so it is a graded ideal. The symmetric algebra is therefore a graded ring. $S^0(M) = T^0(M)/(C(M) \cap T^0(M)) = R$ and $S^1(M) = T^1(M)/(C(M) \cap T^1(M)) = M$. So it is also an R-algebra. The R-module $S^k(M)$ is called the ***k-th symmetric power*** of M.

There is an equivalent way of describing the k-th symmetric power. It is equal to the quotient of $T^k(M)$ by the submodule generated by all elements of the form

$$(m_1 \otimes \cdots \otimes m_k) - (m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(k)}), \tag{2.1}$$

for all $m_i \in M$ and all $\sigma \in S_k$. To see this, note that the k-tensors in $C^k(M)$ in the ideal $C(M)$ are finite sums of elements of the the form

$$m_1 \otimes \cdots \otimes m_{i-1} \otimes (m_i \otimes m_{i+1} - m_{i+1} \otimes m_i) \otimes m_{i+2} \otimes \cdots \otimes m_k, \tag{2.2}$$

with $m_1, \ldots, m_k \in M$ and $k \geq 2$, $1 \leq i < k$. This corresponds to the transposition $(i\ i+1) \in S_k$. On the other hand, since any permutation can be written as a product of transpositions, elements of the form (2.1) can be written as finite sums of elements of the form (2.2). To illustrate, consider

$$m_1 \otimes m_2 \otimes m_3 - m_3 \otimes m_2 \otimes m_1.$$

Since $(m_1 \otimes m_2 - m_2 \otimes m_1) \otimes m_3 \in C^3(M)$, we have

$$m_1 \otimes m_2 \otimes m_3 - m_3 \otimes m_2 \otimes m_1 \equiv m_2 \otimes m_1 \otimes m_3 - m_3 \otimes m_2 \otimes m_1 \mod C(M).$$

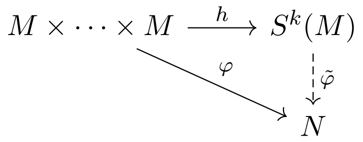
Proceeding similarly we get

$$\begin{aligned} m_1 \otimes m_2 \otimes m_3 - m_3 \otimes m_2 \otimes m_1 &\equiv m_2 \otimes m_1 \otimes m_3 - m_3 \otimes m_2 \otimes m_1 \mod C(M) \\ &\equiv m_2 \otimes m_3 \otimes m_1 - m_3 \otimes m_2 \otimes m_1 \\ &\equiv m_3 \otimes m_2 \otimes m_1 - m_3 \otimes m_2 \otimes m_1 \\ &\equiv 0. \end{aligned}$$

The k-th symmetric power satisfies a ***universal property***: if $\varphi : M \times \cdots \times M \to N$ is a symmetric k-multilinear map between R-modules, there is a unique R-module homomorphism $\tilde{\varphi} : S^k(M) \to N$ such that $\varphi = \tilde{\varphi} \circ h$, where h is the canonical symmetric k-multilinear map

$$\begin{aligned} h : M \times \cdots \times M &\to S^k(M) \\ (m_1, \ldots, m_k) &\mapsto m_1 \otimes \cdots \otimes m_k \mod C(M). \end{aligned}$$

So the following diagram



commutes.

The order doesn't matter in $S(M)$. If $M = V$ is a finite-dimensional vector space over $\mathbb{K}$ with basis $\{e_1, \ldots, e_n\}$. Then $S(V)$ is isomorphic as a graded $\mathbb{K}$-algebra to the ring of polynomials over $\mathbb{K}$ in n indeterminates. $S^d(V)$ is then isomorphic as a vector space to the vector space of forms of degree d in n variables, which has dimension $\binom{d+n-1}{n-1}$.

If we now consider the dual space V^* of V with the dual basis $\{e_1^*, \ldots, e_n^*\}$, then $S(V^*)$ is isomorphic to the ring of polynomials functions on V, which is just $\mathbb{K}[e_1^*, \ldots, e_n^*]$ and takes values in V. In other words, it is the coordinate ring of V. The space of forms of degree d on $\mathbb{P}V$ is just $S^d(V^*)$.

The ***exterior algebra*** $\Lambda(M)$ of an R-module M is the the quotient of the tensor algbera $T(M)$ by the ideal generated by all elements of the form $m \otimes m$, $m \in M$. That is, $\Lambda(M) = T(M)/ < m \otimes m >$. We denote this ideal by $A(M)$. We denote the image of $m_1 \otimes \cdots \otimes m_k$ in $\Lambda(M)$ by $m_1 \wedge \cdots \wedge m_k$. Like in the case of symmetric algebra, an element of this ideal can only be a finite linear combination of k-tensors with $k \geq 2$, so it is a graded ideal. The exterior algebra is therefore a graded ring, where $\Lambda^k(M) = T^k(M)/A^k(M)$. Now $\Lambda^0(M) = T^0(M)/(A(M) \cap T^0(M)) = R$ and $\Lambda^1(M) = T^1(M)/(A(M) \cap T^1(M)) = M$. So it is also an R-algebra. The R-module $\Lambda^k(M)$ is called the ***k-th exterior power*** of M. The multiplication

$$(m_1 \wedge \cdots \wedge m_i) \wedge (n_1 \wedge \cdots \wedge n_j) := m_1 \wedge \cdots \wedge m_i \wedge n_1 \wedge \cdots \wedge n_j$$

in $\Lambda(M)$ is called the ***wedge*** or ***exterior product***. By definition $m_1 \wedge \cdots \wedge m_k$ is 0 if any two m_i's coincide. So $(m + n) \wedge (m + n) = 0$, and it follows that $m \wedge n = -n \wedge m$.

There is an equivalent way of describing the k-th exterior power. It is equal to the quotient of $T^k(M)$ by the submodule generated by all elements of the form

$$(m_1 \otimes \cdots \otimes m_k) \quad \text{where } m_i = m_j \text{ for some } i \neq j. \tag{2.3}$$

Now the elements of $A^k(M)$ are of the form above by definition. As for the reverse inclusion, we know that

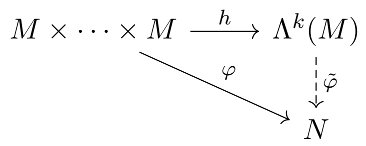
$$m \otimes n \equiv -n \otimes m \mod A(M).$$

So if we interchange the entries in (2.3), so that m_i and m_j are adjacent, we get up to sign an equivalent tensor modulo $A^k(M)$, and so these generators are contained in $A^k(M)$.

Just like in the case of symmetric powers, the k-th exterior power $\Lambda^k(M)$ sastifies a ***universal property***: if $\varphi : M \times \cdots \times M \to N$ is a k-multilinear alternating map between R-modules, there is a unique R-module homomorphism $\tilde{\varphi} : \Lambda^k(M) \to N$ such that $\varphi = \tilde{\varphi} \circ h$, where h is the canonical k-multilinear alternating map

$$\begin{aligned} h : M \times \cdots \times M &\to \Lambda^k(M) \\ (m_1, \ldots, m_k) &\mapsto m_1 \otimes \cdots \otimes m_k \mod A(M). \end{aligned}$$

So the following diagram



commutes.

We will discuss the exterior algbera of a finite-dimensional vector space later when we introduce the Grassmannian in Section 3.

We can also define the symmetric and exterior algebra as subalgebras of the tensor algebra, rather than as quotient algebras if the underlying ring has characteristic 0. For that purpose, we first define symmetric and alternating tensors. For any R-module M there is a left group action of S_k on $T^k(M)$ given by

$$\sigma(m_1 \otimes \cdots \otimes m_k) := m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(k)} \quad \text{for all } \sigma \in S_k.$$

An element $z \in T^k(M)$ is called a ***symmetric k-tensor*** if $\sigma z = z$ for all $\sigma \in S_k$ and an ***alternating k-tensor*** if $\sigma z = \text{sign}\sigma \cdot z$ for all $\sigma \in S_k$. The set of symmetric k-tensors and the set of alternating k-tensors are R-submodules of $T^k(M)$.

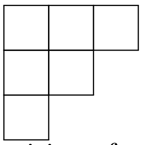
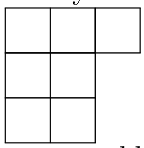
Both $C^k(M)$ and $A^k(M)$ are stable under the action of S_k, so there is an induced action of S_k on $S^k(M)$ and $\Lambda^k(M)$. As we have discussed already, all elements in $S^k(M)$ remain stable under the action of S_k. As for $\Lambda^k(M)$, since all permuations can be written as product of transpositions and $m \wedge n = -n \wedge m$, we have $\sigma w = \text{sign}\sigma \cdot w$ for all $\sigma \in S_k$ and $w \in \Lambda^k(M)$. So the symmetric group acts in the same way on symmetric and exterior powers as it does on symmetric and alternating tensors. For any k-tensor $z \in T^k(M)$, we define the maps

$$\begin{aligned} \text{Sym}(z) &:= \sum_{\sigma \in S_k} \sigma z \\ \text{Alt}(z) &:= \sum_{\sigma \in S_k} \text{sign}\sigma \cdot \sigma z. \end{aligned}$$

Both $\text{Sym}(z)$ and $\text{Alt}(z)$ are symmetric and alternating tensors respectively. If $k!$ is a unit in the ring R, then the map $\frac{1}{k!}\text{Sym} : T^k(M) \to \{\text{symmetric } k\text{-tensors}\}$ induces an R-module isomorphism $S^k(M) \cong \{\text{symmetric } k\text{-tensors}\}$ and $\frac{1}{k!}\text{Alt}$ induces the R-module isomorphism $\Lambda^k(M) \cong \{\text{alternating } k\text{-tensors}\}$.

2.1 Schur Modules

Before we introduce Schur modules we need the notion of a Young diagram.

A ***Young diagram*** is a finite collection of boxes arranged as rows on top of each other such that the number of boxes in each row is weakly decreasing from top to bottom. For example,  (strictly decreasing) and  (weakly decreasing).

Any partition of a positive integer n, written as a weakly decreasing sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ corresponds to a Young diagram. For example, the partition of 6 as $(3, 2, 1)$ corresponds to the Young diagram above on the left. We identify λ with its corresponding diagram and by $|\lambda|$ we denote the number of boxes in λ.

Note: It is sometimes convenient to allow 0's at the end, and identify sequences that differ only by 0's. For example, $(5, 0)$ is the same as (5).

We can fill a Young diagram λ with elements from any set (vector spaces, integers, etc). We choose two different columns in λ and choose one set of boxes from each, such that the number of boxes in each set is the same. For any filling of λ and a chosen set of boxes, the corresponding ***exchange*** is the filling obtained by interganching the entries in the two chosen sets, such that the vertical order is maintained.

For example, if the the Young diagram is $\lambda = (4, 3, 3, 2)$, the two chosen set of boxes are the top two boxes in the third column and the first and the fourth box in the first column, and the filling is

1	7	3	1
1	3	4	
1	8	2	
0	9		

, then the corresponding exchange is

3	7	1	1
1	3	0	
1	8	2	
4	9		

.

Now let N be a R-module over a commutative ring R with 1. For a given Young diagram λ, by $M^{\times\lambda}$ we mean the cartesian product $M \times \cdots \times M$ ($|\lambda|$ times), where an element $m \in M^{\times\lambda}$ is given by specifying a filling of λ with elements from M. We number the entries from top to bottom and left to right. For example, if $\lambda = (3, 2, 1)$, then $m \in M^{\times\lambda}$ is written as

m_1	m_4	m_6
m_2	m_5	
m_3		

, where $m_1, \ldots, m_6 \in M$.

Consider maps from $\varphi : M^{\times\lambda} \to N$, satisfying the following three properties

(i) φ is λ-multilinear. That is, if all but one entry in λ is fixed, then φ is R-linear in that entry.

(ii) φ is alternating in the entries of any column of λ. That is, for any $m \in M^{\times\lambda}$, $\varphi(m) = -\varphi(m')$, if m' is obtained from m by interchanging two entries in a column.

(iii) For any $m \in M^{\times\lambda}$, $\varphi(m) = \sum \varphi(m')$, where the sum is taken over all possible exchanges. That is, if we choose any two columns in λ and chose any set of boxes in the column on the right, then the sum is taken over all possible exchanges between the two columns for this chosen subset of boxes in the right column.

