Nonvanishing of Kronecker coefficients for rectangular shapes

Peter Bürgisser[a,1], Matthias Christandl[b,2], Christian Ikenmeyer[a,1]

[a] *Institute of Mathematics, University of Paderborn, D-33098 Paderborn, Germany*
[b] *Institute for Theoretical Physics, ETH Zurich, CH-8093 Zurich, Switzerland*

Abstract

We prove that for any partition $(\lambda_1, \dots, \lambda_{d^2})$ of size ℓd there exists $k \geq 1$ such that the tensor square of the irreducible representation of the symmetric group $\mathsf{S}_{k\ell d}$ with respect to the rectangular partition $(k\ell, \dots, k\ell)$ contains the irreducible representation corresponding to the stretched partition $(k\lambda_1, \dots, k\lambda_{d^2})$. We also prove a related approximate version of this statement in which the stretching factor k is effectively bounded in terms of d. We further discuss the consequences for geometric complexity theory which provided the motivation for this work.

Keywords: Kronecker coefficients, quantum marginal problem, geometric complexity theory, quantum information theory
2000 MSC: 20C30

1. Introduction

Kronecker coefficients are the multiplicities occurring in tensor product decompositions of irreducible representations of the symmetric groups. These coefficients play a crucial role in geometric complexity theory [15, 16], which is an approach to arithmetic versions of the famous P versus NP problem and

Email addresses: pbuerg@upb.de (Peter Bürgisser), christandl@phys.ethz.ch (Matthias Christandl), ciken@math.upb.de (Christian Ikenmeyer)

[1] Supported by the German Science Foundation (grant BU 1371/3-1 of the SPP 1388 on Representation Theory)

[2] Supported by the Swiss National Science Foundation (grant PP00P2-128455) and the German Science Foundation (grant CH 843/1-1 of the SPP 1388 on Representation Theory and grant CH 843/2-1)

related questions in computational complexity via geometric representation theory. As pointed out in [3] (see Section 4), for implementing this approach, one needs to identify certain partitions $\lambda \vdash_{d^2} \ell d$ with the property that a symmetric version of the Kronecker coefficient associated with $\lambda, \square, \square$ vanishes, where $\square := (\ell, \dots, \ell)$ stands for the rectangle partition of length d. Computer experiments show that such λ occur rarely. Our main result confirms this experimental finding. We prove that for any $\lambda \vdash_{d^2} \ell d$ there exists a stretching factor k such that the Kronecker coefficient of $k\lambda, k\square, k\square$ is nonzero (Theorem 1). Here, $k\lambda$ stands for the partition arising by multiplying all components of λ by k. We also prove a related approximate version of this statement (Theorem 2) that suggests that the stretching factor k may be chosen not too large. Similar results are shown to hold for the symmetric version of the Kronecker coefficient and thus have a bearing on geometric complexity theory (see Lemma 3 and Section 4).

Our proof relies on a recently discovered connection between Kronecker coefficients and the spectra of composite quantum states [11, 5]. Let ρ_{AB} be the density operator of a bipartite quantum system and let ρ_A, ρ_B denote the density operators corresponding to the systems A and B, respectively. It turns out that the set of possible triples of spectra $(\operatorname{spec}\rho_{AB}, \operatorname{spec}\rho_A, \operatorname{spec}\rho_B)$ is obtained as the closure of the set of triples $(\overline{\lambda}, \overline{\mu}, \overline{\nu})$ of normalized partitions λ, μ, ν with nonvanishing Kronecker coefficient, where we set $\overline{\lambda} := \frac{1}{|\lambda|}\lambda$. For proving the main theorem it is therefore sufficient to construct, for any prescribed spectrum $\overline{\lambda}$, a density matrix ρ_{AB} having this spectrum and such that the spectra of ρ_A and ρ_B are uniform distributions.

The set of possible triples of spectra $(\operatorname{spec}\rho_{AB}, \operatorname{spec}\rho_A, \operatorname{spec}\rho_B)$ is interpreted in [11] as the moment polytope of a complex algebraic group variety, thus linking the problem to geometric invariant theory. We do not not use this connection in our paper. Instead we argue as in [5] using the estimation theorem of [9]. The exponential decrease rate in this estimation allows us to derive the bound on the stretching factor in Theorem 2.

