

Incorporating Severity Variations into Credit Risk

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Abstract

We present an approach to modelling credit risk that incorporates the risk of counterparty default and the risk of devaluation of the collateral. The framework is based on a segmentation by industry and collateral type. The systematic risk in both drivers is taken into account by volatilities within and by correlations between the segments. We derive a simple formula for the variance of the loss distribution and describe an algorithm to compute this distribution. Moreover, we show that in the limit of a large portfolio, the loss distribution directly mirrors the assumptions on the economy and depends on the portfolio structure only through the expected loss distribution across the segments.

1 Introduction

In the last years the quantitative modelling of credit risk has received a lot of attention in the financial industry. However, principal problems still exist for the adequate modelling of credit risk as well as with regard to the estimation of the risk parameters. In fact, most of the energy has been devoted to modelling default risk, whereas collateral as a risk factor is mostly neglected. However, variation in loss given default (LGD, severity), the percentage of exposure lost in case of default, is important for two reasons: first, individual severities are highly uncertain and follow skewed, bimodal, or even more complex distributions (see Asarnow and Edwards (1995), Eales and Bosworth (1998), Moody's (2000), de Castle and Keisman (1999)), and second, for highly collateralized portfolios with low risk in the first place, a common shift in collateral value can significantly increase the portfolio risk. This is especially important for middle market portfolios in which loans are typically collateralized by real estate from a limited geographical region where prices change in line with the overall market.

Concern is also raised in a consultative paper by the Bank for International Settlements (1999, p. 37). In particular, it is criticized that variations in loss given default are not properly accounted for in most models: "the assumptions that LGDs between borrowers are mutually independent may represent a serious shortcoming when the bank has significant industry concentrations of credits (e.g., commercial real estate loans within the same geographical region)." Another point brought up relates to "portfolios characterized by distributions of

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exposure sizes that are highly skewed” where “the assumptions that LGDs are known with certainty may tend to bias downward the estimated tail of the probability density function of credit losses.” Only recently have approaches been suggested that account for systematic price movements in collateral; see Gordy (forthcoming), and Frye (2000a), (2000b).

In this article we extend the framework of CreditRisk+ (CSFP (1997)) and Bürgisser et al. (1999) to model credit risk, by taking into account the two major risk drivers: default and severity risk. The framework is based on a segmentation by industry and collateral type, where the industry segments are synonymously used for any kind of segmentation with regard to default risk, e.g., industries, regions, or countries. The systematic effects in both risk drivers are taken into account by volatilities within and by correlations between the segments, whereas in CreditRisk+ the industries are assumed to be independent and the severities are deterministic. The model makes the following three major assumptions: (1) systematic changes in default probabilities due to the economy can be described by scaling factors specific to industry segments, (2) systematic changes in severities due to the economy can be described by scaling factors specific to the collateral segments, and (3) severity varies independently of default. We later outline how to relax these assumptions using factor loadings to describe obligor specific sensitivities to the default and severity scaling variable and by using conditional independence to account for a possible dependence between default and severity risk, a technique also put forward by Finger (1999) and Gordy (2000). Especially for the independence assumption, there is no clear picture on the relation between default and loss severity in the literature. While Moody’s (2000) claims that recovery rates are correlated with the economic cycle, based on a study of bond prices for the period 1980-1999, other authors observe quite stable loss severities over periods of similar length, see Asarnow and Edwards (1995) and Society of Actuaries (1998). Our model naturally includes the description of a portfolio consisting of defaulted loans, which can be described by an artificial industry segment.

The main results of this paper are a simple formula for the variance of the loss distribution and an algorithm to compute this distribution. Moreover, we prove that in the limit of a large portfolio, the loss distribution directly mirrors the assumptions on the economy and depends on the portfolio structure only through the relative expected loss weights. It turns out that in the limit, the distribution is symmetric with respect to default and severity risk. In practice, default risk often dominates severity risk due to the empirically observed smaller volatility of severity variations.

Our approach preserves the analytical tractability of the original CreditRisk+ model (CSFP (1997)). This allows an efficient computation of the full loss distribution, which is a clear advantage over simulation approaches for performing scenario- and sensitivity analyses. The model presented makes only few assumptions and can be further extended to include additional risk drivers. Furthermore, the methodology is easy to implement within a bank’s overall risk framework.

The article is organized as follows: In Section 2 we specify the model in the most general case. We then show in Section 3 and 4 how to determine the first two moments of the loss distribution and present an algorithm for calculating the full loss distribution. Extensions of the model are sketched in Section 5.

In Section 6 we compute the loss distribution in the limit of a large portfolio. Finally, we indicate in Section 7 how the model can be used for active portfolio management, outline how to estimate the risk parameters, and apply the model to specific example portfolios in Section 8.

2 Specification of the Model

In the following, we specify a model for describing credit losses of non-distressed loans², taking into account obligor-specific and systematic risk factors. We assume that systematic default risk and systematic severity risk are independent, and can be described by a single factor each. Furthermore, we assume that the obligor-specific severity is independent of the default event.

Let p_A denote the unconditional default probability of obligor A . The impact of systematic risk in the default behavior is modelled by multiplying each of the individual default probabilities p_A by a common positive random scaling factor Γ , thereby increasing or decreasing the default probability of all obligors proportionally. In other words, conditional on Γ , the default probability of A is Γp_A . This *default scaling factor* Γ describes the relative number of default events in the economy; cf. Bürgisser et al. (1999). We normalize Γ to mean equal to one, which implies that the unconditional default probability of obligor A is p_A . The variance of Γ is denoted with σ^2 and we call it *systematic default variance*³.

We model the loss in case of default by the product $\epsilon_A \mathcal{L}_A \Lambda$, where ϵ_A is the (deterministic) outstanding exposure of obligor A , the positive random variable \mathcal{L}_A is an obligor specific severity, and Λ is a positive random *systematic severity scaling factor*, which affects all obligors in the same way. Similarly as with Γ , we assume Λ to be normalized to mean equal to one. We call the variance δ^2 of Λ the *systematic severity variance*. The *expected loss conditional on default* will be denoted by $\nu_A := \epsilon_A E[\mathcal{L}_A]$. For technical reasons we henceforth work with the normalized variable $\Lambda_A := \mathcal{L}_A / E[\mathcal{L}_A]$ which has mean equal to one. The loss in case of default is then given by $\nu_A \Lambda_A \Lambda$ ⁴. We call the variance δ_A^2

²For the treatment of defaulted loans see 5.3.

³Depending on the distribution specified for Γ one may construct cases with the scaled default probability Γp_A larger than one. If, for example, Γ is Gamma-distributed (an assumption made by CreditRisk+ (CSFP (1997))), and also used later in this paper) with $\sigma = 1$ (a value suggested in CSFP (1997, A7.3)), then $\Gamma = 8.5$ corresponds to the 99.98-percentile which is unproblematic since mostly $p_A < 10\%$. Due to the small probability of large values in Γ and typical default probabilities of well below 10%, the distortion in the model can be safely neglected.

⁴The modelling with a single systematic severity factor is meaningful in case of one collateral type where all obligors have the same expected severity. The combined severity variable $\mathcal{L}_A \Lambda$ is typically bounded in the unit interval. If, for example the severities of all obligors are mainly driven by systematic severities, and $E[\mathcal{L}_A] = 50\%$ for all A , then the distribution of Λ and Λ_A may be modelled with compact support. E.g., Λ (resp. Λ_A) could be chosen to be beta-distributed on the interval $[0, 3/2]$, (resp. on $[0, 4/3]$). The exact composition of $\Lambda_A \Lambda$ needs to be determined according to their underlying economic drivers. In case of different collateral types the portfolio is split into collateral segments with a similar modelling at the segment level as shown below.

However, note that severities larger than one are observed if the severity is meant to measure the economic loss, i.e., accounting for cost of carry and administrative costs; see for instance Eales and Bosworth (1998, Figures 3,4) and Frye (2000b, Figure 2). Then it may be appropriate to model the distribution for instance as lognormal. Similarly, for specific products, negative

of Λ_A the *obligor specific severity variance*⁵. In a first reading, one may assume that all Λ_A are identical to one, thus neglecting the obligor specific severity risk. The portfolio loss X can now be described by the sum

$$X := \sum_A I_A \epsilon_A \mathcal{L}_A \Lambda = \sum_A I_A \nu_A \Lambda_A \Lambda \quad (1)$$

over all obligors A , where I_A denotes the indicator variable describing the default event of A , being one if A defaults and zero otherwise. The conditional expectation $E[I_A | \Gamma]$ equals $p_A \Gamma$ by our specification. Assuming independence of I_A and Λ_A , we obtain for the expected portfolio loss conditional on the economy described by Γ, Λ that

$$E[X | \Gamma, \Lambda] = \Gamma \Lambda \sum_A p_A \nu_A . \quad (2)$$

In a more general situation, each obligor's default risk is not solely determined by the economy in general, but by the economy of the specific industry the obligor is active in. The same applies for the severity risk, since for example mortgages and cash-secured loans have a different behavior in the variation of their severities. The generalization of the model is achieved by introducing N industry segments for modelling systematic default risk, and M collateral segments for systematic severity risk. The assumptions on the obligor specific severity variables Λ_A remain unchanged. We assume that each obligor can be assigned to exactly one of the industries S_1, \dots, S_N and exactly one collateral segment L_1, \dots, L_M ⁶ such that $A \in S_k \cap L_r$ (compare Figure 1). Hence, for

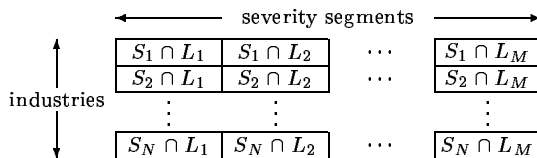


Figure 1: Segmentation of a portfolio

obligor $A \in S_k \cap L_r$, the systematic risk is modelled by segment-specific scaling factors: Γ_k describing the relative number of defaults in industry segment S_k , and Λ_r describing the relative movement of the severity in collateral segment L_r . These random scaling variables are all normalized to mean equal to one.