For example, if $\lambda = (2, 2, 2)$, then we only have two columns, and if we choose the top two boxes in the second (right) column, then for $m \in M^{\times\lambda}$, we have

$$\varphi(m) = \varphi(\begin{array}{|c|c|}\hline m_1 & \colorbox{green}{m_4} \\ \hline m_2 & \colorbox{green}{m_5} \\ \hline m_3 & m_6 \\ \hline\end{array}) = \varphi(\begin{array}{|c|c|}\hline \colorbox{green}{m_4} & \colorbox{green}{m_1} \\ \hline \colorbox{green}{m_5} & \colorbox{green}{m_2} \\ \hline m_3 & m_6 \\ \hline\end{array}) + \varphi(\begin{array}{|c|c|}\hline \colorbox{green}{m_4} & \colorbox{green}{m_1} \\ \hline m_2 & \colorbox{green}{m_3} \\ \hline \colorbox{green}{m_5} & m_6 \\ \hline\end{array}) + \varphi(\begin{array}{|c|c|}\hline m_1 & \colorbox{green}{m_2} \\ \hline \colorbox{green}{m_4} & \colorbox{green}{m_3} \\ \hline \colorbox{green}{m_5} & m_6 \\ \hline\end{array}).$$

If we only choose the top box in the second column, we have

$$\varphi(\begin{array}{|c|c|}\hline m_1 & \colorbox{green}{m_4} \\ \hline m_2 & m_5 \\ \hline m_3 & m_6 \\ \hline\end{array}) = \varphi(\begin{array}{|c|c|}\hline \colorbox{green}{m_4} & \colorbox{green}{m_1} \\ \hline m_2 & m_5 \\ \hline m_3 & m_6 \\ \hline\end{array}) + \varphi(\begin{array}{|c|c|}\hline m_1 & \colorbox{green}{m_2} \\ \hline \colorbox{green}{m_4} & m_5 \\ \hline m_3 & m_6 \\ \hline\end{array}) + \varphi(\begin{array}{|c|c|}\hline m_1 & \colorbox{green}{m_3} \\ \hline m_2 & m_5 \\ \hline \colorbox{green}{m_4} & m_6 \\ \hline\end{array}).$$

If $\lambda = (3, 3, 3)$, and we choose the first and the third column, and we choose the top two boxes in the third (right) column, then for $m \in M^{\times\lambda}$, we have

$$\varphi(\begin{array}{|c|c|c|}\hline m_1 & m_4 & \colorbox{green}{m_7} \\ \hline m_2 & m_5 & \colorbox{green}{m_8} \\ \hline m_3 & m_6 & m_9 \\ \hline\end{array}) = \varphi(\begin{array}{|c|c|c|}\hline \colorbox{green}{m_7} & m_4 & \colorbox{green}{m_1} \\ \hline \colorbox{green}{m_8} & m_5 & \colorbox{green}{m_2} \\ \hline m_3 & m_6 & m_9 \\ \hline\end{array}) + \varphi(\begin{array}{|c|c|c|}\hline \colorbox{green}{m_7} & m_4 & \colorbox{green}{m_1} \\ \hline m_2 & m_5 & \colorbox{green}{m_3} \\ \hline \colorbox{green}{m_8} & m_6 & m_9 \\ \hline\end{array}) + \varphi(\begin{array}{|c|c|c|}\hline m_1 & m_4 & \colorbox{green}{m_2} \\ \hline \colorbox{green}{m_7} & m_5 & \colorbox{green}{m_3} \\ \hline \colorbox{green}{m_8} & m_6 & m_9 \\ \hline\end{array}).$$

The ***Schur module*** $S^\lambda(M)$ is an R-module that satisfies a universal property: if $\varphi : M^{\times\lambda} \to N$ is a map that satisifies the above properties (i)-(iii), there is a unique R-module homomorphism $\tilde{\varphi} : S^\lambda(M) \to N$ such that $\varphi = \tilde{\varphi} \circ h$, where h is a map from $M^{\times\lambda}$ to $S^\lambda(M)$ that satisfies (i)-(iii).

The question remains as to what is the exact description of $S^\lambda(M)$ and h. Let's first consider the cases $\lambda = (1, \ldots, 1)$ (n times) and $\lambda = (n)$. If $\lambda = (1, \ldots, 1)$, property (iii) is trivially satisfied and (i) and (ii) basically mean that we have an n-multilinear alternating map. So the Schur module $S^{(1,\ldots,1)}(M)$ is just the n-th exterior power $\Lambda^n(M)$ and h is the canonical map described earlier. If $\lambda = (n)$, property (ii) is trivially satisified and (i) and (iii) basically mean that we have a symmetric n-multilinear map. So the Schur module $S^{(n)}(M)$ is the n-th symmetric power $S^n(M)$ and h is again the canonical map .

The above two cases help guide our construction of other Schur modules. Property (i) tells us that we need to first construct the tensor product $M^{\otimes\lambda}$. Just like in the case of exterior algebra, property (ii) tells us that we need to take the quotient of $M^{\otimes\lambda}$ by the submodule generated by elements that have two equal entries in the same column. If by μ_i, we denote the length of the i-th column of λ, which say has l columns, the resulting quotient is then $\Lambda^{\mu_1} M \otimes \cdots \otimes \Lambda^{\mu_l} M$. The map h up till now would then be the canonical one. For example,

$$M^{\times(2,2,2)} \to \Lambda^3 M \otimes \Lambda^3 M$$
$$\begin{array}{|c|c|}\hline m_1 & m_4 \\ \hline m_2 & m_5 \\ \hline m_3 & m_6 \\ \hline\end{array} \mapsto (m_1 \wedge m_2 \wedge m_3) \otimes (m_4 \wedge m_5 \wedge m_6).$$

If by $\overline{m}$ we denote the elements in $\Lambda^{\mu_1} M \otimes \cdots \otimes \Lambda^{\mu_l} M$, property (iii) tell us that we need to further quotient this by the submodule generated by all elements of the form $\overline{m} - \sum \overline{m'}$, where the sum is of all possible m' obtained from m by exchanges described in (iii). We denote this submodule by Q^λ. So the Schur module $S^\lambda(M)$ is just $(\Lambda^{\mu_1} M \otimes \cdots \otimes \Lambda^{\mu_l} M)/Q^\lambda$.

For example, $S^{(2,1)}(M)$ is $(\Lambda^2 M \otimes M)/Q^{(2,1)}$, where $Q^{(2,1)}$ is the submodule generated by elements of the form

$$\overline{\begin{array}{|c|c|}\hline m_1 & m_3 \\ \hline m_2 \\ \cline{1-1}\end{array}} - \overline{\begin{array}{|c|c|}\hline \colorbox{green}{m_3} & \colorbox{green}{m_1} \\ \hline m_2 \\ \cline{1-1}\end{array}} + \overline{\begin{array}{|c|c|}\hline m_1 & \colorbox{green}{m_2} \\ \hline \colorbox{green}{m_3} \\ \cline{1-1}\end{array}} = (m_1 \wedge m_2) \otimes m_3 - (m_3 \wedge m_2) \otimes m_1 - (m_1 \wedge m_3) \otimes m_2.$$

Any homomorphism of R-modules M and N determines a module homomorphism between $S^\lambda(M)$ and $S^\lambda(N)$. The map $M \mapsto S^\lambda(M)$ is a functor from the category of modules over a fixed commutative ring to itself. It is even a functor on the smaller category of finite-dimensional vector spaces. These are called ***Schur functors***.

What are the basis elements of the Schur module in the case of finite-dimensional vector spaces? To answer that question we need the notion of a Young tableau.

A ***Young tableau*** is a filling of a Young diagram by positive integers (or any ordered set) such that it is

(i) weakly increasing across each row.

(ii) strictly increasing down each column.

Let $\{e_1, \dots, e_n\}$ be the basis of a vector space V over $\mathbb{K}$. For a Young diagram λ, let T denote a Young tableau with entries from the set $\{1, \dots, n\}$, and let $e_T \in S^\lambda(V)$ denote the image under the map h (described earlier) of the element in $V^{\times\lambda}$, where the entries in T are replaced by the basis elements of V indexed by that entry. For example, if $n = 5$, $\lambda = (2, 1)$ and $T = \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 5 & \\ \cline{1-1}\end{array}$, then $e_T = (e_2 \wedge e_5) \otimes e_2 + Q^{(2,1)}$.

Theorem 2.1. *Let V be a finite-dimensional vector space over $\mathbb{K}$ with basis $\{e_1, \dots, e_n\}$. Let λ be a Young diagram. Then the Schur module $S^\lambda(V)$ is a finite-dimensional vector space, and the set of elements e_T, where T is a Young tableau on λ with entries from the set $\{1, \dots, n\}$, give a basis.*

It turns out all representations of the general and special linear groups over finite-dimensional complex vector spaces can be described using Schur modules, but we will not discuss this here. The interested reader can refer to [**10**][**11**]. Our purpose is to state the following result, which we will use later.

Theorem 2.2. *Let V, W be finite-dimensional complex vector spaces.*

(i) The representation $S^d(V) \otimes S^d(V)$ of $GL(V)$ decomposes into the irreducible components

$$S^d(V) \otimes S^d(V) = \bigoplus_{k=0}^{d} S^{(d+k,d-k)}(V).$$

(ii) The representation $S^d(V \otimes W)$ of $GL(V) \times GL(W)$ decomposes into the irreducible components

$$S^d(V \otimes W) = \bigoplus_{\lambda} S^\lambda(V) \otimes S^\lambda(W),$$

where the sum is over all λ, such that $|\lambda| = d$ and λ has most dim(V)or dim(W) rows.

Remark 2.1. If $\dim(W) = 2$ and d is even, then we have the decomposition

$$S^d(V \otimes W) = \bigoplus_{k=0}^{d/2} S^{(d/2+k,d/2-k)}(V) \otimes S^{(d/2+k,d/2-k)}(W).$$

3 Zero Cycles in Projective Spaces

In this section, we formally introduce the Chow variety of 0-cycles, but first we revise our knowledge of the Grassmannian. More details regarding the Grassmannian can be found in [**12**].

Let n be a positive integer and let $k \in \mathbb{Z}$ with $0 \leq k \leq n$. The ***Grassmannian*** $G(k, n)$ is the set of all k-dimensional linear subspaces of $\mathbb{K}^n$. The Grassmannian $G(1, n)$ is just the projective space $\mathbb{P}^{n-1}$ over $\mathbb{K}$. We can make the Grassmannian into a variety by embedding it in a projective space but we need to do some work before that.

Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{K}^n$. For a positive integer k, let $\Lambda^k\mathbb{K}^n$ be the k-th exterior power of the vector space $\mathbb{K}^n$ (see Section 2). Its basis is given by the elements

$$e_{i_1} \wedge \cdots \wedge e_{i_k}$$

for all multi-indices $(i_1, \ldots, i_k) \in \mathbb{Z}^k_{>0}$ with $1 \leq i_1 < \cdots < i_k \leq n$. It has dimension $\binom{n}{k}$. $\Lambda^1\mathbb{K}^n = \mathbb{K}^n$, and $\Lambda^n\mathbb{K}^n \cong \mathbb{K}$ with the single basis vector $e_1 \wedge \cdots \wedge e_n$. We set $\Lambda^0\mathbb{K}^n := \mathbb{K}$ and if $k > n$, then $\Lambda^k\mathbb{K}^n$ is the zero vector space.

For other multi-indices, $e_{i_1} \wedge \cdots \wedge e_{i_k}$ equal to 0 if any two indices coincide and

$$e_{i_1} \wedge \cdots \wedge e_{i_k} = \text{sign}\sigma \cdot e_{i_{\sigma(1)}} \wedge \cdots \wedge e_{i_{\sigma(k)}}$$

if $(i_1, \ldots, i_k)$ are distinct but not strictly increasing and σ is the unique permutation such that $i_{\sigma(1)} < \cdots < i_{\sigma(k)}$. For $v_1, \ldots, v_k \in \mathbb{K}^n$ with $v_j = \sum_{i=1}^n a_{ij}e_i$ for some $a_{ij} \in \mathbb{K}$, their wedge product is

$$\begin{aligned} v_1 \wedge \cdots \wedge v_k &= (\sum_{i=1}^n a_{i1}e_i) \wedge \cdots \wedge (\sum_{i=1}^n a_{ik}e_i) \\ &= \sum_{1 \leq i_1, \ldots, i_k \leq n} a_{i_1 1} \cdots a_{i_k k} e_{i_1} \wedge \cdots \wedge e_{i_k}. \end{aligned}$$

The coefficient of $e_{j_1} \wedge \cdots \wedge e_{j_k}$ with $j_1 < \cdots < j_k$ in $v_1 \wedge \cdots \wedge v_k$ is then

$$\sum_{\sigma \in S_k} \text{sign } \sigma \cdot a_{j_{\sigma(1)}1} \cdots a_{j_{\sigma(k)}k}.$$

This is just the determinant of the quadratic submatrix obtained by taking only the rows $j_1, \ldots, j_k$ of the matrix $[v_1, \ldots, v_k] = [a_{ij}]$. So the coordinates of $v_1 \wedge \cdots \wedge v_k$ are all the $k \times k$ minors of the matrix whose columns are $v_1, \ldots, v_k$. To illustrate, let $k = 2$ and $n = 3$ and consider $[v_1, v_2] = \begin{bmatrix} 2 & 4 \\ 4 & -1 \\ 0 & 2 \end{bmatrix}$. The 2×2 minors of this matrix are -18 (from the first two rows), 8 (from the last two rows), and 4 (from the first and last row). So $v_1 \wedge v_2 = -18e_1 \wedge e_2 + 8e_2 \wedge +4e_3 e_1 \wedge e_3$.

Remark 3.1. There are several equivalent ways of defining the rank of a matrix, one definition is that it is the largest order of any nonzero minor. So a matrix has full rank if one of its minor of highest order is nonzero. So if $v_1, \ldots, v_n$ are linearly dependent then the matrix $[v_1, \ldots, v_k]$ does not have full rank, which is equivalent to saying all its $k \times k$ minors are zero, and so $v_1 \wedge \cdots \wedge v_k = 0$.

Lemma 3.1. Let $v_1, \ldots, v_k \in \mathbb{K}^n$ and $w_1, \ldots, w_k \in \mathbb{K}^n$ be linearly independent, then $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ are linearly dependent in $\Lambda^k\mathbb{K}^n$ if and only span $\{v_1, \ldots, v_k\}$ $=$ span$\{w_1, \ldots, w_k\}$.