2. Preliminaries

2.1. Kronecker coefficients and their moment polytopes

A *partition* λ of $n \in \mathbb{N}$ is a monotonically decreasing sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of natural numbers such that $\lambda_i = 0$ for all but finitely many i. The length $\ell(\lambda)$ of λ is defined as the number of its nonzero parts and its size as $|\lambda| := \sum_i \lambda_i$. One writes $\lambda \vdash_d n$ to express that λ is a partition of n

with $\ell(\lambda) \le d$. Note that $\bar{\lambda} := \lambda/n = (\lambda_1/n, \lambda_2/n, \ldots)$ defines a probability distribution on $\mathbb{N}$.

It is well known [8] that the complex irreducible representations of the symmetric group S_n can be labeled by partitions $\lambda \vdash n$ of n. We shall denote by $\mathscr{S}_\lambda$ the irreducible representation of S_n associated with λ. The *Kronecker coefficient* $g_{\lambda,\mu,\nu}$ associated with three partitions λ, μ, ν of n is defined as the dimension of the space of S_n-invariants in the tensor product $\mathscr{S}_\lambda \otimes \mathscr{S}_\mu \otimes \mathscr{S}_\nu$. Note that $g_{\lambda,\mu,\nu}$ is invariant with respect to a permutation of the partitions. It is known that $g_{\lambda,\mu,\nu} = 0$ vanishes if $\ell(\lambda) > \ell(\mu)\ell(\nu)$. Equivalently, $g_{\lambda,\mu,\nu}$ may also be defined as the multiplicity of $\mathscr{S}_\lambda$ in the tensor product $\mathscr{S}_\mu \otimes \mathscr{S}_\nu$. If $\mu = \nu$ we define the *symmetric Kronecker coefficient* sg_μ^λ as the multiplicity of $\mathscr{S}_\lambda$ in the symmetric square $\mathrm{Sym}^2(\mathscr{S}_\mu)$. We note that $sg_\mu^\lambda \le g_{\lambda,\mu,\mu}$.

The Kronecker coefficients also appear when studying representations of the general linear groups GL_d over $\mathbb{C}$. We recall that rational irreducible GL_d-modules are labeled by their highest weight, a monotonically decreasing list of d integers, cf. Fulton and Harris [8]. We will only be concerned with highest weights consisting of nonnegative numbers, which are therefore of the form $\lambda \vdash_d k$ for modules of degree k. We shall denote by $\mathscr{V}_\lambda$ the irreducible GL_d-module with highest weight λ.

Suppose now that $\lambda \vdash_{d_1 d_2} k$. When restricting with respect to the morphism $\mathsf{GL}_{d_1} \times \mathsf{GL}_{d_2} \to \mathsf{GL}_{d_1 d_2}, (\alpha, \beta) \mapsto \alpha \otimes \beta$, then the module $\mathscr{V}_\lambda$ splits as follows:

$$\mathscr{V}_\lambda = \bigoplus_{\mu \vdash_{d_1} k, \nu \vdash_{d_2} k} g_{\lambda,\mu,\nu} \mathscr{V}_\mu \otimes \mathscr{V}_\nu. \tag{1}$$

Even though being studied for more than fifty years, Kronecker coefficients are only understood in some special cases. For instance, giving a combinatorial interpretation of the numbers $g_{\lambda,\mu,\nu}$ is a major open problem, cf. Stanley [17, 18] for more information.

We are mainly interested in whether $g_{\lambda,\mu,\nu}$ vanishes or not. For studying this in an asymptotic way one may consider, for fixed $d = (d_1, d_2, d_3) \in \mathbb{N}^3$ with $d_1 \le d_2 \le d_3 \le d_1 d_2$, the set

$$\mathsf{Kron}(d) := \Big\{ \frac{1}{n}(\lambda, \mu, \nu) \mid n \in \mathbb{N}, \lambda \vdash_{d_1} n, \mu \vdash_{d_2} n, \nu \vdash_{d_3} n \;\; g_{\lambda,\mu,\nu} \neq 0 \Big\}.$$

It turns out that $\mathsf{Kron}(d)$ is a rational polytope in $\mathbb{Q}^{d_1+d_2+d_3}$. This follows from general principles from geometric invariant theory, namely $\mathsf{Kron}(d)$ equals the *moment polytope* of the projective variety $\mathbb{P}(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \mathbb{C}^{d_3})$ with respect to the standard action of the group $\mathsf{GL}_{d_1} \times \mathsf{GL}_{d_2} \times \mathsf{GL}_{d_3}$, cf. [14, 7, 11]. For an elementary proof that $\mathsf{Kron}(d)$ is a polytope see [4].