The loss of obligor $A \in S_k \cap L_r$ in this framework is now given by the variable $I_A \nu_A \Lambda_A \Lambda_r$, where the indicator variable I_A of default has conditional expectation

severities may be observed, see Society of Actuaries (1998, Figure 4), requiring a different specification for the distribution.

⁵Instead of modelling idiosyncratic and systematic severity risk multiplicatively by $\Lambda_A \Lambda$, one might as well choose an additive model $\tilde{\Lambda} + \tilde{\Lambda}_A$ with the mean of $\tilde{\Lambda}$ equal to one, and the mean of $\tilde{\Lambda}_A$ equal to zero. However, in this case a change in the obligor specific severity factor $\tilde{\Lambda}_A$ always has the same impact on the loss conditional on default, no matter what value of $\tilde{\Lambda}$. All calculations and results for such an additive model are very similar to the multiplicative approach presented here.

⁶The model is generalized by considering factor loadings in Subsection 5.1.

$E[I_A | \Gamma_k] = p_A \Gamma_k$. Analogous to Equation (2) we obtain for the portfolio loss X conditional on $\Gamma := (\Gamma_1, \dots, \Gamma_N)$ and $\Lambda := (\Lambda_1, \dots, \Lambda_M)$ that

$$E[X | \Gamma, \Lambda] = \sum_{k=1}^N \sum_{r=1}^M \Gamma_k \Lambda_r \sum_{A \in S_k \cap L_r} p_A \nu_A \quad (3)$$

by our assumption of independence.

Let $f_X(x | \Gamma = \gamma, \Lambda = \lambda)$ denote the density function of the random variable X given the states of the economy Γ and Λ with realizations $\gamma = (\gamma_1, \dots, \gamma_N)$ and $\lambda = (\lambda_1, \dots, \lambda_M)$, and let $f_{\Gamma, \Lambda}(\gamma, \lambda)$ be the corresponding multi-dimensional density function of the scaling factors. The *unconditional* probability density function $f(x)$ of the portfolio's losses is then obtained by averaging the conditional densities $f_X(x | \Gamma = \gamma, \Lambda = \lambda)$ over all possible states of the economy $\Gamma = \gamma, \Lambda = \lambda$. Hence, we have

$$f(x) = \int \int f_X(x | \Gamma = \gamma, \Lambda = \lambda) f_{\Gamma, \Lambda}(\gamma, \lambda) d\gamma d\lambda, \quad (4)$$

in the most general case, since a priori severity and default risk are driven by the same macro-economic factors, and are therefore not necessarily independent from each other. A method to account for this dependence is outlined in Subsection 5.2. Here, to simplify the model, we assume that Γ and Λ are independent, and therefore $f_{\Gamma, \Lambda}(\gamma, \lambda)$ decomposes into the product of densities $f_{\Gamma}(\gamma)$ and $f_{\Lambda}(\lambda)$, specific to each risk factor.

3 Moments of the Loss Distribution

The first two moments, i.e., mean and variance, of the loss distribution $f(x)$ are simple risk measures, easy to calculate, and give an understanding of the relevant risk drivers. The mean of the loss distribution, which we call the *expected loss* EL , is calculated as the expectation of the conditional expectations (cf. Equation (3)):

$$EL = E[E[X | \Gamma, \Lambda]] = \sum_{k=1}^N \sum_{r=1}^M E[\Gamma_k] E[\Lambda_r] \sum_{A \in S_k \cap L_r} p_A \nu_A = \sum_A p_A \nu_A. \quad (5)$$

Note that this derivation of $EL = \sum_A p_A \nu_A$ relies on the independence between the risk drivers for default and severity. Similarly, $EL_r^{(k)} = \sum_{A \in S_k \cap L_r} p_A \nu_A$ is the expected loss in industry segment S_k and collateral segment L_r . Moreover, we define $EL^{(k)} = \sum_{r=1}^M EL_r^{(k)}$ as the expected loss in industry segment S_k , and $EL_r = \sum_{k=1}^N EL_r^{(k)}$ as the expected loss in collateral segment L_r .

To determine the standard deviation of the portfolio loss distribution, which we call *unexpected loss* UL in this paper, we use a well known theorem relating unconditional and conditional variances (see e.g., Fristedt and Gray (1997, 23.4) or Goovaerts et al. (1990, V.1.2)):

For random variables Z and Θ , we have

$$\text{var}[Z] = \text{var}[E[Z | \Theta]] + E[\text{var}[Z | \Theta]]. \quad (6)$$

In words: The unconditional variance is the sum of the conditional mean's variance (between variance) and the conditional variance's mean (within variance).

We will see below that the term $E[\text{var}[Z | \Theta]]$ of Equation (6) represents the diversifiable risk when Z describes the portfolio loss. The first term $\text{var}[E[Z | \Theta]]$ is the systematic risk that cannot be eliminated by diversification.

Before presenting the result relating the unexpected loss UL with the systematic risk drivers and the obligor specific quantities, we introduce notations for variances and correlations of the systematic default and severity scaling factors:

$$\begin{aligned}\sigma_k^2 &:= \text{var}[\Gamma_k], & \rho_{k\ell} &:= \text{corr}(\Gamma_k, \Gamma_\ell), \\ \delta_r^2 &:= \text{var}[\Lambda_r], & \psi_{rs} &:= \text{corr}(\Lambda_r, \Lambda_s).\end{aligned}$$

Theorem 3.1 *The unexpected loss UL of the unconditional loss distribution is given by the following relation:*

$$UL^2 = UL_{\text{syst}}^2 + UL_{\text{div}}^2 \quad (7)$$

where

$$\begin{aligned}UL_{\text{syst}}^2 &= \sum_{k,\ell=1}^N \rho_{k\ell} \sigma_k \sigma_\ell EL^{(k)} EL^{(\ell)} + \sum_{k,\ell=1}^N \sum_{r,s=1}^M \rho_{k\ell} \sigma_k \sigma_\ell \psi_{rs} \delta_r \delta_s EL_r^{(k)} EL_s^{(\ell)} \\ &+ \sum_{r,s=1}^M \psi_{rs} \delta_r \delta_s EL_r EL_s\end{aligned}$$

is the variance due to systematic risk (default and loss severity risk), and

$$UL_{\text{div}}^2 = \sum_{r=1}^M (1 + \delta_r^2) \sum_{k=1}^N \sum_{A \in S_k \cap L_r} \left((1 + \delta_A^2) p_A - (1 + \sigma_k^2) p_A^2 \right) \nu_A^2$$

is the variance due to the diversifiable (statistical) nature of default events and obligor-specific severity risk.

The proof is given in Appendix A.1. It is shown there that the decompositions in Equations (6) and (7) match.

For the single sector case with $N = M = 1$ the formulas for UL_{syst}^2 and UL_{div}^2 collapse to:

$$\begin{aligned}UL_{\text{syst}}^2 &= \sigma^2 EL^2 + \sigma^2 \delta^2 EL^2 + \delta^2 EL^2 \\ UL_{\text{div}}^2 &= (1 + \delta^2) \sum_A \left((1 + \delta_A^2) p_A - (1 + \sigma^2) p_A^2 \right) \nu_A^2\end{aligned} \quad (8)$$

The first term of the systematic variance UL_{syst}^2 represents the systematic default risk. The third term represents the systematic severity risk, and the second one is due to the mixed systematic risk of default- and severity variation.

The decomposition of UL^2 into UL_{syst}^2 and UL_{div}^2 is motivated by separating the risk into the component solely determined by systematic risk parameters, and the risk diversified away for a large portfolio with many obligors. The fact that UL_{div}^2 is negligible for a large portfolio is easily seen as follows: UL_{syst}^2 consists

of n^2 terms when EL is written as the sum over all n obligors. On the other hand, UL_{div}^2 only has n terms and is therefore of relatively minor importance for portfolios with many obligors. Moreover, we show in Theorem 6.1 that in the limit of a large portfolio, the loss distribution does not depend upon any obligor-specific quantities.