Proof. If $\text{span}\{v_1, \ldots, v_k\} = \text{span}\{w_1, \ldots, w_k\}$, then $w_j = \lambda_{1j}v_1 + \cdots + \lambda_{kj}v_k$ for some $\lambda_{ij} \in \mathbb{K}$. So $w_1 \wedge \cdots \wedge w_k = (\lambda_{11}v_1 + \cdots + \lambda_{k1}v_k) \wedge \cdots \wedge (\lambda_{1k}v_1 + \cdots + \lambda_{kk}v_k)$, and therefore $w_1 \wedge \cdots \wedge w_k = \lambda(v_1 \wedge \cdots \wedge v_k)$, for some $\lambda \in \mathbb{K}$, as the other wedge products are 0 because of repeating indices. Now if we start with the assumption $w_1 \wedge \cdots \wedge w_k = \lambda(v_1 \wedge \cdots \wedge v_k)$ for some $\lambda \in \mathbb{K}$, then $\text{span}\{v_1, \ldots, v_k\}$ must be equal to $\text{span}\{w_1, \ldots, w_k\}$, otherwise we can assume $v_1 \notin \text{span}\{w_1, \ldots, w_k\}$ and $v_1, w_1, \ldots, w_k$ are then linearly independent and so their wedge product is not 0, but $v_1 \wedge w_1 \wedge \cdots \wedge w_k = v_1 \wedge \lambda(v_1 \wedge \cdots \wedge v_k) = 0$ □

Let $0 \leq k \leq n$. Consider the map $f : G(k, n) \to \mathbb{P}^{\binom{n}{k}-1}$ which sends a k-dimensional linear subspace $\text{span}\{v_1, \ldots, v_k\}$ to $[v_1 \wedge \cdots \wedge v_k]$, the equivalance class of $v_1 \wedge \cdots \wedge v_k$. By the above lemma, this map is well-defined and injective. It is called the ***Plücker embedding*** of $G(k, n)$. The homogenous coordinates of $[v_1 \wedge \cdots \wedge v_k]$ in $\mathbb{P}^{\binom{n}{k}-1}$ are called the ***Plücker coordinates*** of the k-dimensional linear subspace $\text{span}\{v_1, \ldots, v_k\}$. So using this map we can think of the Grassmannian as a subset of a projective space. But this does not make it a projective variety, to show that we need another lemma.

Lemma 3.2. Let $k < n$. For a nonzero $w \in \Lambda^k\mathbb{K}^n$ consider the $\mathbb{K}$-linear map

$$f : \mathbb{K}^n \to \Lambda^{k+1}\mathbb{K}^n, \ v \mapsto v \wedge w.$$

The kernel of this map has dimension at most k, with equality if and only w is a simple tensor.

Proof. Let $e_1, \ldots, e_n$ be a basis for $\mathbb{K}^n$, such that $e_1, \ldots, e_r$ is a basis of the kernel of f. Let

$$w = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k} e_{i_1} \wedge \cdots \wedge e_{i_k},$$

for some $a_{i_1} \cdots a_{i_k} \in \mathbb{K}$. Now since $e_1, \ldots, e_r \in \ker(f)$, we have

$$f(e_i) = 0 = \sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1} \cdots a_{i_k} e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}, \quad \text{for all } 1 \leq i \leq r.$$

The wedge product $e_i \wedge e_{i_1} \wedge \cdots \wedge e_{i_k}$, where $i \notin \{i_1, \ldots, i_k\}$ is nonzero, so the corresponding coefficient has to be equal to 0. The coefficient $a_{i_1} \cdots a_{i_k}$ can only be nonzero if $i \in \{i_1, \ldots, i_k\}$. Since this is true for all $1 \leq i \leq r$, the coefficient $a_{i_1} \cdots a_{i_k}$ can only be nonzero if $\{1, \ldots, r\} \subset \{i_1, \ldots, i_k\}$. Since $w \neq 0$, at least one coefficient is nonzero and it follows $r \leq k$. If $r = k$, then $w = a_1 \cdots a_k e_1 \wedge \cdots \wedge e_k$.

Conversely, if $w = v_1 \wedge \cdots \wedge v_k$, then $v_1, \ldots, v_k$ are linearly independent since $w \neq 0$ (see Remark 3.1). Now $f(v_i) = 0, \ 1 \leq i \leq k$, so the dimension of the kernel is at least k, but by what we have just argued it is also bounded by k, so we have equality. □

Remark 3.2. The entries of the matrix of the linear map f consist of only 0 or are up to sign the coordinates $a_{i_1} \cdots a_{i_k}$ of w.

Theorem 3.1. *The Grassmannian $G(k, n)$ is a closed subset of $\mathbb{P}^{\binom{n}{k}-1}$.*

Proof. Let $k < n$. Using the Plücker embedding, we can think of $G(k, n)$ as a subset of $\mathbb{P}^{\binom{n}{k}-1}$. An element $w \in \mathbb{P}^{\binom{n}{k}-1}$ is in $G(k, n)$ if it is simple tensor, and this is case if the corresponding linear map f has a kernel of dimension k, or equivalently, rank of f is $n-k$. Since the rank of this map is always at least $n-k$, to check that it is equal to $n-k$, we have to check whether all the $(n-k+1) \times (n-k+1)$ minors of the matrix of f are 0. This is a matrix in the coordinates of w, and therefore the minors are polynomials in the coordinates of w. So $G(k, n)$ is just the solution set of these polynomials. □

Note: From now by $\mathbb{P}^n$ we mean $\mathbb{P}(V^*)$, where V is an $n+1$-dimensional complex vector space.

Projective subspaces are algebraic subvarieties of degree[3] one, so the Grassmannian just parametrizes algebraic subvarieties of degree one. It is natural to look for parameter spaces parametrizing algebraic subvarieties of higher degrees. For this purpose we need the notion of an algebraic cycle.

A $(k-1)$-dimensional ***algebraic cycle*** in $\mathbb{P}^{n-1}$ is a formal finite linear combination $X = \sum m_i X_i$, where $m_i \in \mathbb{Z}_{\geq 0}$ and where the X_i's are $(k-1)$-dimensional irreducible closed subvarieties in $\mathbb{P}^{n-1}$. The degree of X is defined to be $\deg(X) := \sum m_i \deg(X_i)$. We denote the set of all $(k-1)$-dimensional algebraic cycles in $\mathbb{P}^{n-1}$ by $G(k,d,n)$.

The Grassmannian $G(k,1,n)$ is one particular case of something called a Chow variety, which allows us to view the set of algebraic cycles $G(k,d,n)$ as a variety by embedding it into a projective space. In the case of the Grassmannian, this embedding coincides with the Plücker embedding. As the title of this section suggests, we will now introduce the Chow variety $\mathrm{Chow}(d,n) := G(1,d,n+1)$ of 0-cycles, which can be thought of as the *even* analog of the Grasmmannian as we will explain in a short while. Also, just like the case of the Grassmannian, this case also helps illustrate what we mean by embedding the algebraic cycles into a projective space. For discussion of the general Chow variety, we refer the reader to [**13**].

A 0-***cycle*** of degree d in $\mathbb{P}^n$ is a formal finite linear combination $X = x_1 + \cdots + x_d$ of d points in $\mathbb{P}^n$. These points may not be necessarily distinct. Consider the projectivisation $\mathbb{P}(S^d(V^*))$ of the vector space of forms of degree d in $n+1$ variables. The ***Chow form*** R_X of the 0-cycle X is the form $x_1 x_2 \cdots x_d$. The map $X \mapsto R_X$ defines a set-theoretic embedding of the set of 0-cycles of degree d in $\mathbb{P}^n$ into $\mathbb{P}(S^d(V^*))$. The collection of all such Chow forms is called the ***Chow variety*** of 0-cycles in $\mathbb{P}^n$. The above embedding is called the ***Chow embedding*** and the coordinates of the vector R_X are called the ***Chow coordinates***.

The Chow variety of 0-cycles is indeed a variety as it is the image of the morphism

$$(\mathbb{P}^n)^d \to \mathbb{P}(S^d(V^*))$$
$$[v_1] \times \cdots \times [v_d] \mapsto [v_1 \cdots v_d].$$

In short, it is the projectivisation of the space of forms of degree d in $n+1$ variables which can written as products of linear forms, that is, which are factorisable. In this sense it acts as even analog of the Grassmannian which consists of elements in $\Lambda^d \mathbb{C}^{n+1}$ which are decomposable into a wedge product of d vectors.

For a variety X, by $\mathrm{Sym}^d(X)$ we mean the quotient of the cartesian product X^d by S_d. which is again a variety, A detailed explanation of this can be found in [**17**]. If X is an affine variety and R is its coordinate ring, then the coordinate ring of the cartesian product R^d is given by the tensor product $R \otimes \cdots \otimes R$ (d times). The coordinate ring of $\mathrm{Sym}^d(X)$ is then by definition the S_d-invariants in $R \otimes \cdots \otimes R$. In other words, it is the ring of regular functions $f(x_1, \ldots, x_d)$, $x_i \in X$, which remain unchanged under the permuation of x_i.

Consider the morphism of algebraic varieties

$$\phi : \mathrm{Sym}^d(\mathbb{P}^n) \to \mathrm{Chow}(d,n)$$
$$\{x_1, \ldots, x_d\} \mapsto x_1 + \cdots + x_d$$

[3] A definition of the degree of a variety can be found in [**17**].

This is set-theoretically a bijection but this does not imply it is an isomophism.

Theorem 3.2. *The morphism ϕ defined above is an isomorphism of algebraic varities over $\mathbb{C}$*

Proof. Without loss of generality we can restrict our attention to an affine space $\mathbb{C}^n$ inside $\mathbb{P}^n$. Let $x_1, \ldots, x_d$ be any d vectors in $\mathbb{C}^n$ with coordinates $x_i = (x_{i1}, \ldots, x_{in})$. The homogenous coordinates of x_i are $\mathbf{x}_i := (x_{i1}, \ldots, x_{in}, 1)$. Let $X = \mathbf{x}_1 + \cdots + \mathbf{x}_d \in \phi(\mathrm{Sym}^d(\mathbb{C}^n))$. So the Chow form R_X is the form

$$R_X(t_1, \ldots, t_n) = \prod_{i=1}^{d}(x_{i1}t_1 + \cdots + x_{in}t_n + 1 \cdot t_{n+1}).$$

The coefficients of this form are the same as that of the non-homogenous polynomial (1.1), that is, the Chow coordinates of X are the elementary symmetric polynomials $e_{k_1,\ldots,k_n}(x_1, \ldots, x_d)$, where $0 \le k_1 + \cdots + k_n \le d$. So the ring of regular funtions on $\phi(\mathrm{Sym}^d(\mathbb{C}^n))$ is the ring generated by the elementary symmetric polynomials, which by Theorem 1.1 is all the symmetric polynomials. So we have an isomorphism of coordinate rings, which implies the map

$$\phi : \mathrm{Sym}^d(\mathbb{C}^n) \to \phi(\mathrm{Sym}^d(\mathbb{C}^n))$$

is an isomorphism. □

Remark 3.3. This proof only works in fields with characteristic 0. Over fields with characteristic p, if $d \ge p + 1$ and $n \ge p + 1$, the map ϕ is not an isomoprhism [**27**].

Every form of degree d in two variables decomposes into linear forms, since

$$\begin{aligned} f(x, y) &= \sum_{i=0}^{d} a_{ij}x^i y^{d-i} = y^d \sum_{i=0}^{d} a_{ij}(\frac{x}{y})^i \\ &= c_0 y^d(\frac{x}{y} - c_1)(\frac{x}{y} - c_d) = c_0(x - c_1 y)(x - c_d y) \end{aligned}$$

for some $c_0, \ldots, c_d \in \mathbb{C}$ by the fundamental theorem of algebra. So the Chow variety of 0-cycles in this case is $\mathbb{P}(S^d(\mathbb{C}^2)) \cong \mathbb{P}^d(\mathbb{C})$. In other words, it is rational. It turns out all Chow varieties of 0-cycles are rational.

Theorem 3.3. *The Chow variety of* 0*-cycles in* $\mathbb{P}^n$ *of degree* d *is rational. It is birationally isomorphic to the projective space* $\mathbb{P}^{dn}$.

Proof. By Theorem 3.2, $\mathrm{Chow}(d, n) \cong \mathrm{Sym}^d(\mathbb{P}^n)$. Consider $\mathrm{Sym}^d(\mathbb{C}^n) \subset \mathrm{Sym}^d(\mathbb{P}^n)$. By Theorem 1.3, the field of fractions of the coordinate ring of $\mathrm{Sym}^d(\mathbb{C}^n)$ is isomoprhic to the field of rational functions in dn variables over $\mathbb{C}$, which implies $\mathrm{Sym}^d(\mathbb{C}^n)$ is birationally isomorphic to $\mathbb{C}^{dn}$. So $\mathrm{Sym}^d(\mathbb{P}^n)$ and $\mathbb{P}^{dn}$ have isomorphic open subsets and it follows that they are also isomorphic. □

4 When Is a Form Factorizable?

In the section we will only give the explicit description of the covariant and not explain how we arrived at it, which we will discuss in Section 4.