2.2. Spectra of density operators

Let $\mathcal{H}$ be a d-dimensional complex Hilbert space and denote by $\mathcal{L}(\mathcal{H})$ the space of linear operators mapping $\mathcal{H}$ into itself. For $\rho \in \mathcal{L}(\mathcal{H})$ we write $\rho \geq 0$ to denote that ρ is positive semidefinite. By the *spectrum* $\operatorname{spec}\rho$ of ρ we will understand the vector $(r_1, \ldots, r_d)$ of eigenvalues of ρ in decreasing order, that is, $r_1 \geq \cdots \geq r_d$. The set of *density operators* on $\mathcal{H}$ is defined as

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{L}(\mathcal{H}) \mid \rho \geq 0, \mathrm{tr}\rho = 1\}.$$

Density operators are the mathematical formalism to describe the states of quantum objects. The spectrum of a density operator is a probability distribution on $[d] := \{1, \ldots, d\}$.

The state of a system composed of particles A and B is described by a density operator on a tensor product of two Hilbert spaces, $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The partial trace $\rho_A = \mathrm{tr}_B(\rho_{AB}) \in \mathcal{L}(\mathcal{H}_A)$ of ρ_{AB} obtained by tracing over B then defines the state of particle A. We recall that the *partial trace* tr_B is the linear map $\mathrm{tr}_B \colon \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A)$ uniquely characterized by the property $\mathrm{tr}(R\,\mathrm{tr}_B(\rho_{AB})) = \mathrm{tr}(\rho_{AB} R \otimes \mathrm{id})$ for all $\rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ and $R \in \mathcal{L}(\mathcal{H}_A)$.

2.3. Admissible spectra and Kronecker coefficients

The *quantum marginal problem* asks for a description of the set of possible triples of spectra $(\operatorname{spec}\rho_{AB}, \operatorname{spec}\rho_A, \operatorname{spec}\rho_B)$ for fixed $d_A = \dim \mathcal{H}_A$ and $d_B = \dim \mathcal{H}_B$. In [5, 11, 4] it was shown that this set equals the closure of the moment polytope for Kronecker coefficients, so

$$\overline{\mathsf{Kron}(d_A, d_B, d_A d_B)} = \Big\{(\operatorname{spec}\rho_{AB}, \operatorname{spec}\rho_A, \operatorname{spec}\rho_B) \mid \rho_{AB} \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)\Big\}.$$

We remark that this result is related to *Horn's problem* that asks for the compatibility conditions of the spectra of Hermitian operators A, B, and $A + B$ on finite dimensional Hilbert spaces. Klyachko [10] gave a similar characterization of these triples of spectra in terms of the Littlewood-Richardson coefficients. The latter are the multiplicities occurring in tensor products of irreducible representations of the general linear groups. For Littlewood-Richardson coefficients one can actually avoid the asymptotic description since the so called saturation conjecture is true [12].

2.4. Estimation theorem

We will need a consequence of the estimation theorem of [9]. The group $\mathsf{S}_k \times \mathsf{GL}_d$ naturally acts on the tensor power $(\mathbb{C}^d)^{\otimes k}$. Schur-Weyl duality describes the isotypical decomposition of this module as

$$(\mathbb{C}^d)^{\otimes k} = \bigoplus_{\lambda \vdash_d k} \mathscr{S}_\lambda \otimes \mathscr{V}_\lambda. \tag{2}$$

We note that this is an orthogonal decomposition with respect to the standard inner product on $(\mathbb{C}^d)^{\otimes k}$. Let P_λ denote the orthogonal projection of $(\mathbb{C}^d)^{\otimes k}$ onto $\mathscr{S}_\lambda \otimes \mathscr{V}_\lambda$. The estimation theorem of Keyl and Werner [9] states that for any density operator $\rho \in \mathcal{L}(\mathbb{C}^d)$ with spectrum r we have

$$\mathrm{tr}(P_\lambda\, \rho^{\otimes k}) \le (k+1)^{d(d-1)/2} \exp\big(-\frac{k}{2}\|\overline{\lambda} - r\|_1^2\big) \tag{3}$$

(see [5] for a simple proof). This shows that the probability distribution $\overline{\lambda} \mapsto \mathrm{tr}(P_\lambda\, \rho^{\otimes k})$ is concentrated around r with exponential decay in the distance $\|\overline{\lambda} - r\|_1$.