We remark that Bürgisser et al. (1999, Equation (12)) is a special case of Theorem 3.1, obtained by setting $\delta_r = 0$, $\delta_A = 0$, and replacing the Bernoulli by a Poisson event. This replacement results in omitting the second order term $(1 + \sigma_k^2)p_A^2$ in the diversifiable variance UL_{div}^2 . This allows a quantification of the error in UL_{div}^2 due to the Poisson approximation. The error in the loss quantiles is calculated in Gordy (2000) for specific parameter settings, using a Monte Carlo simulation approach of CreditRisk+, modelling default as a Bernoulli random variable⁷.

The presented formulas for the mean and the variance do not depend upon any specific choice of the distributions f_Γ and f_Λ . Assuming that the distribution f_Γ for the systematic default variable is Gamma as in CreditRisk+ (CSFP (1997)), and omitting the severity risk, Gordy (1999) computes higher moments of the corresponding loss distribution.

In the situation, where systematic severity variations are mostly due to price movements in the real-estate market, affecting the severities for mortgages, one can obtain an improvement upon a modelling of pure default risk with just one additional risk parameter δ_1^2 . To do so, one specifies two uncorrelated collateral segments, one for mortgages with variance δ_1^2 , and one for all other collateral types with variance $\delta_2^2 = 0$.

We will discuss the estimation of the risk parameters occurring in Theorem 3.1 in Subsection 7.2.

4 Computation of the Loss Distribution

The goal of this section is the explicit and efficient computation of the portfolio loss distribution and *economic capital*, which is defined as the amount by which the percentile of the loss distribution corresponding to the chosen confidence level exceeds the expected loss. In Subsections 4.1-4.3 we treat the case of a single industry and collateral segment in three steps: First, we derive the loss distribution conditional on the state of the economy and including obligor specific severity variation. This distribution will be described concisely in terms of its probability generating function. Then we extend this to include the systematic default variation by assuming that the default scaling factors is Gamma-distributed, and give a closed analytic form for the probability generating function of the loss distribution as in CreditRisk+ (CSFP (1997)). In a third step we incorporate the systematic severity variation, and provide a numerical procedure to compute the loss distribution given by a multiplicative convolution. In Subsection 4.4 we finally extend our discussion to multiple segments.

⁷Idiosyncratic default variable: Analogous to modelling the severities, one could think of introducing an idiosyncratic default scaling factor Γ_A independent from all other variables. However, it turns out that this would not change the variance UL^2 . This is because the idiosyncratic risk is already included in the notion of default probability.

4.1 Loss Distribution Conditioned on the State of the Economy

It is convenient to express a discrete loss distribution by its *probability generating function*, which is a power series of the form

$$G(z) = \sum_{\nu \geq 0} \pi(\nu) z^\nu ,$$

where $\pi(\nu)$ is the probability of losing the amount ν and z is a formal variable. Probability generating functions are a compact way of dealing with discrete probability distributions and have some useful properties, see e.g., Fisz (1963).

In order to represent the losses as discrete amounts, the expected losses conditional on default ν_A are rounded up to the next integer multiples of a predefined unit (e.g., \$ 1 million). The default probabilities are correspondingly changed such that the expected loss of obligor A remains the same. This discretization of the exposures has negligible impact on the results as shown in Gordy (2000, 4) and CSFP (1997, A4.2), and can be quantified by a sensitivity analysis.

To illustrate the concept of probability generating functions, consider the loss distribution of a single obligor A with a given fixed severity. In case A defaults, the loss equals the expected loss conditional on default ν_A (here, we need independence between default event and severity), which, in the following, is assumed to be a positive integer value. We have two possible states: $\nu = 0$ (no default) and $\nu = \nu_A$ (default). The corresponding probability generating function can be expressed as

$$G_A(z) = (1 - p_A)z^0 + p_A z^{\nu_A} = 1 + p_A(z^{\nu_A} - 1) . \quad (9)$$

If the default events between obligors are independent, then the probability generating function $G(z)$ of the total loss distribution is the product of $G_A(z)$ over all obligors A and we obtain

$$G(z) = \prod_A G_A(z) = \exp \sum_A \log(1 + p_A(z^{\nu_A} - 1)) \approx \exp(Q(z) - Q(1)) , \quad (10)$$

where we have used the approximation $\log(1 + x) \approx x$, which is valid for small default probabilities. We call the quantity

$$Q(z) := \sum_{j \geq 1} \mu'_j z^j := \sum_A p_A z^{\nu_A}$$

the *portfolio polynomial*. This polynomial contains all relevant information on the portfolio for our purposes. The coefficient $\mu'_j = \sum_{A: \nu_A=j} p_A$ represents the expected number of defaults for obligors having loss j conditional on default. Note that $Q(1)$ is the expected number of default events in the portfolio.

The linear approximation of the logarithm is equivalent to assuming that default events are Poisson distributed. This is a valid approximation as long as the default probabilities p_A are small, cf. e.g. Fristedt and Gray (1997, 14.1). The purpose of this approximation is the fact that the mixture of a compound Poisson with a Gamma distribution has closed analytical form which is amenable to an efficient computation of the coefficients $\pi(\nu)$ (see Equations (13) and (14)).

As will be shown in Section 6, the loss distribution of a large portfolio with many obligors is mainly determined by systematic effects, which puts little weight

on specifying the exact shape of the distribution of the obligor specific severity variables Λ_A . Only for small portfolios is a significant amount of the risk contributed by obligor specific severity variations; see Section 8. When we incorporate individual severity variations, the form of Equation (10) remains unchanged, but $Q(z)$ is redefined as

$$Q(z) := \sum_{j \geq 0} \mu_j z^j, \quad \mu_j := \sum_A p_A f_{\Lambda_A}(j), \quad (11)$$

where f_{Λ_A} denotes the discrete density function of the idiosyncratic severity variable Λ_A . In other words, $f_{\Lambda_A}(j)$ is the probability of losing the amount j on loan A conditional on default of A . If we assume, for simplicity, that all Λ_A have the same type of distribution, then we can write $f^{(\nu_A, \Delta)} = f_{\Lambda_A}$ with mean ν_A and volatility $\nu_A \Delta$, for some fixed relative volatility Δ . In this case

$$\mu_j = \sum_{\nu} \mu'_{\nu} f^{(\nu, \Delta)}(j),$$

where $\mu'_{\nu} = \sum_{A: \nu_A = \nu} p_A$ is as before⁸.

4.2 Incorporating Systematic Default Variation

We refine the model to include the systematic risk of coherent changes in the default probabilities by introducing the default scaling factor Γ , as done in Section 2. Recall that $f_{\Gamma}(\gamma)$ denotes the corresponding probability density function with systematic default variance σ^2 .

To obtain the generating function $G(z)$ of the loss distribution (still excluding systematic variations in severities) we average the conditional generating functions $G(z | \Gamma = \gamma)$ over the possible states of the economy, characterized by $\Gamma = \gamma$:

$$G(z) = \int_0^{\infty} G(z | \Gamma = \gamma) f_{\Gamma}(\gamma) d\gamma = \int_0^{\infty} e^{\gamma(Q(z) - Q(1))} f_{\Gamma}(\gamma) d\gamma. \quad (12)$$

Here we have used that the generating function of the loss distribution conditional on the state $\Gamma = \gamma$ is $G(z | \Gamma = \gamma) = \exp(\gamma(Q(z) - Q(1)))$, following Equation (10). The corresponding distribution of the generating function $G(z | \Gamma = \gamma)$ is compound Poisson, see Bowers et al. (1997, 12.3.1).

In the following we assume, as in CreditRisk+ (CSFP (1997)), that the default scaling factor Γ is Gamma-distributed with density

$$f_{\Gamma}(\gamma) = \frac{1}{\sigma^2/\sigma^2 \Gamma_0(1/\sigma^2)} e^{-\gamma/\sigma^2} \gamma^{1/\sigma^2 - 1},$$

with mean equal to one, volatility σ , and Γ_0 denoting the Gamma function. Under this distributional assumption, the probability generating function $G(z)$ can be expressed in closed analytical form which is caused by the assumption that the (conditional) loss behavior is compound Poisson, and by mixing it with a Gamma distribution. For a Poisson variable (i.e. $Q(z) = z$) the economic reasoning for mixing distributions can be explained in intuitive terms: The probability

⁸See the introduction of Section 8 and footnote 13 for an explicit parameterization to model the density function $f^{(\nu, \Delta)}$ of idiosyncratic severity variations.

of default (i.e. the mean of the Poisson variable) is not constant over time, but depends on the economic situation, and is therefore made stochastic by modelling it with a Gamma distribution. It turns out that

$$G(z) = \left(\frac{1}{1 - \sigma^2(Q(z) - Q(1))} \right)^{1/\sigma^2}. \quad (13)$$

(This is easily seen by substituting $t = \gamma(1/\sigma^2 - (Q(z) - Q(1)))$ in Equation (12), and since $\int_0^\infty e^{-t} t^{1/\sigma^2 - 1} dt = \Gamma_0(1/\sigma^2)$; cf. Bowers et al. (1997, Example 12.3.1).