Given a form f of degree d in n variables, its Brill's covariant is a form of multidegree $[d, d, d(d-1)]$ in three sets of n variables $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n)$ and $z = (z_1, \ldots, z_n)$, whose coefficients are forms of degree $d + 1$ in the coefficients of f. That is,

$$B_f(x, y, z) = \sum c_{\alpha,\beta,\gamma}(f) x^\alpha y^\beta z^\gamma,$$

where $\alpha, \beta, \gamma \in \mathbb{Z}^n_{\geq 0}$. We will show that f is factorizable if and only if $B_f(x, y, z) = 0$. Brill's equations are the equations

$$c_{\alpha,\beta,\gamma}(f) = 0.$$

To explicitly describe it, we first need to inroduce two notions: polars and apolar covariant of forms.

4.1 Polars of Forms

Let f be a form of degree d in n variables. For $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ consider the polynomial $f(tx + y)$, where we have multiplied x by a scalar variable t. This is a polynomial of degree d in t, so we can write is as $f(tx + y) = \sum_{m=0}^{d} g_m(x, y)t^m$. Since f is a form of degree d, the coefficient $g_m(x, y)$ of t^m must be a form of degree m in x and a form of degree $d - m$ in y. Such a form is said to be a ***bihomogenous form.*** To illustrate, let $f(x_1, x_2) = x_1^2 + x_2^2$, then

$$\begin{aligned} f(tx_1 + y_1, tx_2 + y_2) &= (tx_1 + y_1)^2 + (tx_2 + y_2)^2 \\ &= (y_1^2 + y_2^2) + (2x_1y_1 + 2x_2y_2)t + (x_1^2 + x_2^2)t^2. \end{aligned}$$

We can rewrite $f(tx + y)$ as

$$f(tx + y) = \sum_{m=0}^{d} \binom{d}{m} f_{x^m}(x, y)t^m,$$

where $f_{x^m}(x, y) = \dfrac{g_m(x, y)}{\binom{d}{m}}$ and is called the ***m-th polar of f***. Now differentiating k-times, $0 \leq k \leq d$, and evaluating at $t = 0$, we get

$$\frac{d^k}{dt^k} f(tx + y)\Big|_{t=0} = \binom{d}{k} k! f_{x^k}(x, y) = \frac{d!}{(d - k)!} f_{x^k}(x, y).$$

Note: By the derivative we mean the formal derivative, which satisfies the same properties as the analytic derivative, like the chain rule and the Leibniz rule.

By the chain rule we have $\dfrac{d}{dt} f(tx + y)\Big|_{t=0} = \sum_{i=1}^{n} x_i \dfrac{\partial f(y)}{\partial y_i}$. So

$$f_{x^k}(x, y) = \frac{(d - k)!}{d!} \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i} \Big)^k f(y). \tag{4.1}$$

Continuing with the above example, let the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C})$ act on the polynomial f as follows

$$\begin{aligned} \tilde{f}(x_1, x_2) = f(ax_1 + bx_2, cx_1 + dx_2) &= (ax_1 + bx_2)^2 + (cx_1 + dx_2)^2 \\ &= (a^2 + c^2)x_1^2 + (b^2 + d^2)x_2^2 + (2ab + 2cd)x_1x_2. \end{aligned}$$

The polars of $\tilde{f}$ are

$$\begin{aligned} \tilde{f}_{x^0}(x, y) &= (a^2 + c^2)y_1^2 + (b^2 + d^2)y_2^2 + (2ab + 2cd)y_1y_2, \\ \tilde{f}_{x^1}(x, y) &= (a^2 + c^2)x_1y_1 + (b^2 + d^2)x_2y_2 + (ab + cd)(x_1y_2 + x_2y_2), \\ \tilde{f}_{x^2}(x, y) &= (a^2 + c^2)x_1^2 + (b^2 + d^2)x_2^2 + (2ab + 2cd)x_1x_2. \end{aligned}$$

The result is the same if we first calculate the polars of f and then let the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ act on them.

In general, it is a straightforward verification that the linear map $\mathcal{P} : f(x) \mapsto f_{x^k}(x, y)$ is $\mathrm{GL}(n, \mathbb{C})$-equivariant.

Remark 4.1. For $x = u$ and $y = v$ in V, with $u \neq 0$, $\frac{d}{dt} f(tu + v)\Big|_{t=0}$ can be thought as the directional derivative of f at v in the direction u and $\sum_{i=1}^{n} x_i \frac{\partial f(y)}{\partial y_i}\Big|_{y=u}$ is the equation of the tangent to the hypersurface $\{f = 0\}$ in $\mathbb{P}V$ (in x) at $y = [u]$.

Let $f = hl$ where l is a linear form and h is a form of degree $d - 1$, then

$$f_{x^k}(x, y) = \frac{(d-k)!}{d!} \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}\Big)^k (h(y)l(y)).$$

By the general Leibniz rule

$$f_{x^k}(x, y) = \frac{(d-k)!}{d!} \Big[\sum_{j=0}^{k} \binom{k}{j} \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}\Big)^{k-j} h(y) \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}\Big)^{j} l(y)\Big].$$

For $j > 1$ the terms vanish since l is linear form, so we have

$$= \frac{(d-k)!}{d!} \Big[l(y) \binom{k}{0} \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}\Big)^{k} h(y) + \binom{k}{1} \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}\Big)^{k-1} h(y) \Big(\sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}\Big) l(y)\Big].$$

After simplifying we get

$$f_{x^k}(x, y) = (\frac{d-k}{d}) l(y) h_{x^k}(x, y) + \frac{k}{d} l(x) h_{x^{k-1}}(x, y). \tag{4.2}$$

If f is the d-th power of a linear form l, that is, $f = l^d$, then $f(tx + y) = l^d(tx + y) = [tl(x) + l(y)]^d = \sum_{k=0}^{d} t^k l^k(x) l^{d-k}(y)$. Therefore

$$l^d_{x^k}(x, y) = l^k(x) l^{d-k}(y). \tag{4.3}$$

4.2 Apolar Covariant of Forms

For two forms f, g of degree d, their ***apolar covariant*** or their ***vertical (Young) product*** is defined by

$$(f \odot g)(x, y) := \frac{1}{d+1} \sum_{k=0}^{d} (-1)^k \binom{d}{k} f_{y^k}(y, x) g_{x^k}(x, y). \tag{4.4}$$

Proposition 4.1. *If f is a form of degree d and l is a linear form, then f is divisible by l if and only if $f \odot l^d = 0$.*

Proof. Let $f \odot l^d = 0$. We have

$$\begin{aligned}
(f \odot l^d)(x, y) &= \frac{1}{d+1} \sum_{k=0}^{d} (-1)^k \binom{d}{k} f_{y^k}(y, x) l^d_{x^k}(x, y) \\
&= \frac{1}{d+1} \sum_{k=0}^{d} (-1)^k \binom{d}{k} f_{y^k}(y, x) l^k(x) l^{d-k}(y) \quad \text{by (4.3)} \\
&= \underbrace{\frac{1}{d+1} \sum_{k=1}^{d} (-1)^k \binom{d}{k} f_{y^k}(y, x) l^k(x) l^{d-k}(y)}_{\text{divisible by } l(x)} + \frac{1}{d+1} (f(x) l^d(y)).
\end{aligned}$$

So $f(x)$ must be divisible by $l(x)$.

Conversely, let $f(x) = l(x)h(x)$ for some linear form $l(x)$. By (4.2)

$$f_{y^k}(y,x) = \frac{(d-k)}{d} l(x) h_{y^k}(y,x) + \frac{k}{d} l(y) h_{y^{k-1}}(y,x).$$

Substituting this in $f \odot l^d$, we get

$$(f \odot l^d)(x,y) = \frac{1}{d+1} \sum_{k=0}^{d} (-1)^k \binom{d}{k} \Big[\frac{(d-k)}{d} l(x) h_{y^k}(y,x) + \frac{k}{d} l(y) h_{y^{k-1}}(y,x) \Big] l^k(x) l^{d-k}(y).$$

So we have

$$\begin{aligned} &\frac{1}{d+1} \sum_{k=0}^{d} (-1)^k \binom{d}{k} \frac{(d-k)}{d} h_{y^k}(y,x) l^{k+1}(x) l^{d-k}(y) \quad \text{(at } k=d \text{ this is zero)} \\ &= \frac{1}{d+1} \sum_{k=0}^{d-1} (-1)^k \frac{(d-1)!}{k!(d-k-1)!} h_{y^k}(y,x) l^{k+1}(x) l^{d-k}(y) \end{aligned}$$

plus

$$\begin{aligned} &\frac{1}{d+1} \sum_{k=0}^{d} (-1)^k \binom{d}{k} \frac{k}{d} h_{y^{k-1}}(y,x) l^k(x) l^{d-k+1}(y) \quad \text{(at } k=0 \text{ this is zero)} \\ &= \frac{1}{d+1} \sum_{k=0}^{d-1} (-1)^{k+1} \binom{d}{k+1} \frac{k+1}{d} h_{y^k}(y,x) l^{k+1}(x) l^{d-(k+1)+1}(y). \end{aligned}$$

Both of the above summations differ by a sign, so their sum is zero, that is, $f \odot l^d = 0$. $\square$

4.3 Brill's Covariant and Factorizable Forms

Let f be a form of degree d. For $x = (x_1, \dots, x_n)$ and $z = (z_1, \dots, z_n)$ consider the following generating function in the scalar variable t, which is a quotient of polynomials

$$E(t) := \frac{f(tf(z)x + z)}{f(z)}.$$

We can expand the numerator in terms of polars and we get

$$f(tf(z)x + z) = \sum_{k=0}^{d} \binom{d}{k} f_{x^k}(x,z)(tf(z))^k.$$

Therefore

$$E(t) = \sum_{k=0}^{d} e_k t^k, \quad \text{where } e_k = \binom{d}{k} f_{x^k}(x,z) f(z)^{k-1}.$$

Each e_k is a form in x, z and the coefficients of f. It has degree k in x, degree $d-k+d(k-1) = k(d-1)$ in z, and k in the coefficients of f.

As we saw in Section 1, using the Newton-Girard formula we can express the power sum $p_d(x_1, \dots, x_d) = x_1^d + \dots + x_d^d$ in terms of the elementary symmetric polynomials. Now if we replace the elementary symmetric polynomials e_k in Equation (1.2) by the coefficient e_k of t^k in $E(t)$ defined above, we get

$$p_d(e_1, \dots, e_d) := (-1)^d d \sum_{i_1 + 2i_2 + \dots + di_d = d} \frac{(i_1 + \dots + i_d - 1)!(-1)^{k_1 + \dots + k_d}}{i_1! \cdots i_d!} e_1^{i_1} \cdots e_d^{i_d}. \quad (4.5)$$

We think of it as a polynomial in x depending on f and z as paremeters. We call it the ***d-th power sum*** and denote it by $P_{f,z}(x) := p_d(e_1, \dots, e_d)$.

Remark 4.2. We can also take the d-th power sum of forms of degree $m < d$ by setting $e_{m+1}, \ldots, e_d = 0$ and then taking $p_d(e_1, \ldots, e_m, 0, \ldots, 0)$.

Remark 4.3. $P_{f,z}(x)$ is a form of degree $1 \cdot i_1 + 2 \cdot i_2 + \cdots + d \cdot i_d = d$ in x, degree $(d-1) \cdot i_1 + 2(d-1) \cdot i_2 + \cdots + d(d-1) \cdot i_d = d(d-1)$ in z and degree $1 \cdot i_1 + 2 \cdot i_2 + \cdots d \cdot i_d = d$ in the coefficients of f.

Remark 4.4. The generating function $E(t)$ of a product f of two polynomials f_1 and f_2 is

$$E(t) = \frac{f(tf(z)x + z)}{f(z)} = \frac{f_1\big(tf_1(z)(f_2(z)x) + z\big)}{f_1(z)} \cdot \frac{f_2\big(tf_2(z)(f_1(z)x) + z\big)}{f_2(z)}.$$

It follows that the d-th power sum of a product of two polynomials f_1 and f_2, with degree less than or equal to d is given by

$$P_{f_1 f_2, z} = f_1(z)^d P_{f_2,z} + f_2(z)^d P_{f_1,z}.$$

Let us consider an example of a power sum in the case $d = 2$. Let $f(x_1, x_2) = ax_1^2 + bx_2^2$. Then

$$\begin{aligned} f(tx_1 + y_1, tx_2 + y_2) &= a(tx_1 + y_1)^2 + b(tx_2 + y_2)^2 \\ &= (ay_1^2 + by_2^2) + (2ax_1y_1 + 2bx_2y_2)t + (ax_1^2 + bx_2^2)t^2. \end{aligned}$$

Now $e_1 = \binom{2}{1} f_{x^1}(x, z) f(z)^0 = 2ax_1z_1 + 2bx_2z_2$ and $e_2 = \binom{2}{2} f_{x^2}(x, z) f(z) = (ax_1^2 + bx_2^2)(az_1^2 + bz_2^2)$. Substituting this in the power sum we get

$$\begin{aligned} P_{f,z}(x) = e_1^2 - 2e_2 &= (2ax_1z_1 + 2bx_2z_2)^2 - 2(ax_1^2 + bx_2^2)(az_1^2 + bz_2^2) \\ &= 2[a^2x_1^2z_1^2 + b^2x_2^2z_2^2 - abx_1^2z_2^2 - abx_2^2z_1^2 + 4abx_1z_1x_2z_2]. \end{aligned}$$

Brill's covariant is the apolar covariant or vertical (Young) product of f and its d-th power sum $P_{f,z}$, that is,

$$B_f(x, y, z) = (f \odot P_{f,z})(x, y).$$

This is a form in (x, y, z) of multidegree $[d, d, d(d-1)]$, whose coefficients are forms of degree $d + 1$ in the coefficients of f.

We continue with the above example. Let g denote $P_{f,z}$. Then

$$\begin{aligned} B_f(x, y, z) = (f \odot g)(x, y) &= \frac{1}{3}\Big[\sum_{k=0}^{2} (-1)^k \binom{2}{k} f_{y^k}(y, x) g_{x^k}(x, y)\Big] \\ &= \frac{1}{3}\Big[f_{y^0}(y, x) g_{x^0}(x, y) - 2f_{y^1}(y, x) g_{x^1}(x, y) + f_{y^2}(y, x) g_{x^2}(x, y)\Big]. \end{aligned}$$

We calculate each of the three terms with the help of (4.2) and (4.3), and we get

$$\begin{aligned} f_{y^0}(y, x) g_{x^0}(x, y) &= 2[ax_1^2 + bx_2^2][a^2y_1^2z_1^2 + b^2y_2^2z_2^2 - aby_1^2z_2^2 - aby_2^2z_1^2 + 4aby_1z_1y_2z_2] \\ f_{y^1}(y, x) g_{x^1}(x, y) &= 2[ay_1x_1 + by_2x_2][a^2x_1y_1z_1^2 + b^2x_2y_2z_2^2 - abx_1y_1z_2^2 - abx_2y_2z_1^2 \\ &\qquad + 2(abx_1y_2 + x_2y_1)z_1z_2] \\ f_{y^2}(y, x) g_{x^2}(x, y) &= 2[ay_1^2 + by_2^2][a^2x_1^2z_1^2 + b^2x_2^2z_2^2 - abx_1^2z_2^2 - abx_2^2z_1^2 + 4abx_1z_1x_2z_2]. \end{aligned}$$

In the end we do get a form in (x, y, z) of multidegree $[2, 2, 2]$, whose coefficients are forms of degree 3 in the coefficients of f. But we also see that it vanishes. So Brill's covariant of the factorizable form $f(x_1, x_2) = ax_1^2 + bx_2^2$ is 0.

With the help of following lemma due to Briand [**3**] [4] we will able to prove the main result of Brill.

Lemma 4.1. Let f be a form of degree d with $B_f(x, y, z) = 0$. Let $[w]$ be a smooth point on the reduced hypersurface $\{f = 0\}$. Then the irreducible component of $\{f = 0\}$ containing $[w]$ is a hyperplane.

Proof. Let $g = 0$ be an equation of the irreducible component of $\{f = 0\}$ containing $[w]$. We first assume that g is not a multiple factor of f. In $P_{f,z}(x)$ (Equation 4.5), let $i_1 = d$ and we see that $e_1^d = \binom{d}{1} f_x^d(x, z)$ occurs in $P_{f,z}(x)$ with coefficient 1. So

$$P_{f,z} = \text{ constant} \cdot f_x^d(x, z) + \text{ terms divisible by } f(z).$$

For $z = w$, we have $P_{f,w} = \text{constant} \cdot f_x^d(x, w)$. Now $f_x(x, w) = \frac{1}{d} \sum_{i=1}^{n} \frac{\partial f(z)}{\partial z_i} x_i$ (see Remark 3.1) is an equation of the tangent hyperplane to $\{f = 0\}$ at $[w]$. Since $[w]$ is smooth on $\{f = 0\}$ and cancels no multiple factor of f, $f_x(x, w)$ is non-zero. Now B_f vanishes identically, so in particular, it vanishes identically with respect to x and y, and we have $f \odot P_{f,w} = \text{const} \cdot f \odot f_x^d(x, w) = 0$. By Propostion 4.1, this means f is divisible by $f_x(x, w)$. Therefore the hyperplane $\{f_x(x, w) = 0\}$ is the irreducible component of $\{f = 0\}$ containing $[w]$. Since $[w]$ is smooth on $\{f = 0\}$, it does not belong to any other component of $\{f = 0\}$.

Now we consider the case where g is a multiple factor of f, that is , $f = g^k h$, where g does not divide h. By Remark 4.4, the d-the power sum of a product of two polynomials f_1 and f_2, with degree less than or equal to d, is given by

$$P_{f_1 f_2, z} = f_1(z)^d P_{f_2, z} + f_2(z)^d P_{f_1, z}.$$

Using this we get

$$P_{g^k, z} = P_{g^{k-1} g, z} = (g^{k-1}(z))^d P_{g,z} + g(z)^d P_{g^{k-1}, z}.$$

and similarly

$$P_{g^{k-1}, z} = P_{g^{k-2} g, z} = (g^{k-2}(z))^d P_{g,z} + g(z)^d P_{g^{k-2}, z}.$$

Substituting $P_{g^{k-1}, z}$ in $P_{g^k, z}$ we get

$$\begin{aligned} P_{g^k, z} &= (g^{k-1}(z))^d P_{g,z} + g(z)^d \Big[(g^{k-2}(z))^d P_{g,z} + g(z)^d P_{g^{k-2}, z} \Big] \\ &= 2g^{d(k-1)}(z) P_{g,z} + g(z)^{2d} P_{g^{k-2}, z}. \end{aligned}$$

Proceeding recursively, in the end we have

$$\begin{aligned} P_{g^k, z} &= k g^{d(k-1)}(z) P_{g,z} + g(z)^{kd} \underbrace{P_{g^{k-k}, z}}_{=0} \\ &= k g^{d(k-1)}(z) P_{g,z}. \end{aligned}$$

Therefore

$$\begin{aligned} P_{g^k h, z} &= (g^k(z))^d P_{h,z} + h(z)^d P_{g^k, z} \\ &= (g^k(z))^d P_{h,z} + k h(z)^d g^{d(k-1)}(z) P_{g,z} \\ &= g^{d(k-1)}(z) \Big[(g(z))^d P_{h,z} + k h(z)^d P_{g,z} \Big]. \end{aligned}$$

[4] As Briand explains in [**3**], the proof given in [**4**][**13**][**14**] is incomplete, since these texts ignore the possibility that Brill's covariant could vanish for a non-totally decomposable form with multiple factors.

Now

$$\begin{aligned} B_f = f \odot P_{f,z} &= f \odot P_{g^k h,z} \\ &= f \odot \Big[g^{d(k-1)}(z) \big[(g(z))^d P_{h,z} + kh(z)^d P_{g,z} \big] \Big] \\ &= g^{d(k-1)}(z) \Big[(g(z))^d \big(f \odot P_{h,z} \big) + kh(z)^d \big(f \odot P_{g,z} \big) \Big]. \end{aligned}$$

Since B_f vanishes identically, so does

$$(g(z))^d \big(f \odot P_{h,z} \big) + kh(z)^d \big(f \odot P_{g,z} \big).$$

For $z = w$, we then have

$$kh(w)^d (f \odot P_{g,w}) = 0,$$

and $h(w) \neq 0$ implies

$$f \odot P_{g,w} = 0.$$

Now we can conclude as in the first case that $g_x(x, w)$ divides f. $\square$

Theorem 4.1. *A form of degree d is a product of linear forms if and only if* $B_f(x, y, z) = 0$.

Proof. Let $f(x) = l_1(x) \cdots l_d(x)$ then

$$\begin{aligned} E(t) = \frac{f(tf(z)x + z)}{f(z)} &= \frac{l_1(tf(z)x + z) \cdots l_d(tf(z)x + z)}{f(z)} \\ &= \frac{[tf(z)l_1(x) + l_1(z)] \cdots [tf(z)l_d(x) + l_d(z)]}{f(z)} \\ &= \frac{l_1(z)l_2(z) \cdots l_d(z)}{f(z)} [1 + \frac{tf(z)}{l_1(z)} l_1(x)] \cdots [1 + \frac{tf(z)}{l_d(z)} l_d(x)] \\ &= \prod_{i=1}^{d} \Big[1 + (\frac{f(z)}{l_i(z)} l_i(x))t \Big]. \end{aligned}$$

If you expand this, the coefficients of t^i are elementary symmetric polynomials in $\frac{f(z)}{l_i(z)} l_i(x)$. So the d-the power sum is

$$P_{f,z}(x) = \sum_{i=1}^{d} \frac{f(z)^d}{l_i^d(z)} l_i^d(x)$$

and Brill's covariant is

$$B_f(x, y, z) = (f \odot P_{f,z})(x, y) = \sum_{i=1}^{d} \frac{f(z)^d}{l_i^d(z)} f \odot l_i^d(x).$$

Since f is divisible by l_i, $f \odot l_i^d = 0$ for all i by Proposition 4.1, and therefore $B_f(x, y, z) = 0$.
Conversely, any irreducible component of $\{f = 0\}$ has a smooth point $[w]$ not contained in any other irreducible component of $\{f = 0\}$ and since B_f vanishes identically, by Lemma 4.1 this irreducible component is necessarily a hyperplane. Therefore f is totally decomposable. $\square$

Using Brill's covariant we will recover the condition for factorizability for a quadratic form stated in Theorem 0.1.

Let f be a quadratic form. Let $\phi(x,y) \coloneqq f_x(x,y)$ be the symmetric bilinear form such that $f(x) = \phi(x,x)$. This implies $\phi(x,z) = f_x(x,z)$, $\phi(y,z) = f_y(y,z)$, $\phi(y,x) = f_y(y,x)$ and $f(z) = \phi(z,z)$.

Now $e_1 = \binom{2}{1} f_x(x,y) = 2f_x(x,z) = 2\phi(x,z)$ and $e_2 = \binom{2}{2} f_{x^2}(x,z)f(z) = f(x)f(z) = \phi(x,x)\phi(z,z)$. Let $g(x)$ denote $P_{f,z}(x)$. So the power sum is

$$g(x) = e_1^2 - 2e_2 = 4f_x^2(x,z) - 2f(x)f(z),$$

which implies

$$g_x(x,y) = 4f_x(x,z)f_y(y,z) - 2f_x(x,y)f(z).$$

Brill's covariant of f is

$$\begin{aligned} B_f(x,y,z) = (f \odot g)(x,y) &= \frac{1}{3}\sum_{k=0}^{2}(-1)^k \binom{2}{k} f_{y^k}(y,x)g_{x^k}(x,y) \\ &= \frac{1}{3}\Big[\binom{2}{0} f_{y^0}(y,x)g_{x^0}(x,y) - \binom{2}{1} f_y(y,x)g_x(x,y) \\ &\qquad + \binom{2}{2} f_{y^2}(y,x)g_{x^2}(x,y)\Big] \\ &= \frac{1}{3}\Big[f(x)g(y) - 2f_y(y,x)g_x(x,y) + f(y)g(x)\Big]. \end{aligned}$$

We now calculate the three terms above and we get

$$\begin{aligned} f(x)g(y) = \phi(x,x)g(y) &= \phi(x,x)[4\phi^2(y,z) - 2\phi(y,y)\phi(z,z)] \\ &= 4\phi(x,x)\phi^2(y,z) - 2\phi(x,x)\phi(y,y)\phi(z,z), \end{aligned}$$

$$\begin{aligned} -2f_y(y,x)g_x(x,y) &= -2\phi(y,x)[4\phi(x,z)\phi(y,z) - 2\phi(x,y)\phi(z,z)] \\ &= -8\phi(x,y)\phi(y,z)\phi(x,z) + 4\phi^2(x,y)\phi(z,z), \end{aligned}$$

and

$$\begin{aligned} f(y)g(x) &= \phi(y,y)[4\phi^2(x,z) - 2\phi(x,x)\phi(z,z)] \\ &= 4\phi(y,y)\phi^2(x,z) - 2\phi(x,x)\phi(y,y)\phi(z,z). \end{aligned}$$

Adding all three we get

$$\begin{aligned} f(x)g(y) - 2f_y(y,x)g_x(x,y) + f(y)g(x) = &-4\phi(x,x)\phi(y,y)\phi(z,z) \\ &- 8\phi(x,y)\phi(y,z)\phi(x,z) \\ &+ 4[\phi^2(x,y)\phi(z,z) \\ &+ \phi(y,y)\phi^2(x,z) + \phi(x,x)\phi^2(y,z)]. \end{aligned}$$

Substituting this in $B_f(x,y,z)$ we get

$$\begin{aligned} B_f(x,y,z) &= \frac{-4}{3}\Big[\phi(x,x)\phi(y,y)\phi(z,z) + 2\phi(x,y)\phi(y,z)\phi(x,z) \\ &\qquad - [\phi^2(x,y)\phi(z,z) + \phi(y,y)\phi^2(x,z) + \phi(x,x)\phi^2(y,z)]\Big] \\ &= \frac{-4}{3}\Big[\phi(x,x)\begin{vmatrix}\phi(y,y) & \phi(y,z)\\ \phi(z,y) & \phi(z,z)\end{vmatrix} - \phi(x,y)\begin{vmatrix}\phi(y,x) & \phi(y,z)\\ \phi(z,x) & \phi(z,z)\end{vmatrix} \\ &\qquad + \phi(x,z)\begin{vmatrix}\phi(y,x) & \phi(y,y)\\ \phi(z,x) & \phi(z,y)\end{vmatrix}\Big] \quad \text{as } \phi \text{ is symmetric.} \end{aligned}$$

So up to a nonzero multiple

$$B_f(x, y, z) = \det \begin{pmatrix} \phi(x,x) & \phi(x,y) & \phi(x,z) \\ \phi(y,x) & \phi(y,y) & \phi(y,z) \\ \phi(z,x) & \phi(z,y) & \phi(z,z) \end{pmatrix}.$$

The coefficients of $B_f(x, y, z)$ are forms of degree 3 in the coefficients of f. f is a product of two linear forms if and only if the above determinant vanishes. This means that the restriction of ϕ on every 3-dimensional subspace of V is degenerate, that is, $\text{rank}(\phi) \leq 2$.