Based on the above closed form representation of the probability generating function $G(z) = \sum_{\nu \geq 0} \pi(\nu) z^\nu$, one can efficiently compute its coefficients $\pi(\nu)$ using a recursive relationship due to Panjer (1981). In fact, the probability $\pi(\nu)$ of losing the amount $\nu \geq 1$ is given by

$$\pi(\nu) = \frac{1}{\nu(1 + Q(1)\sigma^2)} \sum_{j=1}^{\min\{m, \nu\}} \{\sigma^2\nu + (1 - \sigma^2)j\} \mu_j \pi(\nu - j). \quad (14)$$

where the start value is given by $\pi(0) = (1 + Q(1)\sigma^2)^{-1/\sigma^2}$, and $m = \deg Q$ is the largest exposure in the portfolio⁹.

Remark 4.1 By comparing generating functions it is easily seen that the distribution corresponding to $G(z)$ (i.e., mixing of compound Poisson with Gamma) coincides with a compound negative binomial distribution; cf. Bowers et al. (1997, Table 12.3.1). This means that the random loss variable X corresponding to $G(z)$ allows the representation $X = \sum_{i=1}^B X_i$ where X_i are independent, identically distributed with generating function $Q(z)/Q(1)$, and B is negative binomial representing the number of defaults in the portfolio. The mean of B is equal to $Q(1)$ and its variance equals $\sigma^2 Q(1)^2 + Q(1)$.

Observe that $Q(z)$ includes the obligor specific severity variations as in Equation (11), but systematic severity variations (incorporated by the variable Λ) have not yet been taken into account. When doing so, the new loss variable X still allows the form $X = \sum_{i=1}^B Y_i$, where $Y_i = X_i \cdot \Lambda$, i.e., the Y_i are still identically distributed, but no longer independent.

4.3 Incorporating Systematic Severity Variation

In the last step we merge the loss distribution $\pi(\nu)$, accounting for systematic default and obligor specific severity variations, with the systematic variation of severities. There are few data for systematic severity variations, hence it is a priori not clear which distribution for the systematic severity scaling factor Λ should be chosen. Distinct internal loss measurement processes, such as handling of extra recovery costs (administrative costs, cost of carry), different types of collateral, and a changing economic environment may lead to various shapes for the systematic severity behavior (bimodal, skewed, extreme severities exceeding one; cf. Eales and Bosworth (1998, Figures 3,4) and Frye (2000b, Figure 2)).

⁹In order to derive the recursive relation (14), use that $G'(z)\{(1 - \sigma^2(Q(z) - Q(1)))\} = Q'(z)G(z)$.

Typical choices for the distribution of Λ are a beta- or lognormal distribution¹⁰. However, the presented calculation works for any distribution on the positive axis.

Recall now the specification of the model in Section 2. The unconditional loss distribution, now including systematic severity variations, is obtained by averaging the conditional loss distributions over all states $(\Gamma, \Lambda) = (\gamma, \lambda)$ using the weights $f_\Gamma(\gamma)f_\Lambda(\lambda)$ (see Equation (14)). This averaging process can be performed in two steps: first for Γ and then for Λ . In the previous subsection, we have already computed the average with respect to Γ resulting in the discrete distribution $\pi(\nu)$. The unconditional loss distribution is therefore the distribution of the product $\nu \cdot \Lambda$, where ν and Λ are independent. In other words, the unconditional loss distribution is the *multiplicative convolution* of the distributions of ν and Λ . Its cumulative distribution $F(n)$ can be easily expressed as follows. If $F_\Lambda(\lambda)$ denotes the cumulative probability distribution of the variable Λ , then we have

$$\begin{aligned} F(n) &= \text{Prob}\{\nu\lambda \leq n\} = \text{Prob}\{\nu = 0\} + \sum_{\nu>0} \pi(\nu)\text{Prob}\{\lambda \leq n/\nu\} , \\ &= \pi(0) + \sum_{\nu>0} \pi(\nu)F_\Lambda(n/\nu) . \end{aligned} \tag{15}$$

In order to actually compute the probabilities $F(n)$ and corresponding percentiles, we replace this infinite sum by a finite summation over ν , with the upper summation bound depending on n . The resulting approximation error can be controlled, as demonstrated in Appendix A.2.

4.4 Computation of the Loss Distribution for Multiple Segments

In the situation of a single segment, we have assumed that Γ follows a Gamma distribution and that Λ is lognormally distributed. It would be natural to extend these assumptions to the situation of many segments. However, whereas a multivariate lognormal distribution is fully specified by its marginal distributions and the correlations, there is no canonical way to define a multivariate distribution whose marginals are Gamma-distributed. Note that the assumption of a Gamma-distribution is crucial for the calculation of the loss distribution with the Panjer-recursion. Moreover, there are few data to determine the shape of a multidimensional default or severity distribution.

An article of Embrechts et al. (1999a, 1.2) (short version Embrechts et al. (1999b)) strikingly shows that based on correlations only, there are quite different ways to define a multivariate distribution with marginal Gamma-distribution. The multivariate structure with maximal correlations does not even produce the highest quantile losses, see Embrechts et al. (1999a, 5. Remark 2). The multivariate recursion for several independent segments as done in CSFP (1997, A10) is therefore inappropriate for our situation of correlated industry segments, since an orthogonalization imposes a particular multivariate structure which may misspecify the risk structure.

¹⁰If we assume as usual that returns are normally distributed, and if we regard the changes in real estate prices as returns, then the values of the collateral follow a lognormal distribution. Since the loss severity and the value of the collateral follow a close relationship, the systematic severity scaling factor Λ can also be modelled by a lognormal distribution.

As a consequence, we take a pragmatic approach in which we reduce a multivariate structure to a single segment model with the same *EL* and *UL*. The calculation procedure is as follows: First, we calculate the unexpected loss of the portfolio by Theorem 3.1, thereby taking into account the segment structure. Then we estimate single systematic default and severity volatilities σ and δ such that the unexpected loss of the portfolio, computed with the single segment formula (Equation (8)), matches the unexpected loss computed before. Finally, the loss distribution is calculated as in the single segment situation, where the systematic default behavior is Gamma- and the systematic severity variation is lognormally distributed¹¹.

There are two natural possibilities to estimate the implied overall systematic volatilities in the number of defaults σ and in the severities δ :

1. We may determine σ by equating the *UL*-formulas of the single and multi-segment situation for the case of pure default risk, i.e., by setting $\delta_k = 0$, $\delta_A = 0$, cf. Bürgisser et al. (1999, Eq. (13)).

2. Alternatively, we may focus on pure severity risk and determine δ by equating the *UL*-formulas of the single and multi-segment situation by setting $\sigma_k = 0$.

Because of the non-uniqueness in the parameter estimation, one has to assess the impact of the choice of parameters on the overall loss distribution. We have performed various experiments but found no indication of a problem resulting from this issue. Even in the rather risky case of volatilities σ_k, δ_r equal to one, the shape of the loss distribution hardly depends on this choice: in extreme cases the relative difference in the 99.98-percentile was less than 3% in our examples considered.

5 Extensions of the Model

In this section we outline extensions of the model. These are (1) a more flexible approach to account for a different sensitivity of the default- and severity-parameters to the economy by factor loadings, (2) a method to model default and severity variations which are not independent, and (3) an extension to include a portfolio of defaulted loans.

5.1 Factor Loadings

So far we have presented a specification where the systematic variables Γ and Λ are fully applied to the obligor's risk drivers, i.e., modelling the default probability of obligor A conditional on Γ_k is given by $p_A^{\Gamma_k} := \Gamma_k p_A$, and correspondingly for the severity. Now, as proposed in CreditRisk+ (CSFP (1997, A12.3)), each obligor A can be assigned to several industry segments by apportioning its default probability via factor loadings ω_{Ak} , i.e.,

$$p_A^\Gamma = p_A \left(1 + \sum_{k=1}^N \omega_{Ak} (\Gamma_k - 1) \right) \quad \text{with } 0 \leq \omega_{Ak} \leq 1 \text{ and } \sum_{k=1}^N \omega_{Ak} \leq 1 . \quad (16)$$

Note that $\omega_{Ak} = 1$ yields the original model with $p_A^{\Gamma_k} = \Gamma_k p_A$, i.e., assigning obligor A to industry segment S_k . The condition $\sum_{k=1}^N \omega_{Ak} < 1$ corresponds to a

¹¹As pointed out earlier, any other severity distribution works as well for this computation.

reduced sensitivity of the default probability p_A on the economy. The weights ω_{Ak} may be obtained by an industry analysis of each individual firm, e.g., using the composition of the turnovers.

Equation (16) can be written as a combination $p_A^\Gamma = p_A \sum_{k=0}^N \omega_{Ak} \Gamma_k$ by setting $\omega_{A0} := 1 - \sum_{k=1}^N \omega_{Ak}$ and $\Gamma_0 := 1$. This can be interpreted as introducing a specific industry segment accounting for the idiosyncratic risk of default events. Calibration of the factor loadings using rating agency data in Gordy (2000, 3.2) indicates that the weight ω_{A0} of the specific segment increases with increasing default probability p_A .

The question now arises whether the decomposition of the default probability changes the default behavior of an individual obligor in the model since now theoretically parts of an obligor may default separately. No error is introduced in the systematic variance UL_{sys}^2 by apportioning default probabilities of individual obligors to different industry segments. However, it does lead to a bias in the diversifiable variance, namely in the term involving p_A^2 which is of second order and can be safely neglected (cf. Equation (18) below).