5 Interpretation of Brill's covariant

In this section we arrive at the defining formula (4.4) for the apolar covariant of two forms using the language of representation theory.

By Theorem 2.2 (i) we have the following decomposition of $S^d(V^*) \otimes S^d(V^*)$ as a $GL(V^*)$-module

$$\begin{aligned} S^d(V^*) \otimes S^d(V^*) &= \bigoplus_{k=0}^{d} S^{(d-k,d+k)}(V^*) \\ &= S^{(d,d)}(V^*) \oplus \Big(\bigoplus_{k=1}^{d} S^{(d-k,d+k)}(V^*) \Big). \end{aligned}$$

$S^d(V^*)$ is the vector space of forms of degree d, so we can idenitfy $S^d(V^*) \otimes S^d(V^*)$ with the space of bihomogenous forms of bidegree (d, d) by the map $f \otimes g \mapsto f(x)g(y)$. Under this identification $S^d(V^*) \otimes S^d(V^*)$ can thought of as the subspace of $S^{2d}(V^* \oplus V^*)$, which is the vector space of bihomogenous forms $F(x, y)$ of total degree $2d$, where $x, y \in V^*$.

For two forms $f, g \in S^d(V^*)$, we claim that that the component of $f(x)g(y)$ lying in $S^{(d,d)}(V^*)$ is given by the apolar covariant $(f \odot g)(x, y)$ of f and g. But before we prove this claim we need to introduce the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$.

5.1 The Lie Algebra $\mathfrak{sl}(2, \mathbb{C})$ and its Generators

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ consists of complex 2×2 invertible matrices that have trace 0 and where the Lie bracket is given by the commutator $[x, y] = xy - yx,\ x, y \in \mathfrak{sl}(2, \mathbb{C})$. Its underlying matrix Lie group is the special linear group $SL(2, \mathbb{C})$, which consists of complex 2×2 invertibles matrices with determinant 1. As a complex vector space, $\mathfrak{sl}(2, \mathbb{C})$ has a basis given by

$$e_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \qquad e_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

These basis elements satistfy the commutation relations $[e_+, e_-] = h,\ [h, e_+] = 2e_+$, and $[h, e_-] = -2e_-$.

Consider now the complex vector space $S^{2d}(V^* \oplus V^*)$ which, as we already discussed, can be identified with the space of bihomogenous forms of total degree $2d$. Let $\mathfrak{gl}(S^{2d}(V^* \oplus V^*))$ denote the Lie algebra of endomorphisms on $S^{2d}(V^* \oplus V^*)$, which consists of linear maps from $S^{2d}(V^* \oplus V^*)$ to itself and where the Lie bracket is given by the commutator. Consider now the following endomorphisms on $S^{2d}(V^* \oplus V^*)$

$$E_+ = \sum_{i=1}^{n} x_i \frac{\partial}{\partial y_i}; \qquad E_- = \sum_{j=1}^{n} y_j \frac{\partial}{\partial x_j}; \qquad H = \sum_{i=1}^{n} (x_i \frac{\partial}{\partial x_i} - y_i \frac{\partial}{\partial y_i}).$$

Using a straightforward computation one can show that these endomorphisms satisfy the same commutation relations as the basis elements of $\mathfrak{sl}(2, \mathbb{C})$.

The map from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{gl}(S^{2d}(V^*\oplus V^*))$ given by $e_+\mapsto E_+,\ e_-\mapsto E_-,\ h\mapsto H$ is then a Lie algebra homomorphism, in other words, $S^{2d}(V^*\oplus V^*)$ is a representation of $\mathfrak{sl}(2,\mathbb{C})$.

We now examine the action of E_+, E_- and H on polars of forms, which will later help us in computing the covariant.

Lemma 5.1. The action of H on a bihomogenous form $f(y,x)$ of degree k in y and degree $d-k$ in x is given by

$$H[f(y,x)] = (d-2k)f(y,x).$$

Proof. Without loss of generality let

$$f(y,x) = x_1^{j_1}\cdots x_n^{j_n}y_1^{i_1}\cdots y_n^{i_n},$$

where $j_1+\cdots j_n = d-k$ and $i_1+\cdots+i_n = k$. Now

$$H[f(y,x)] = \sum_{i=1}^{n}\Big[x_i\frac{\partial}{\partial x_i}f(y,x)\Big] - \sum_{i=1}^{n}\Big[y_i\frac{\partial}{\partial y_i}f(y,x)\Big],$$

where

$$\begin{aligned}\sum_{i=1}^{n}\Big[x_i\frac{\partial}{\partial x_i}f(y,x)\Big] &= \sum_{i=1}^{n}\Big[x_i\frac{\partial}{\partial x_i}(x_1^{j_1}\cdots x_n^{j_n}y_1^{i_1}\cdots y_n^{i_n})\Big]\\ &= \sum_{i=1}^{n}[y_1^{i_1}\cdots y_n^{i_n}j_1x_1^{j_1}\cdots x_n^{j_n}+\cdots+j_nx_n^{j_n}\cdots x_n^{j_n}]\\ &= \sum_{i=1}^{n}[y_1^{i_1}\cdots y_n^{i_n}x_1^{j_1}\cdots x_n^{j_n}\underbrace{(j_1+\cdots j_n)}_{=d-k}]\\ &= (d-k)x_1^{j_1}\cdots x_n^{j_n}y_1^{i_1}\cdots y_n^{i_n} = (d-k)f(y,x),\end{aligned}$$

and by similar calculations

$$\sum_{i=1}^{n}\Big[y_i\frac{\partial}{\partial y_i}f(y,x)\Big] = kf(y,x).$$

After taking the difference we get the result. □

Lemma 5.2. Let g be a form of degree d in n variables, then

$$E_-^{m-k}E_+^m g(y) = \frac{m!(d-k)!}{k!(d-m)!}E_+^k g(y).$$

Proof. We use induction on m. For $m=0$, k is also 0, and we have equality. Now

$$\begin{aligned}E_-^{m+1-k}E_+^{m+1} &= E_-^{m-k}E_-E_+E_+^m\\ &= E_-^{m-k}[-H+E_+E_-]E_+^m \quad \text{by the commutation relation}\\ &= -E_-^{m-k}HE_+^m + E_-^{m-k}E_+E_-E_+^m.\end{aligned}$$

So we have

$$\begin{aligned}E_-^{m-k}E_+E_-E_+^m &= E_-^{m-k}E_+E_-^{m-(m-1)}E_+^m\\ &= E_-^{m-k}E_+m(d-m+1)E_+^{m-1}\\ &= m(d-m+1)E_-^{m-k}E_+^m \quad \text{by inductive hypothesis}\end{aligned}$$

and

$$\begin{aligned}
-E_-^{m-k}HE_+^m &= -E_-^{m-k}[2E_+ + E_+H]E_+^{m-1} \quad \text{by the commutation relation}\\
&= -2E_-^{m-k}E_+^m - E_-^{m-k}(E_+H)E_+^{m-1}\\
&= -2E_-^{m-k}E_+^m - E_-^{m-k}E_+(HE_+)E_+^{m-2}\\
&= -2E_-^{m-k}E_+^m - E_-^{m-k}E_+(2E_+ + E_+H)E_+^{m-2}\\
&= -2\cdot 2E_-^{m-k}E_+^m - E_-^{m-k}E_+^2HE_+^{m-2}\\
&\vdots\\
&= -2\cdot mE_-^{m-k}E_+^m - E_-^{m-k}E_+^mHE_+^{m-m}\\
&= -2\cdot mE_-^{m-k}E_+^m - E_-^{m-k}E_+^mH.
\end{aligned}$$

Adding the two we get

$$E_-^{m+1-k}E_+^{m+1} = m(d-m+1)E_-^{m-k}E_+^m - 2\cdot mE_-^{m-k}E_+^m - E_-^{m-k}E_+^mH$$

and therefore

$$\begin{aligned}
E_-^{m+1-k}E_+^{m+1}g(y) &= m(d-m+1)E_-^{m-k}E_+^mg(y) - 2\cdot mE_-^{m-k}E_+^mg(y) - E_-^{m-k}E_+^mHg(y)\\
&= m(d-m+1)E_-^{m-k}E_+^mg(y) - 2\cdot mE_-^{m-k}E_+^mg(y)\\
&\qquad + dE_-^{m-k}E_+^m \qquad \text{as } Hg(y) = -dg(y) \text{ by Lemma 5.1}\\
&= (m+1)(d-m)E_-^{m-k}E_+^mg(y)\\
&= \frac{(m+1)!(d-k)!}{k!(d-(m+1))!}E_+^kg(y) \quad \text{by inductive hypothesis.}
\end{aligned}$$

□

Lemma 5.3. For forms f and g of degree d in n variables, the action of E_+ and E_- on their polars is given by

$$\begin{aligned}
E_+^m[g_{x^k}(x,y)] &= \frac{(d-k)!}{[d-(k+m)]!}g_{x^{k+m}}(x,y),\\
E_-^m[f_{y^k}(y,x)] &= \frac{(d-k)!}{[d-(k+m)]!}f_{y^{k+m}}(y,x),
\end{aligned}$$

if $k+m \leq d$ and 0 otherwise. Moreover

$$\begin{aligned}
E_-^m[g_{x^k}(x,y)] &= \frac{k!}{(k-m)!}g_{x^{k-m}}(x,y),\\
E_+^m[f_{y^k}(y,x)] &= \frac{k!}{(k-m)!}f_{y^{k-m}}(y,x),
\end{aligned}$$

if $k-m \geq 0$ and 0 otherwise.

Proof. By definiton of the polar (4.1) we have

$$E_+^kg(y) = \frac{d!}{(d-k)!}g_{x^k}(x,y).$$

This implies

$$E_+^{k+1}g(y) = \frac{d!}{[d-(k+1)]!}g_{x^{k+1}}(x,y).$$

Also,

$$E_+[E_+^k g(y)] = E_+[\frac{d!}{(d-k)!} g_{x^k}(x,y)].$$

Comparing the two we get

$$E_+ g_{x^k}(x,y) = (d-k) g_{x^{k+1}}(x,y) = \frac{(d-k)!}{[d-(k+1)]} g_{x^{k+1}}(x,y).$$

Proceeding recursively we get the result in the case $k+m \leq d$. For $k+m > d$ the power of the exponents reduces to 0 and the derivative of the resulting constant terms is 0. Similar arguments work for $E_-^m[f_{y^k}(y,x)]$.

Now

$$E_- E_+^m g(y) = E_-[\frac{d!}{(d-m)!} g_{x^m}(x,y)].$$

Also by Lemma 5.2

$$\begin{aligned} E_- E_+^m g(y) = E_-^{m-(m-1)} E_+^m g(y) &= \frac{m![d-(m-1)]!}{(m-1)!(d-m)!} E_+^{m-1}[g(y)] \\ &= m(d-m+1) E_+^{m-1}[g(y)] = \frac{md!}{(d-m)!} g_{x^{m-1}}(x,y). \end{aligned}$$

Comparing the two we get

$$E_-[g_{x^m}(x,y)] = m g_{x^{m-1}}(x,y),$$

which written differently is

$$E_-[g_{x^k}(x,y)] = k g_{x^{k-1}}(x,y) = \frac{k!}{(k-1)!} g_{x^{k-1}}(x,y).$$

Proceeding recursively we get the result in the case $k-m \geq 0$. In the case $k-m \ngeq 0$, we get 0 by the same reasoning as before. Similar arguments work for $E_+^m[f_{y^k}(y,x)]$. □

5.2 Projection onto the Space of Covariants

As discussed in the previous subsection, the Lie Algebra $\mathfrak{sl}(2,\mathbb{C})$ acts on $S^{2d}(V^* \oplus V^*)$ via E_+, E_- and H. The formula for H implies

$$S^d(V^*) \otimes S^d(V^*) = \{F \in S^{2d}(V^* \oplus V^*) \mid HF = 0\}.$$

$V^* \oplus V^*$ can be identified with $V^* \otimes \mathbb{C}^2$. By Remark 2.1 we get the following decomposition of $S^{2d}(V^* \otimes \mathbb{C}^2)$ as a module over $GL(V^*) \times GL(2,\mathbb{C})$

$$\begin{aligned} S^{2d}(V^* \otimes \mathbb{C}^2) &= \bigoplus_{k=0}^{d} S^{(d+k,d-k)}(V^*) \otimes S^{(d+k,d-k)}(\mathbb{C}^2) \\ &= \left(S^{(d,d)}(V^*) \otimes S^{(d,d)}(\mathbb{C}^2)\right) \oplus \bigoplus_{k=1}^{d} S^{(d+k,d-k)}(V^*) \otimes S^{(d+k,d-k)}(\mathbb{C}^2). \end{aligned}$$

By Theorem 2.1 $\dim(S^{(d,d)}(\mathbb{C}^2)) = 1$. From the above decomposition it follows immediately that

$$\begin{aligned} S^{(d,d)}(V^*) &= \{F \in S^{2d}(V^* \oplus V^*) \mid IF = 0 \text{ for all } I \in \mathfrak{sl}(2,\mathbb{C})\} \\ &= \{F \in S^{2d}(V^* \oplus V^*) \mid HF = E_+F = E_-F = 0\} \subset S^d(V^*) \otimes S^d(V^*). \end{aligned}$$

In short, $S^{(d,d)}(V^*)$ is precisely the submodule in $S^{2d}(V^* \oplus V^*)$ which is annihilated by $\mathfrak{sl}(2,\mathbb{C})$.

We now introduce a sum that appears when we are computing the covariant that we will simplify using the Lemma below. We define

$$b_{k,d} := \sum_{m=k}^{d} \frac{(-1)^{m-k}(d-k)!}{(m+1)(m-k)!(d-m)!}.$$

Lemma 5.4. $b_{k,d} = \int_0^1 t^k(1-t)^{d-k}dt = \dfrac{k!(d-k)!}{(d+1)!}$.

Proof. Let $n = m-k$, then

$$b_{k,d} = \sum_{n=0}^{d-k} \frac{(-1)^n(d-k)!}{(n+k+1)n!(d-(n+k))!}.$$

Now $(1-t)^{d-k} = \sum_{n=0}^{d-k} \binom{d-k}{n}(-1)^n t^n$. Multiplying by t^k we get

$$t^k(1-t)^{d-k} = \sum_{n=0}^{d-k} \binom{d-k}{n}(-1)^n t^{n+k}.$$

Integrating this from 0 to 1 we get

$$\begin{aligned}\int_0^1 t^k(1-t)^{d-k} &= \Big[\sum_{n=0}^{d-k} \binom{d-k}{n}\frac{(-1)^n t^{n+k+1}}{n+k+1}\Big]\Big|_0^1 \\ &= \sum_{n=0}^{d-k} \binom{d-k}{n}\frac{(-1)^n}{n+k+1} \\ &= \sum_{n=0}^{d-k} \frac{(-1)^n(d-k)!}{(n+k+1)n![(d-k)-n)]!} = b_{k,d}.\end{aligned}$$

The integeral is the beta function $B(k+1, d-k+1) = \frac{\Gamma(k+1)\Gamma(d-k+1)}{\Gamma(d+2)}$, where Γ is the gamma function, which for a positive integer n is $\Gamma(n) = (n-1)!$. So

$$b_{k,d} = \sum_{m=k}^{d} \frac{(-1)^{m-k}(d-k)!}{(m+1)(m-k)!(d-m)!} = \frac{k!(d-k)!}{(d+1)!}.$$

□

Theorem 5.1. *The map from $S^d(V^*) \otimes S^d(V^*)$ onto $S^{(d,d)}(V^*)$ given by*

$$\begin{aligned}\pi : S^d(V^*) \otimes S^d(V^*) &\to S^{(d,d)}(V^*) \\ F &\mapsto \sum_{m=0}^{d} \frac{(-1)^m}{m!(m+1)!} E_-^m E_+^m(F),\end{aligned}$$

has the following properties

(i) For forms $f, g \in S^d(V^*)$, we have

$$\pi(f(x)g(y)) = (f \odot g)(x,y).$$

(ii) π is a $GL(V^*)$-equivariant projection map.

Proof. (i) We first show that for $F = f(x)g(y) \in S^d(V^*) \otimes S^d(V^*)$, $\pi(F) = (f \odot g)(x,y)$. Now

$$\begin{aligned}\pi(F) &= \sum_{m=0}^{d} \frac{(-1)^m}{m!(m+1)!} E_-^m E_+^m (f(x)g(y)) \\ &= \sum_{m=0}^{d} \frac{(-1)^m}{m!(m+1)!} E_-^m (f(x) E_+^m g(y)).\end{aligned}$$

By the general Leibniz rule

$$E_-^m(f(x)E_+^m g(y)) = \sum_{k=0}^{m} \binom{m}{k} (E_-^k(f(x)) E_-^{m-k}(E_+^m g(y)).$$

Substituting this in $\pi(F)$ we get

$$\begin{aligned}\pi(F) &= \sum_{m=0}^{d} \frac{(-1)^m}{m!(m+1)!} [\sum_{k=0}^{m} \binom{m}{k} (E_-^k(f(x)) E_-^{m-k}(E_+^m g(y))] \\ &= \sum_{m=0}^{d} \frac{(-1)^m}{m!(m+1)!} [\sum_{k=0}^{m} \binom{m}{k} (E_-^k(f(x)) E_-^{m-k}(E_+^m g(y))] \\ &= \sum_{m=0}^{d} \frac{(-1)^m}{m!(m+1)!} [\sum_{k=0}^{m} \binom{m}{k} \frac{d!}{(d-k)!} f_{y^k}(y,x) \frac{m!d!}{k!(d-m)!} g_{x^k}(x,y)] \quad \text{by Lemma 5.3} \\ &= \sum_{m=0}^{d} \sum_{k=0}^{m} \frac{(-1)^m}{m!(m+1)!} \binom{m}{k} \frac{d!}{(d-k)!} f_{y^k}(y,x) \frac{m!d!}{k!(d-m)!} g_{x^k}(x,y) \\ &= \sum_{k=0}^{d} \sum_{m=k}^{d} \frac{(-1)^m}{m!(m+1)!} \frac{m!}{k!(m-k)!} \frac{d!}{(d-k)!} f_{y^k}(y,x) \frac{m!d!}{k!(d-m)!} g_{x^k}(x,y) \\ &= \sum_{k=0}^{d} \sum_{m=k}^{d} \frac{(-1)^{m-k}(-1)^k (d-k)!}{(m+1)(m-k)(d-m)!} \binom{d}{k}^2 f_{y^k}(y,x) g_{x^k}(x,y) \\ &= \sum_{k=0}^{d} b_{k,d} (-1)^k \binom{d}{k}^2 f_{y^k}(y,x) g_{x^k}(x,y) \\ &= \sum_{k=0}^{d} \frac{k!(d-k)!}{(d+1)!} (-1)^k \binom{d}{k}^2 f_{y^k}(y,x) g_{x^k}(x,y) \quad \text{by Lemma 5.4} \\ &= \frac{1}{(d+1)} \sum_{k=0}^{d} (-1)^k \binom{d}{k} f_{y^k}(y,x) g_{x^k}(x,y) = (f \odot g)(x,y).\end{aligned}$$

□

Proof. (ii) To show that π is a $GL(V^*)$-equivariant projection map, we need to show four things

(a) π is $GL(V^*)$-equivariant.

(b) The image of π is annihilated by $\mathfrak{sl}(2,\mathbb{C})$.

(c) $(\pi)^2 = \pi$.

(d) $\pi \neq 0$ on $S^d(V^*) \otimes S^d(V^*)$.

(a) As mentioned in Section 4.1, the linear map $\mathcal{P} : f(x) \mapsto f_{x^k}(x, y)$ is $GL(n, \mathbb{C})$-equivariant and this implies π is $GL(V^*)$-equivariant.

(b) To show that $\pi(F)$ is annihilated by $\mathfrak{sl}(2, \mathbb{C})$, it is enough to show that it is annihilated by E_+, E_- and H. Now

$$\begin{aligned} E_+(f_{y^k}(y,x)g_{x^k}(x,y)) &= E_+(f_{y^k}(y,x))g_{x^k}(x,y) + f_{y^k}(y,x)E_+(g_{x^k}(x,y)) \\ &= kf_{y^{k-1}}(y,x)g_{x^k}(x,y) + f_{y^k}(y,x)(d-k)g_{x^{k+1}}(x,y) \quad \text{by Lemma 5.3.} \end{aligned}$$

Subsituting this in $E_+(\pi(F))$ we get

$$\begin{aligned} E_+(\pi(F)) &= \frac{1}{(d+1)} \sum_{k=0}^{d} (-1)^k \binom{d}{k} E_+[f_{y^k}(y,x)g_{x^k}(x,y)] \\ &= \frac{1}{(d+1)} \sum_{k=0}^{d} (-1)^k \binom{d}{k} kf_{y^{k-1}}(y,x)g_{x^k}(x,y) \\ &\quad + \frac{1}{(d+1)} \sum_{l=0}^{d} (-1)^l \binom{d}{l} f_{y^l}(y,x)(d-l)g_{x^{l+1}}(x,y). \end{aligned}$$

Both of these summations are 0 at $k = 0$ and $l = d$ respectively, so we have

$$\begin{aligned} E_+(\pi(F)) &= \frac{1}{(d+1)} \sum_{k=1}^{d} (-1)^k \binom{d}{k} kf_{y^{k-1}}(y,x)g_{x^k}(x,y) \\ &\quad + \frac{1}{(d+1)} \sum_{l=0}^{d-1} (-1)^l \binom{d}{l} f_{y^l}(y,x)(d-l)g_{x^{l+1}}(x,y) \\ &= \frac{1}{(d+1)} \sum_{l=0}^{d-1} (-1)^{l+1} \binom{d}{l+1} (l+1)f_{y^l}(y,x)g_{x^{l+1}}(x,y) \\ &\quad + \frac{1}{(d+1)} \sum_{l=0}^{d-1} (-1)^l \binom{d}{l} f_{y^l}(y,x)(d-l)g_{x^{l+1}}(x,y) \\ &= \frac{1}{(d+1)} \sum_{l=0}^{d-1} \underbrace{\left[(-1)^{l+1} \binom{d}{l+1} (l+1) + (-1)^l \binom{d}{l} (d-l)\right]}_{=0, \text{ since both terms differ by a sign}} f_{y^l}(y,x)g_{x^{l+1}}(x,y) \\ &= 0. \end{aligned}$$

Similar calculations also show that $E_-(\pi(F)) = 0$. In the case of H we have

$$\begin{aligned} H(\pi(F)) &= \sum_{k=0}^{d} (-1)^k \binom{d}{k} H[f_{y^k}(y,x)g_{x^k}(x,y)] \\ &= \sum_{k=0}^{d} (-1)^k \binom{d}{k} [H(f_{y^k}(y,x))g_{x^k}(x,y) + f_{y^k}(y,x)H(g_{x^k}(x,y))] \\ &= \sum_{k=0}^{d} (-1)^k \binom{d}{k} [(d-2k+2k-d)f_{y^k}(y,x)g_{x^k}(x,y)] \quad \text{by Lemma 5.1} \\ &= 0. \end{aligned}$$

(c) We have to show that π is idempotent, that is, $\pi(\pi(F)) = \pi(F)$. Now

$$\pi(F) = \sum_{k=0}^{d} \frac{(-1)^k}{k!(k+1)!} E_-^k E_+^k(F)$$

and

$$\pi(\pi(F)) = \sum_{l=0}^{d} \frac{(-1)^l}{l!(l+1)!} E_-^l E_+^l [\sum_{k=0}^{d} \frac{(-1)^k}{k!(k+1)!} E_-^k E_+^k (F)]$$
$$= \sum_{k=0}^{d} \frac{(-1)^k}{k!(k+1)!} E_-^k E_+^k (F) + \sum_{l=1}^{d} \frac{(-1)^l}{l!(l+1)!} E_-^l E_+^l [\sum_{k=0}^{d} \frac{(-1)^k}{k!(k+1)!} E_-^k E_+^k (F)]$$
$$= \pi(F) + \sum_{l=1}^{d} \frac{(-1)^l}{l!(l+1)!} E_-^l E_+^l [\sum_{k=0}^{d} \frac{(-1)^k}{k!(k+1)!} E_-^k E_+^k (F)].$$

We will prove that π idempotent if we show that the term on the right, which we denote by $\pi'(F)$, is 0. By the calculation we did at the beginning of the proof with regards to the covariant we have

$$\pi'(F) = \sum_{l=1}^{d} \frac{(-1)^l}{l!(l+1)!} E_-^l E_+^l [\sum_{k=0}^{d} \frac{(-1)^k}{k!(k+1)!} E_-^k E_+^k (F)]$$
$$= \sum_{l=1}^{d} \frac{(-1)^l}{l!(l+1)!} E_-^l E_+^l [\frac{1}{d+1} \sum_{k=0}^{d} \frac{(d-k)!}{k!d!} E_-^k (f(x)) E_+^k (g(y))]$$
$$= \sum_{l=1}^{d} \sum_{k=0}^{d} \frac{(-1)^{l+k}}{l!(l+1)!} \frac{1}{d+1} \frac{(d-k)!}{k!d!} E_-^l E_+^l [E_-^k (f(x)) E_+^k (g(y))].$$

By consecutive application of the general Leibniz rule we get

$$E_-^l E_+^l [E_-^k (f(x)) E_+^k (g(y))] = \sum_{m=0}^{l} \sum_{n=0}^{l} \binom{l}{m} \binom{l}{n} E_-^n E_+^m E_-^k [f(x)] E_-^{l-n} E_+^{k+l-m} [g(y)].$$