Similarly, we may introduce factor loadings ϑ_{Ar} to apportion loans to different collateral segments by modelling the expected loss of A conditional on default by $\nu_A^\Lambda = \nu_A \Lambda_A \sum_{r=0}^M \vartheta_{Ar} \Lambda_r$ with $0 \leq \vartheta_{Ar} \leq 1$ and $\sum_{r=0}^M \vartheta_{Ar} = 1$ with the segment $r = 0$ corresponding to the specific segment with only idiosyncratic severity variations and $\Lambda_0 = 1$. Recall that so far we have modelled this by $\nu_A^\Lambda = \nu_A \Lambda_A \Lambda_r$. The factor loadings may be derived by individual analysis of the obligors' collateral types.

By taking both factor loadings into account, the expected portfolio loss conditional on the economy described by Γ, Λ takes the following form (cf. Equation (3)):

$$\mathbb{E}[X|\Gamma, \Lambda] = \sum_{k=0}^N \sum_{r=0}^M \Gamma_k \Lambda_r \sum_A \omega_{Ak} \vartheta_{Ar} p_A \nu_A. \quad (17)$$

The diversifiable variance UL_{div}^2 of Theorem 3.1 generalizes to the formula

$$UL_{\text{div}}^2 = \sum_A \left(1 + \sum_{r,s=0}^M \vartheta_{Ar} \vartheta_{As} \psi_{rs} \delta_r \delta_s \right) \left((1 + \delta_A^2) p_A - \left(1 + \sum_{k,\ell=0}^N \omega_{Ak} \omega_{A\ell} \rho_{k\ell} \sigma_k \sigma_\ell \right) p_A^2 \right) \nu_A^2, \quad (18)$$

whose derivation is analogous to the proof of Theorem 3.1 (see Appendix A.1).

5.2 Independence between Default- and Severity Behavior

Another extension concerns the assumption of independence between default and severity variations, which can be bypassed by using the technique of conditional independence. In such a model we calculate the loss density function $f_X(x | \Gamma = \gamma, \Lambda = \lambda)$ conditional on the states $\Gamma = \gamma, \Lambda = \lambda$; see Section 2 for notations. Such realizations of (Γ, Λ) can be either taken from historical data, from judgement or from a linear factor model. The unconditional loss distribution is then obtained by averaging over all conditional loss distributions with

the corresponding weights of occurrence of the set (Γ, Λ) ; cf. Equation (4). We thereby include any dependence structure between the risk drivers, without distributional assumptions, at the cost of computational efficiency. This concept is well known in market risk.

More specifically, for a linear factor model the possible underlying macro-economic variables for modelling default rates by industry and severities by collateral types are interest rates, gross domestic product, real estate price indices, unemployment rates, exchange rates, and others. The dependence between default and severity can be modelled by relating them to common factors, e.g., interest rates. Frye (2000a), (2000b) applies this concept with default modelled by a Merton-approach, whereas here the model uses stochastic default rates.

5.3 Inclusion of Defaulted Loans

Commercial loan books typically contain a subset of loans that have defaulted but not yet been resolved. Default is no longer the issue, but there is the risk that the realized losses may differ from the provisions. Such loans can be accommodated within the model by introducing an additional industry segment with default volatility zero. All loans in this segment have default probability one. The allocation of the customers to the collateral segments is as if they had not defaulted.

The variance of the loss distribution of the combined portfolio (i.e., including defaulted loans) is calculated according to Theorem 3.1. The mean of the loss distribution is given by

$$\text{Mean} = EL + EW, \text{ where } EL = \sum_{A \text{ not defaulted}} p_A \nu_A, \quad EW = \sum_{A \text{ defaulted}} \nu_A,$$

where EW denotes the expected write-off on defaulted loans. The probability generating function $G_{n-d}(z)$ for the subportfolio of non-defaulted loans with corresponding expected loss EL is computed by the Panjer recursion (Equation (14)). Here, $G_{n-d}(z)$ accounts for systematic default risk and obligor specific severity variations (cf. Equations (10), (11)).

If the defaulted loans were included, they would be treated improperly since the Poisson approximation is not valid for default probabilities equal to one. The generating function $G_d(z)$ of the subportfolio of defaulted loans, accounting for obligor specific severity variations, is given by a product of polynomials of the form $G_A(z) = \sum_{j \geq 0} f_{\Lambda_A}(j) z^j$ (cf. Equation (11)). As for the subportfolio of non-defaulted loans the distribution of f_{Λ_A} can be assumed to depend on ν_A and some fixed relative volatility only (as in Subsection 4.1). For a portfolio with many defaulted obligors, where idiosyncratic severity variations are negligible, this computation may be skipped. In this case the generating function has the form $G_d(z) = z^{EW}$. The generating function $G(z) = \sum_{\nu \geq 0} \pi(\nu) z^\nu$ of the combined portfolio is then given by the product $G(z) = G_{n-d}(z) G_d(z)$ due to independence since, at this stage, only idiosyncratic severity variations are taken into account for defaulted loans, see e.g., Fisz (1963). Note that the corresponding mean of $G(z)$ is given by $EL + EW$. Including in addition systematic severity variations, the cumulative loss distribution of the combined portfolio can be computed by a multiplicative convolution (cf. Equation (15)).

If provisions are raised by the amount of EW , which is considered to be the best estimate for the actual losses of the defaulted loans, then EW should be credited against realized loss to avoid double-counting. This results in a left-shift of the loss distribution by the amount EW ; UL is left unchanged.

6 Loss Distribution in the Limit of a Large Portfolio

Loan portfolios of banks active in the middle market segment usually consist of a large number of loans of order 10'000 or more. We show that the loss distribution of such large portfolios approaches an explicit limit distribution depending on the portfolio only through the segment expected losses, while the obligor specific severity variables and default probabilities are not relevant. The limit distribution directly mirrors the assumed behavior of the economy, modelled by the distributions f_Γ and f_Λ describing the systematic default and severity impact. These insights reveal the importance of the economy and the associated distributional assumptions for quantifying credit risk. Banks active in the middle market segment should therefore focus on the two main issues: first, getting the correct expected loss and second, specifying the economic risk drivers and their distributions.

Before stating the result we have to explain what we mean by the limit of a large portfolio. We start with a loan portfolio \mathcal{P} with expected loss EL , and segment expected losses $EL_r^{(k)}$ in industry S_k and collateral segment L_r . Now we replicate each position of this portfolio n times by taking identical risk parameters. The resulting hypothetical portfolio is denoted by \mathcal{P}_n . We intend to study the limit of the loss distribution of the portfolio \mathcal{P}_n as $n \rightarrow \infty$. In order to study this limit, we normalize the loss of \mathcal{P}_n to the expected loss of EL by scaling down the loss with $1/n$. In Appendix A.3 we use the strong law of large numbers to prove:

Theorem 6.1 *The limit of the normalized loss distribution of the portfolio \mathcal{P}_n as $n \rightarrow \infty$ has the following cumulative distribution function*

$$F_{lim}(x) = \int_{E(x)} f_\Gamma(\gamma) f_\Lambda(\lambda) d\gamma d\lambda$$

with $E(x) := \{(\gamma, \lambda) \in \mathbb{R}_+^N \times \mathbb{R}_+^M \mid \sum_{k,r} EL_r^{(k)} \gamma_k \lambda_r \leq x\}$ as the integration set, which is bounded by a quadratic hypersurface.

The theorem states that in the limit, the probability of losing at most x is obtained by integrating over all states of the economy (γ, λ) having the property that the sum of the segment expected losses $EL_r^{(k)}$, weighted with $\gamma_k \lambda_r$, is less than x . In the single segment case, the integration set is bounded by the hyperbola $\{(\gamma, \lambda) \in \mathbb{R}_+^2 \mid EL\gamma\lambda \leq x\}$.

It is worthwhile noting that in the limit, we have symmetry with regard to the systematic default variable Γ and severity variable Λ . In general, there is no such symmetry, see for instance the UL -formula in Theorem 3.1, which shows that only the systematic part of the UL is symmetric.

Next we are going to have a closer look at the special case of one segment $N = M = 1$. To simplify notations assume here $EL = 1$. We obtain the density

of the loss distribution by the first derivative $f_{lim}(x) = F'_{lim}(x)$, i.e.,

$$f_{lim}(x) = \frac{d}{dx} \left[\int_0^\infty \left(\int_0^{x/\lambda} f_\Gamma(\gamma) d\gamma \right) f_\Lambda(\lambda) d\lambda \right] = \int_0^\infty \frac{1}{\lambda} f_\Gamma\left(\frac{x}{\lambda}\right) f_\Lambda(\lambda) d\lambda . \quad (19)$$

Substituting $\lambda = e^\eta$, $x = e^y$ one observes that $f_{lim}(e^y) = \int_{-\infty}^\infty f_\Gamma(e^{y-\eta}) f_\Lambda(e^\eta) d\eta$, which is the convolution with respect to the additive group of real numbers. Thus we can think of f_{lim} as the *multiplicative convolution*, also called Mellin convolution, of the functions f_Γ and f_Λ defined on the group of positive real numbers. Since the (additive) convolution of the densities of normal distributions is again normal, we conclude that the multiplicative convolution of the densities of lognormal distributions is also lognormal. This implies that if we assume that the systematic default and severity variables have lognormal distribution, then the limit of the loss distribution for large portfolios is lognormal too.

If we do not impose systematic severity variations, the limit of the loss distribution of a large portfolio is the assumed distribution $f_\Gamma(\gamma)$ for the relative number of defaults. In CreditRisk+ (CSFP (1997)) the limit distribution is therefore a Gamma-distribution. Gordy (forthcoming) derives the variance and the quantiles of the credit loss distribution of an 'asymptotically fine-grained' portfolio where the loss is modelled by a single risk factor.

7 Application to Active Portfolio Management

Many large banks are changing their approach to their loan business, moving from a buy-and-hold strategy towards active portfolio management. The credit risk of sub-portfolios is sold to investors or credit risk is taken over from other banks with the goal of diversifying existing portfolios. In this section we give a description for the calculation of the risk contribution for individual loans as well as for sub-portfolios.

7.1 Risk Contributions

The contribution to the economic capital of the portfolio from an individual obligor A can be computed in first order as the sensitivity of the portfolio's economic capital with respect to small changes in the exposure ϵ_A of obligor A . However, since the portfolio economic capital cannot be written in closed form, we use the portfolio unexpected loss UL as given in Theorem 3.1 to define risk contributions at the level of individual obligors:

$$RC_A = \epsilon_A \frac{\partial UL}{\partial \epsilon_A} = \frac{\epsilon_A}{2UL} \frac{\partial UL^2}{\partial \epsilon_A} . \quad (20)$$

These risk contributions add up to the UL of the portfolio. Risk contributions are typically used for pricing the cost of capital to individual loans, applying an overall factor to match the unexpected loss to the economic capital of the overall portfolio.

However, the use of risk contributions as defined in (20) is limited to exposures that are small compared to the overall exposure in the portfolio. This condition is not necessarily met for secondary market transactions. Therefore, for pricing

such deals, one would take the risk contribution as defined above only as an indicator to structure the deal. The final contribution to the portfolio's economic capital however would be calculated by using the loss distribution of the entire portfolio and the one of the portfolio with the loans to be securitized excluded.

Based on the UL -formula developed in Theorem 3.1 we derive the following expression for the risk contribution of obligor A in industry segment S_k and collateral segment L_r :

$$RC_A = \frac{p_A \nu_A}{UL} \left(\sum_{\ell=1}^N \rho_{k\ell} \sigma_k \sigma_\ell EL^{(\ell)} + \sum_{\ell=1}^N \sum_{s=1}^M \rho_{k\ell} \sigma_k \sigma_\ell \psi_{rs} \delta_r \delta_s EL_s^{(\ell)} + \sum_{s=1}^M \psi_{rs} \delta_r \delta_s EL_s + (1 + \delta_r^2) \left((1 + \delta_A^2) - (1 + \sigma_k^2) p_A \right) \nu_A \right). \quad (21)$$

Risk contributions at a segment level are obtained by adding up the risk contributions for all obligors within the segment in question.

7.2 Estimation of Risk Parameters

In order to apply the framework developed in this article, it is necessary to estimate the risk parameters in a reliable way, namely the volatility in default rates as well as the idiosyncratic and systematic volatility of severities. Here we will only touch this important topic.

A general issue in parameter estimation is the time horizon: one may choose a fixed period of say, one year, or match the period to the liquidity of the portfolio. Here we will refer to unconditional figures with a one year time horizon.

To estimate the default risk parameters, the systematic default variance σ_k^2 and the default correlations $\rho_{k\ell}$, one may choose between the following three approaches:

1. using industry specific time series of historically observed default events,
2. using asset values of firms to derive pairwise asset correlations, which are then linked to the relevant default correlations using the Merton model (cf. Merton (1974)), and
3. using a factor model that relates the relative number of default events to macro-economic drivers.

The method to be applied depends on the type of portfolio: while time series of defaults are suited for middle market portfolios with mostly non-quoted firms, the asset value approach may be preferred for portfolios with large corporates.

Empirical estimates for the volatility of default rates can be found for example in Moody's (2000, exhibit 17 and 27) for the period 1970-1999 or in Gordy (2000, appendix B), based on default data from S&P, for the period 1981-1997. Both analyses are based on US data and show relative default rate volatilities of the order of one. However, if differentiated by rating, lower rating categories show less volatility in the range of 0.4 to 1.1 for ratings BBB to CCC, see Gordy (2000, Appendix B). Since middle market portfolios have typical ratings in the lower categories we will use a value of 0.7 for the examples in Section 8. It is well known that periods of relatively low default rates alternate with periods of high

default rates. Consequently, the results depend on the time period used. Similarly, unconditional approaches will yield different values compared to estimates conditional on the economy.

The parameters for the severity risk may be estimated from prices of defaulted bonds, bank internal data, from time series of real estate indices or indices of real estate funds for mortgage loans. Studies on severities can for instance be found in Moody's (2000, exhibit 19, 20 and 24), Asarnow and Edwards (1995), Eales and Bosworth (1998) and in de Castle and Keisman (1999). These investigations typically show highly skewed severity distributions. The analysis of Moody's (2000) yields relative severity volatilities of 34% for secured bank loans, and volatilities in the range of 48%-70% for bonds of different seniority. Although idiosyncratic and systematic contributions to severity variations are not separated in these data, variations of severities over time (see below) suggest that the idiosyncratic component is dominant in these data.

To estimate the magnitude of systematic severity volatilities one may look at time series of average annual severities. Based on the study from Moody's (2000, exhibit 20) using bond prices and from another study in Asarnow and Edwards (1995) using commercial and industrial corporate loans, we estimate systematic severity volatilities in the range of 10%-20%. These data may serve as benchmarks for larger corporate loans but are probably not a good description for medium and small loans in middle market portfolios since the collateral types are different and since problem loans are managed more aggressively, especially in comparison to bonds.

In general, risk parameters found in the literature are mostly based on bond default, which very likely differ from parameters determined from bank internal data. Reasons for differences include different client types, a different definition of default, and the bank internal processes for workout. Therefore, if possible, bank internal data should be used. However a common limiting factor is the time period over which internal data exist. Here a factor model has some advantage because time series of the drivers of default are available far back in time. Recent projects initiated by the banking industry and other institutions to pool default data from international banks may also help to relax the data constraint.

8 Illustrative Examples

We will now use the model as derived in the previous sections to calculate the loss distribution for two portfolios: (1) a portfolio with a few obligors ("small portfolio") and (2) a portfolio with many obligors ("large portfolio"). Since the main ideas of this article can be illustrated for a single segment portfolio, we limit ourselves to this situation, following the calculation procedure as given in Section 4. We assume the default scaling factor to be Gamma, obligor specific severity variations are modelled with a normal distribution¹², and systematic

¹²For the explicit implementation we have chosen to model the obligor specific severity density function $f^{(\nu, \Delta)}$ by discretizing a normal distribution to the predefined exposure bands. Negative values are cut off, maintaining the symmetric shape and the mean of the density by also cutting off the corresponding part on the positive axis. The mass, taken away from each side, is proportionally allocated to the remaining support.

severity variations are assumed to be lognormal distributed¹³.

For both portfolios, we will consider default risk in the first step and then account for obligor specific and systematic variation in collateral value. Finally we compare the theoretical distribution in the limit of a large portfolio described in Theorem 6.1 with the one calculated by our model.

8.1 Loss distribution for a small portfolio

We consider a small portfolio of 102 customers with a total exposure of CHFm 360 and expected loss CHFm 2.5. All loans have the same type of collateral and belong to the same industry segment. The detailed portfolio description is given by the following table:

#customers	Exposure in CHFm per obligor	Default probability per obligor	expected loss conditional on default
50	2	1%	1
50	4	1%	2
1	20	2%	10
1	40	4%	20

Table 1: Specification of the small sample portfolio.

We show the graphs of the loss distributions of this small portfolio using the technique specified in Section 4 for the following four cases:

1. for pure default risk: systematic default volatility $\sigma = 70\%$,
2. by adding small obligor specific severity volatilities $\delta_A = 5\%$ (assumed to be equal for all obligors),
3. by adding larger obligor specific severity volatilities $\delta_A = 15\%$,
4. by adding obligor specific severity volatilities $\delta_A = 15\%$, and a systematic severity volatility of $\delta = 15\%$.

The parameters have been chosen to perform a realistic modelling, see Section 7.2. In addition, we calculate the effect of more extreme severity parameters, which are shown in Table 2.

The interesting part of the loss distributions and the corresponding cumulative distributions is shown in Figure 2. Note that the peaks in the loss distribution, best visible in the plot for pure default risk, are caused by the inhomogeneous portfolio structure (large loans of expected loss conditional on default CHFm 10 and 20, respectively). The severity variations have a smoothing effect on the loss distributions. We observe that the obligor specific severity variations have a considerable impact on the loss distribution for such a small portfolio.