Now using Lemma 5.3 we have

$$E_-^n E_+^m E_-^k [f(x)] = E_-^n E_+^m [\frac{d!}{(d-k)!} f_{y^k}(y,x)] = \frac{d!}{(d-k)!} E_-^n E_+^m [f_{y^k}(y,x)]$$
$$= \frac{d!}{(d-k)!} E_-^n [\frac{k!}{(k-m)!} f_{y^{k-m}}(y,x)] = \frac{d!k!}{(d-k)!(k-m)!} E_-^n [f_{y^{k-m}}(y,x)]$$
$$= \frac{d!k!}{(d-k)!(k-m)!} [\frac{[d-(k-m)]!}{[d-(k-m+n)]!} f_{y^{k-m+n}}(y,x)]$$
$$= \frac{d!k![d-(k-m)]!}{(d-k)!(k-m)![d-(k-m+n)]!} f_{y^{k-m+n}}(y,x)$$

and

$$E_-^{l-n} E_+^{k+l-m} [g(y)] = E_-^{l-n} [\frac{d!}{[d-(k+l-m)]!} g_{x^{k+l-m}}(x,y)]$$
$$= \frac{d!}{[d-(k+l-m)]!} E_-^{l-n} [g_{x^{k+l-m}}(x,y)]$$
$$= \frac{d!(k+l-m)!}{[d-(k+l-m)]!(k-m+n)!} g_{x^{k-m+n}}(x,y).$$

So $E_-^l E_+^l [E_-^k (f(x)) E_+^k (g(y))]$ is equal to

$$\sum_{m=0}^{l} \sum_{n=0}^{l} \binom{l}{m} \binom{l}{n} B_{k,l,m,n} f_{y^{k-m+n}}(y,x) g_{x^{k-m+n}}(x,y),$$

where

$$B_{k,l,m,n} := \frac{d!^2 k![d-(k-m)]!(k+l-m)!}{(d-k)!(k-m)![d-(k-m+n)]![d-(k+l-m)]!(k-m+n)!}$$
$$= \frac{d!k![d-(k-m)]!(k+l-m)!}{(d-k)!(k-m)![d-(k+l-m)]!}\binom{d}{k-m+n}.$$

Substituting this in $\pi'(F)$ and then by cancelling terms in the numerator and denominator we get

$$\pi'(F) = \frac{1}{d+1}\sum_{l=1}^{d}\sum_{k=0}^{d}\sum_{m=0}^{l}\sum_{n=0}^{l} B'_{k,l,m,m} f_{y^{k-m+n}}(y,x) g_{x^{k-m+n}}(x,y),$$

where

$$B'_{k,l,m,m} := \frac{(-1)^{l+k}}{l!(l+1)!}\binom{l}{m}\binom{l}{n}\frac{[d-(k-m)]!(k+l-m)!}{(k-m)![d-(k+l-m)]!}\binom{d}{k-m+n}.$$

We bring the summation indexed by k on the inside, switch the ones indexed by m and n, and since k must be greater than or equal to m, we get

$$\pi'(F) = \frac{1}{d+1}\sum_{l=1}^{d}\sum_{n=0}^{l}\sum_{m=0}^{l}\sum_{k=m}^{d} B'_{k,l,m,m} f_{y^{k-m+n}}(y,x) g_{x^{k-m+n}}(x,y).$$

Setting $u = k - m$ we get

$$\pi'(F) = \frac{1}{d+1}\sum_{l=1}^{d}\sum_{n=0}^{l}\sum_{m=0}^{l}\sum_{u=0}^{d-m} B'_{l,u,m,n} f_{y^{u+n}}(y,x) g_{x^{u+n}}(x,y),$$

where

$$B'_{l,u,m,n} := \frac{(-1)^{l+u+m}}{l!(l+1)!}\binom{l}{m}\binom{l}{n}\frac{[d-u]!(u+l)!}{u![d-(u+l)]!}\binom{d}{u+n}.$$

Note that $u + l$ must be less than or equal to d. Define

$$A_{l,n,u} := \frac{1}{d+1}\frac{(-1)^{l+u}}{l!(l+1)!}\binom{l}{n}\frac{[d-u]!(u+l)!}{u![d-(u+l)]!}\binom{d}{u+n} f_{y^{u+n}}(y,x) g_{x^{u+n}}(x,y).$$

By substituting $A_{l,n,u}$ in $\pi'(F)$, we get

$$\pi'(F) = \sum_{l=1}^{d}\sum_{n=0}^{l}\sum_{m=0}^{l}\sum_{u=0}^{d-m}(-1)^m\binom{l}{m}A_{l,n,u}.$$

Up till now we could interchange the order of summation because we did not have dependent limits but now we do. We claim

$$\sum_{m=0}^{l}\sum_{u=0}^{d-m} = \sum_{u=0}^{d}\sum_{m=0}^{d-u} - \sum_{u=0}^{d-(l+1)}\sum_{m=l+1}^{d-u} \quad \text{if } l < d$$
$$\sum_{m=0}^{d}\sum_{u=0}^{d-m} = \sum_{u=0}^{d}\sum_{m=0}^{d-u} \quad \text{if } l = d.$$

This formula can be verified visually if we look at the tuples (m, u) in the following tabular fashion from the bottom to top and left to right

$$
\begin{array}{cccccccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) & (0,5) & \cdots & (0,d) \\
(1,0) & (1,1) & (l,2) & (l,3) & \cdots & \cdots & (1,d-1) & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \iddots & & \\
(l,0) & (l,1) & (l,2) & \cdots & (l,d-l) & & & \\
(l+1,0) & (l+1,1) & \cdots & (l+1,d-[l+1]) & & & & \\
\vdots & \vdots & \iddots & & & & & \\
(d-1,0) & (d-1,1) & & & & & & \\
(d,0) & & & & & & &
\end{array}
$$

If $l = d$, then since $u + l \leq d$, u must be equal to 0. So we have

$$\pi'(F) = \sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{m=0}^{l} \sum_{u=0}^{d-m} (-1)^m \binom{l}{m} A_{l,n,u} + \sum_{n=0}^{d} \sum_{m=0}^{d} (-1)^m \binom{d}{m} A_{d,n,0}.$$

Then by using the formula for interchanging the order of summation in the case $l < d$, we get

$$\begin{aligned}
\pi'(F) = \sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{u=0}^{d} \sum_{m=0}^{d-u} (-1)^m \binom{l}{m} A_{l,n,u} - \sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{u=0}^{d-(l+1)} \sum_{m=l+1}^{d-u} (-1)^m \binom{l}{m} A_{l,n,u} \\
+ \sum_{n=0}^{d} \sum_{m=0}^{d} (-1)^m \binom{d}{m} A_{d,n,0}.
\end{aligned}$$

The first term splits into

$$\sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{u=0}^{d-(l+1)} \sum_{m=0}^{d-u} (-1)^m \binom{l}{m} A_{l,n,u} + \sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{u=d-l}^{d} \sum_{m=0}^{d-u} (-1)^m \binom{l}{m} A_{l,n,u},$$

so after cancellation in $\pi'(F)$ we get

$$\pi'(F) = \sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{u=d-l}^{d} \sum_{m=0}^{d-u} (-1)^m \binom{l}{m} A_{l,n,u} + \sum_{n=0}^{d} \sum_{m=0}^{d} (-1)^m \binom{d}{m} A_{d,n,0}.$$

As we noted before $u + l \leq d$, so $u \not> d - l$. Therefore

$$\begin{aligned}
\pi'(F) &= \sum_{l=1}^{d-1} \sum_{n=0}^{l} \sum_{m=0}^{[d-(d-l)]=l} (-1)^m \binom{l}{m} A_{l,n,d-l} + \sum_{n=0}^{d} \sum_{m=0}^{d} (-1)^m \binom{d}{m} A_{d,n,0} \\
&= \sum_{l=1}^{d-1} \sum_{n=0}^{l} A_{l,n,d-l} \sum_{m=0}^{l} (-1)^m \binom{l}{m} + \sum_{n=0}^{d} A_{d,n,0} \sum_{m=0}^{d} (-1)^m \binom{d}{m}.
\end{aligned}$$

Both the terms on the right are 0 since the inner summations are equal to $[1 + (-1)]^l$ and $[1 + (-1)]^d$ respectively. So $\pi'(F) = 0$ and it follows that $\pi(\pi(F)) = \pi(F)$.

(d) We now show that $\pi(F)$ is not the zero map. Let $f(y) = y_1^d$ and $g(x) = x_2^d$, then $f_{y^k}(y, x) = y_1^k x_1^{d-k}$ and $g_{x^k}(x, y) = x_2^k y_2^{d-k}$ by (4.3). So

$$\begin{aligned}
\pi(x_2^d y_1^d) &= \sum_{k=0}^{d} \frac{1}{(d+1)} (-1)^k \binom{d}{k} f_{y^k}(y, x) g_{x^k}(x, y) \\
&= \sum_{k=0}^{d} \frac{1}{(d+1)} (-1)^k \binom{d}{k} (x_1 y_2)^{d-k} (-x_2 y_1)^k \\
&= \frac{1}{(d+1)} (x_1 y_2 - x_2 y_1)^d \neq 0.
\end{aligned}$$

□

6 Zusammenfassung

Unter welchen Bedingungen ist eine komplexe Form vollständig in linearen Formen zerlegbar? Brill lieferte eine Antwort auf diese Frage mit den Methoden der Invariantentheorie [**4**]. Er leitete gewisse Kovarianten her, deren gleichzeitiges Verschwinden eine notwendige und hinreichende Bedingung für die Zerlegbarkeit einer Form ist. Gordan [**14**] erkennte in diesen Kovarianten eine bestimmte, von nun an Brill'sche Kovariante genannt, die bereits die notwendige und hinreichende Bedingung für die Zelegbarkeit bietet. Eine kurze Zusammenfassung von Gordans Präsentation der Brill'schen Kovariante ist in [**3**] gegeben. In dieser Arbeit präsentieren wir die in [**13**] gegebene moderne Darstellung-theoretische Interpretation der Brill'schen Kovariante.

Aus der Perspektiv der algebraischen Geometrie betrachten wir die Projektivisierung von dem Raum aller Formen vom Grad d in $n+1$ Variablen mit komplexen Koeffizienten ist eine Varietät, nähmlich, die Chow-Varietät von 0-Zyklen vom Grad d in einem komplex-projektivem Raum der Dimension n. Wir bezeichnen es mit $\mathrm{Chow}(d,n)$. Wir interessieren uns für die Gleichungen, die $\mathrm{Chow}(d,n)$ Mengentheoretisch definieren und dies ist äquivalent zu der Untersuchung der Bedingungen, unter denen eine Form vollständig zerlegbar ist.

In Abschnitt 1 stellen wir multisymmetrische Polynome über Körper der Charakteristik 0 vor, die wir später in Abschnitt 3 brauchen. Sie bieten eine Verallgemeinerung der üblichen symmetrischen Polynome und können als Polynome in Vektorvariablen gedacht werden, die unter Permutation dieser Vektorvariablen unverändert bleiben. Wir beweisen eine Verallgemeinerung des Fundamentalsatzes der symmetrischen Polynome. Wir untersuchen dann den Quotientenkörper des Rings der multisymmetrischen Polynome, der sich als eine rein transzendentale Erweiterung von dem Grundkörper herausstelle. In dem komplexen Fall können wir eine genauere Aussage über die Transzendenzbasis machen.

In Abschnitt 2 stellen wir Schur-Module vor, die später bei der Interpretation der apolaren Kovariante eine wichtige Rolle spielen, da sie dazu beitragen, Darstellungen von linearen Gruppen zu zerlegen. Zu diesem Zweck zunächst geben wir eine kurze Zusammmenfassung der Konstruktion der Tensoralgebra und der daraus abgeleiteten symmetrischen und äußeren Algebra.

In Abschnitt 3 stellen wir zunächst die Grassmann-Varietät vor, die das Konzept der algebraischen Zykel und der Chow-Varietät motiviert. Die Chow-Varietät erlaubt uns, die Menge der algebraischen Zyklen als eine Varietät zu betrachten, indem wir sie in einen projektiven Raum einbetten, aber wir werden uns nicht damit im Allgemeinen beschäftigen. Wir interessieren uns nur für den spezifischen Fall $\mathrm{Chow}(d,n)$. Durch was wir in Abschnitt 1 bewiesen haben, werden wir zeigen, dass $\mathrm{Chow}(d,n)$ isomorph zu der Varietät $\mathrm{Sym}^d(\mathbb{P}^n)$ ist und dass es eine rationale Varietät ist.

Letztendlich in Abschnitt 4 beschreiben wir explizit die Brill'sche Kovariante . Zunächst stellen wir Polare einer Form und die apolare Kovariante vor, die wesentliche Werkzeuge bei der Konstruktion der Brill'schen Kovariante sind.

In Abschnitt 5 interpretieren wir die Formel für die apolare Kovariante in der Sprache der modernen Darstellungstheorie. Mit Hilfe der in Abschnitt 4 angegebenen Schur-Modulzerlegung und unter Betrachtung der Wirkung von den Erzeuger der Lie-Algebra $\mathfrak{sl}(2,\mathbb{C})$ auf den Raum bihomogener Formen können wir über eine Projektionsabbildung zur Formel für die apolare Kovariante gelangen.

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