As seen in Figure 2, the loss distribution corresponding to $\delta = 15\%$ smooths out when including all risk drivers, suggesting a homogeneous portfolio. However, the lumpy loans are still present in the portfolio, even though they are not visible in the graph of the loss distribution.

¹³As discussed in Section 4 other distributions for modelling of severity variations can be implemented without any problems.

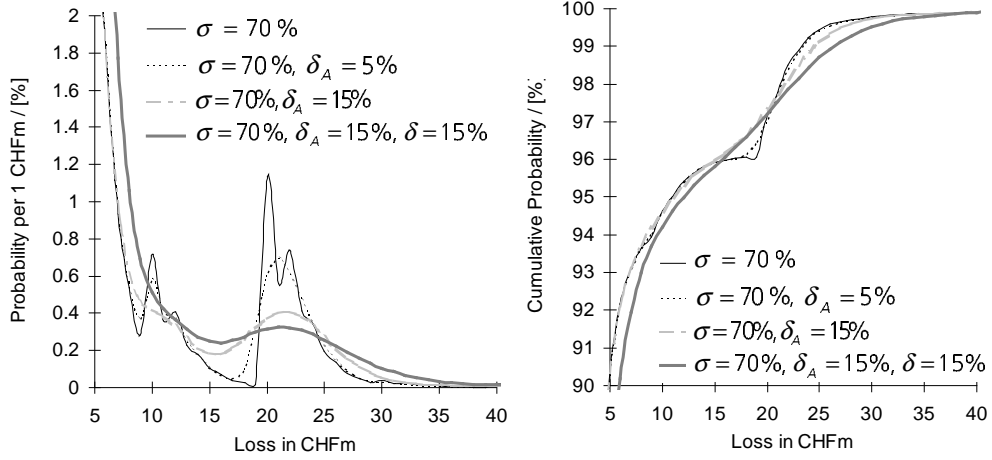


Figure 2: Probability densities (left) and corresponding cumulative loss distributions (right) of the small portfolio with $EL = \text{CHFm } 2.5$ for different assumptions on the volatility parameters.

Looking at the loss percentiles in Table 2 it might seem surprising that the 95- and 97.5-percentiles decrease when increasing the amount of severity risk. The explanation for this fact is that both percentiles are in the range of the expected loss conditional on default 10 and 20 of the lumpy loans, which cause the peaks in the loss distribution (compare Figure 2). By introducing severity risk (obligor specific or systematic) these peaks are smeared out, thereby moving mass to both sides of the peaks lowering the percentile losses on the left of these peaks. This phenomenon is present for portfolios with lumpy exposures.

σ	0%	0%	0%	0%	0%	70%	70%	70%	70%	70%
δ	0%	0%	15%	15%	30%	0%	0%	15%	15%	30%
δ_A	0%	15%	0%	15%	30%	0%	15%	0%	15%	30%
UL	4.45	4.50	4.51	4.56	4.91	4.74	4.79	4.81	4.86	5.20
UL_{sys}	0	0	0.38	0.38	0.75	1.75	1.75	1.81	1.81	1.98
UL_{div}	4.45	4.50	4.50	4.55	4.85	4.41	4.46	4.46	4.51	4.81
Perc 95%	10.40	10.52	10.41	10.35	9.44	11.00	11.05	11.00	11.01	10.38
Perc 97.5%	20.07	19.91	19.65	16.29	18.01	20.53	20.44	20.53	19.90	18.77
Perc 99%	21.98	23.56	23.50	24.29	26.61	23.26	24.58	23.26	25.37	27.76
Perc 99.98%	41.95	44.32	45.45	46.47	59.00	45.62	48.13	45.62	51.46	63.93

Table 2: Unexpected loss and percentiles in CHFm of the small portfolio for different assumptions on the risk parameters.

The table shows that the diversifiable risk is the dominant driver for this small portfolio, as seen by comparing UL_{div} with UL_{sys} .

8.2 Loss distribution for a large portfolio

We now create a large portfolio of 10'200 customers by replicating all positions of the small portfolio a hundred times. Exposure sizes are rescaled to keep total portfolio exposure and expected loss unchanged.

Figure 3 exhibits the loss distributions and the corresponding cumulative distributions in three cases: Pure default risk with systematic default volatility $\sigma = 70\%$, and then adding obligor specific severity volatilities $\delta_A = 15\%$ (resp.

$\delta_A = 30\%$), and a systematic severity volatility of $\delta = 15\%$ (resp. $\delta = 30\%$). In Table 3 we calculate the effect of more extreme severity parameters. As seen in Table 3, the relative impact of UL now favors the systematic risk. We observe that the obligor specific severity variations have almost no impact for such a large portfolio with $\sigma = 70\%$, $\delta = 15\%$. We also see that the inhomogeneity of the portfolio has no impact on the shape of the loss distributions: there are no peaks as in the situation of the small portfolio.

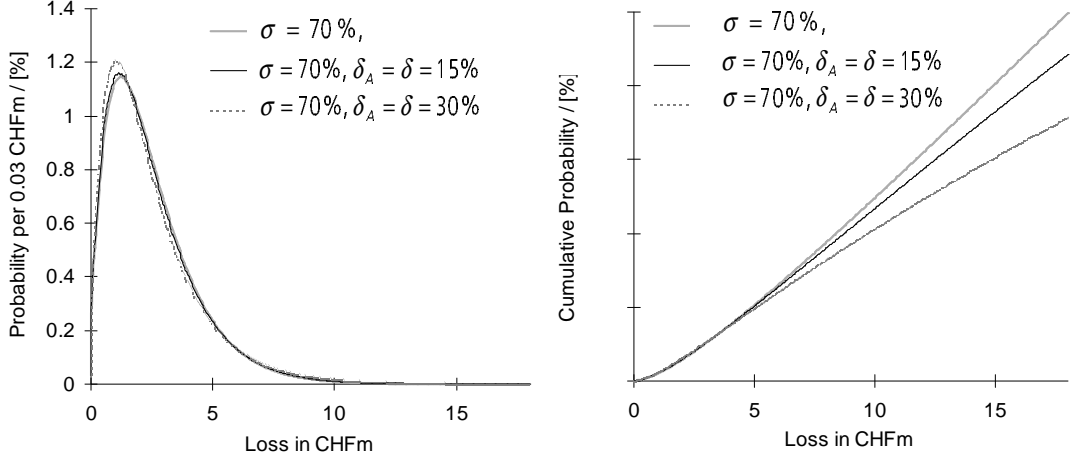


Figure 3: Probability densities (left) and corresponding cumulative loss distributions (right, log-scale) of the large portfolio with $EL = \text{CHFm } 2.5$ for different assumptions on the risk parameters.

σ	0%	0%	0%	0%	0%	70%	70%	70%	70%	70%
δ	0%	0%	15%	15%	30%	0%	0%	15%	15%	30%
δ_A	0%	15%	0%	15%	30%	0%	15%	0%	15%	30%
UL	0.44	0.45	0.59	0.59	0.89	1.80	1.81	1.86	1.86	2.03
UL_{sys}	0.00	0.00	0.38	0.38	0.75	1.75	1.75	1.81	1.81	1.98
UL_{div}	0.44	0.45	0.45	0.45	0.49	0.44	0.45	0.45	0.45	0.48
Perc 95%	3.29	3.30	3.56	3.57	4.19	6.00	6.00	6.11	6.12	6.44
Perc 97.5%	3.46	3.47	3.82	3.83	4.68	7.05	7.06	7.25	7.26	7.82
Perc 99%	3.67	3.68	4.15	4.16	5.32	8.41	8.42	8.75	8.75	9.72
Perc 99.98%	4.40	4.43	5.38	5.41	7.88	13.96	13.97	15.18	15.19	18.77

Table 3: Unexpected loss and percentiles in CHFm of the large portfolio for different assumptions on the risk parameters.

Finally, we compare the loss distribution of the large portfolio with the limit distribution as derived in Theorem 6.1. Recall that this limit distribution is essentially the multiplicative convolution of the systematic default distribution with the systematic severity distribution. The results are shown in Table 4 for the values $\sigma = 70\%$, $\delta = 30\%$ and $\delta_A = 30\%$. We observe quite a good coincidence if we do not go out too far in the tail.

As seen from the comparison of Tables 3 and 2, the diversifiable risk due to obligor specific severity volatility is only important for small portfolios, and the risk may be neglected for large portfolios. The ratio of $\sum_A \delta_A^2 p_A \nu_A^2$ over $\sigma^2 EL^2$, describing the diversifiable variance induced by idiosyncratic severity variations relative to the systematic variance due to default risk, can be taken to judge

	model	limit distribution	relative error
Perc 95%	6.44	6.31	-2.0%
Perc 97.5%	7.82	7.63	-2.4%
Perc 99%	9.72	9.46	-2.7%
Perc 99.98%	18.77	18.16	-3.2%
Perc 99.99%	20.55	19.86	-3.4%

Table 4: Comparison of loss percentiles in CHFm of the large portfolio with the limit distribution of Theorem 6.1 for $\sigma = 70\%$, $\delta = 30\%$ and $\delta_A = 30\%$.

whether idiosyncratic variations in severity can be neglected. This ratio is 60% for the small portfolio but only 0.6% for the large portfolio with parameters $\delta_A = 15\%$ and $\sigma = 70\%$.

Using similar arguments on the ratio above, we find that for small portfolios the modelling of idiosyncratic risk is of major importance, whereas less weight has to be put on the specification of the systematic risk parameters.

A Appendix

A.1 Proof of Theorem 3.1

We make use of the variance decomposition in Equation (6) and separately determine its terms. By Equation (3) we obtain for the variance of the conditional expectation:

$$\begin{aligned} \text{var}[\mathbb{E}[X \mid \Gamma, \Lambda]] &= \text{var}\left[\sum_{k,r} \Gamma_k \Lambda_r EL_r^{(k)}\right] = \sum_{k,r} \sum_{\ell,s} \text{cov}(\Gamma_k \Lambda_r, \Gamma_\ell \Lambda_s) EL_r^{(k)} EL_s^{(\ell)} \\ &= \sum_{k,r} \sum_{\ell,s} (\rho_{k\ell} \sigma_k \sigma_\ell \psi_{rs} \delta_r \delta_s + \rho_{k\ell} \sigma_k \sigma_\ell + \psi_{rs} \delta_r \delta_s) EL_r^{(k)} EL_s^{(\ell)}, \end{aligned}$$

which equals UL_{sys}^2 , and where we have used the following relation based on the independence of Γ and Λ :

$$\begin{aligned} \text{cov}(\Gamma_k \Lambda_r, \Gamma_\ell \Lambda_s) &= \mathbb{E}[\Gamma_k \Lambda_r \Gamma_\ell \Lambda_s] - 1 \\ &= (\mathbb{E}[\Gamma_k \Gamma_\ell] - 1)(\mathbb{E}[\Lambda_r \Lambda_s] - 1) + (\mathbb{E}[\Gamma_k \Gamma_\ell] - 1) + (\mathbb{E}[\Lambda_r \Lambda_s] - 1) \\ &= \rho_{k\ell} \sigma_k \sigma_\ell \psi_{rs} \delta_r \delta_s + \rho_{k\ell} \sigma_k \sigma_\ell + \psi_{rs} \delta_r \delta_s. \end{aligned}$$

Hence, it remains to show that the second term $\mathbb{E}[\text{var}[X \mid \Gamma, \Lambda]]$ equals UL_{div}^2 . By the conditional independence of obligors' losses $X_A = I_A \nu_A \Lambda_A \Lambda_r$ (cf. Equation (1)) and the independence between Γ and Λ we can write:

$$\begin{aligned} \mathbb{E}[\text{var}[X \mid \Gamma, \Lambda]] &= \mathbb{E}\left[\sum_A \text{var}[X_A \mid \Gamma, \Lambda]\right] \\ &= \mathbb{E}\left[\sum_{k,r} \sum_{A \in S_k \cap L_r} (\Gamma_k p_A \mathbb{E}[\Lambda_A^2] - \Gamma_k^2 p_A^2 \mathbb{E}[\Lambda_A]^2) \nu_A^2 \Lambda_r^2\right] \\ &= \sum_{k,r} \mathbb{E}[\Lambda_r^2] \sum_{A \in S_k \cap L_r} (\mathbb{E}[\Gamma_k] p_A \mathbb{E}[\Lambda_A^2] - \mathbb{E}[\Gamma_k^2] p_A^2) \nu_A^2 \\ &= \sum_{k,r} (1 + \delta_r^2) \sum_{A \in S_k \cap L_r} (p_A (1 + \delta_A^2) - p_A^2 (1 + \sigma_k^2)) \nu_A^2, \end{aligned}$$

which is indeed UL_{div}^2 , and hence Theorem 3.1 is shown.

A.2 Numerical error estimation for computation of percentiles

Recall Equation (15) expressing the cumulative distribution $F(n)$ of the unconditional loss. In order to actually compute the probabilities $F(n)$, we replace this infinite sum by a finite summation over ν , with the upper summation bound depending on n . The resulting approximation error can be controlled by a parameter ε as explained in the following.

For $\varepsilon > 0$ let λ_ε be the lower ε -percentile of $F_\Lambda(\lambda)$, that is, $F_\Lambda(\lambda_\varepsilon) = \varepsilon$. By ν_ε we denote the upper ε -percentile of the distribution $\pi(\nu)$, which means $\sum_{\nu \geq \nu_\varepsilon} \pi(\nu) \leq \varepsilon$ and $\sum_{\nu > \nu_\varepsilon - 1} \pi(\nu) > \varepsilon$. We define the upper summation bound $N_\varepsilon(n) := \min\{\frac{n}{\lambda_\varepsilon}, \nu_\varepsilon\}$ and denote by

$$F_\varepsilon(n) := \pi(0) + \sum_{0 < \nu < N_\varepsilon(n)} \pi(\nu) F_\Lambda(n/\nu)$$

the corresponding approximation of $F(n)$. We then have for all n

$$F_\varepsilon(n) \leq F(n) \leq F_\varepsilon(n) + \varepsilon. \quad (22)$$

In fact, the left hand inequality is clear. To see the inequality on the right hand side note that

$$F(n) - F_\varepsilon(n) = \sum_{\nu \geq N_\varepsilon(n)} \pi(\nu) F_\Lambda(n/\nu) \leq F_\Lambda\left(\frac{n}{N_\varepsilon(n)}\right) \cdot \sum_{\nu \geq N_\varepsilon(n)} \pi(\nu).$$

The right hand product is at most ε , since both factors are bounded by one and at least one of them can be bounded by ε . Indeed, we either have $N_\varepsilon(n) = n/\lambda_\varepsilon$, in which case $F_\Lambda(n/N_\varepsilon(n)) = F_\Lambda(\lambda_\varepsilon) = \varepsilon$; or $N_\varepsilon(n) = \nu_\varepsilon$, which tells us that $\sum_{\nu \geq N_\varepsilon(n)} \pi(\nu) \leq \varepsilon$.

The economic capital for a given confidence level ℓ (e.g. $\ell = 99\%$) is defined by $n(\ell)$ such that $F(n(\ell)-1) < \ell$, $F(n(\ell)) \geq \ell$. The approximation F_ε introduces a certain error in the percentiles, which can be estimated as follows: If $n_\varepsilon(\ell)$ is defined as $n(\ell)$, but with F replaced by F_ε , we have

$$n_\varepsilon(\ell - \varepsilon) \leq n(\ell) \leq n_\varepsilon(\ell) \quad (23)$$

for any confidence level ℓ and error bound ε . In fact, the right hand inequality is obvious. To verify the left hand inequality note that $\ell - \varepsilon \leq F(n(\ell)) - \varepsilon \leq F_\varepsilon(n(\ell))$ by Equation (22) and the definition of percentiles. Thus, we are able to compute the percentiles $n(\ell)$ up to the desired accuracy by choosing $\varepsilon \leq 1 - \ell$ suitably small. Examples can be found in Section 8.

A.3 Proof of Theorem 6.1

The proof is based on the strong law of large numbers. Let X be the random loss variable of a loan portfolio \mathcal{P} with expected loss EL . We define $Z_n := \frac{1}{n} \sum_{j=1}^n X_j$ where X_j is an independent copy of X . For simplicity we consider the single segment case. Theorem 6.1 claims that Z_n converges in distribution to the random variable $\Gamma \Lambda EL$ which is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{E}[g(Z_n)] = \mathbb{E}[g(\Gamma \Lambda EL)]$$

for all bounded continuous functions g on \mathbb{R} , cf. Taylor (1997, Chap. VI).

By the strong law of large numbers (e.g. Taylor (1997, 4.4.2.)) we may conclude that conditionally (i.e., under the conditional probability measure induced by the condition $\Gamma = \gamma, \Lambda = \lambda$) the following convergence holds: $\lim_{n \rightarrow \infty} Z_n = \gamma\lambda EL$ almost surely (see Equation (2)). So, for any continuous bounded function g on \mathbb{R} we conditionally obtain $g(Z_n) \rightarrow g(\gamma\lambda EL)$ almost surely. The function g is bounded, so is the random variable $g(Z_n)$. By the theorem of dominated convergence (see e.g. Taylor (1997, 2.1.38)), it follows that conditionally

$$\lim_{n \rightarrow \infty} E[g(Z_n)] = E[g(\gamma\lambda EL)] = g(\gamma\lambda EL) ,$$

and therefore $\lim_{n \rightarrow \infty} E[g(Z_n) \mid \Gamma, \Lambda] = g(\Gamma\Lambda EL)$. Since the random variable $E[g(Z_n) \mid \Gamma, \Lambda]$ is also bounded, it again follows by the theorem of dominated convergence that

$$\lim_{n \rightarrow \infty} E[E[g(Z_n) \mid \Gamma, \Lambda]] = E[g(\Gamma\Lambda EL)] .$$

Since $E[g(Z_n)] = E[E[g(Z_n) \mid \Gamma, \Lambda]]$ (see Goovaerts et al. (1990, V.1.2)), the claim follows.

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