3. Main results

By a decreasing probability distribution r on $[d^2]$ we understand $r \in \mathbb{R}^{d^2}$ such that $r_1 \ge \cdots \ge r_{d^2} \ge 0$ and $\sum_i r_i = 1$. We denote by $\mathbf{u}_d = (\frac{1}{d}, \ldots, \frac{1}{d})$ the uniform probability distribution on $[d]$.

Theorem 1. *The following statements are true:*

(1) *For all decreasing probability distributions r on $[d^2]$, the triple $(r, \mathbf{u}_d, \mathbf{u}_d)$ is contained in $\overline{\mathsf{Kron}(d^2, d, d)}$.*

(2) *Let $\lambda \vdash \ell d$ be a partition into at most d^2 parts for $\ell, d \ge 1$ and let $\square := (\ell, \ldots, \ell)$ denote the rectangular partition of ℓd into d parts. Then there exists a stretching factor $k \ge 1$ such that $g_{k\lambda, k\square, k\square} \neq 0$.*

The next result indicates that the stretching factor k may be chosen not too large.

Theorem 2. *Let $\lambda \vdash_{d^2} \ell d$ and $\epsilon > 0$. Then there exists a stretching factor $k = O(\frac{d^4}{\epsilon^2} \log \frac{d}{\epsilon})$ and there exist partitions $\Lambda \vdash_{d^2} k\ell d$ and $R_1, R_2 \vdash_d k\ell d$ of $k\ell d$ such that $g_{k\lambda, R_1, R_2} \neq 0$ and*

$$\|\Lambda - k\lambda\|_1 \le \epsilon|\Lambda|, \quad \|R_i - k\square\|_1 \le \epsilon|R_i| \quad \text{for } i = 1, 2.$$

Suppose that $g_{\lambda,\mu,\mu} \neq 0$. By stretching the partitions λ, μ with two, we can guarantee that the corresponding symmetric Kronecker coefficients does not vanish either.

Lemma 3. *Let $\lambda, \mu \vdash n$. If $\mathscr{S}_\lambda$ occurs in $\mathscr{S}_\mu \otimes \mathscr{S}_\mu$, then $\mathscr{S}_{2\lambda}$ occurs in $\mathrm{Sym}^2(\mathscr{S}_{2\mu})$. In other words, $g_{\lambda,\mu,\mu} \neq 0$ implies $sg_{2\mu}^{2\lambda} \neq 0$.*

This lemma, when combined with Theorems 1 and 2, shows that finding partitions λ with $sg_{\square}^{\lambda} = 0$, as required for the purposes of geometric complexity theory (see below), requires a careful search.

4. Connection to geometric complexity theory

The most important problem of algebraic complexity theory is Valiant's Hypothesis [19, 20], which is an arithmetic analogue of the famous P versus NP conjecture (see [2] for background information). Valiant's Hypothesis can be easily stated in precise mathematical terms.

Consider the determinant $\det_d = \det[x_{ij}]_{1\le i,j\le d}$ of a d by d matrix of variables x_{ij}, and for $m < d$, the *permanent* of its m by m submatrix defined as

$$\mathrm{per}_m := \sum_{\sigma\in S_m} x_{1,\sigma(1)} \cdots x_{m,\sigma(m)}.$$

We choose $z := x_{dd}$ as a homogenizing variable and view $\det_d$ and $z^{d-m}\mathrm{per}_m$ as homogeneous functions $\mathbb{C}^{d^2} \to \mathbb{C}$ of degree d. How large has d to be in relation to m such that there is a linear map $A\colon \mathbb{C}^{d^2} \to \mathbb{C}^{d^2}$ with the property that

$$z^{d-m}\mathrm{per}_m = \det_d \circ A? \qquad (*)$$

It is known that such A exists for $d = O(m^2 2^m)$. Valiant's Hypothesis states that (*) is impossible for d polynomially bounded in m.

Mulmuley and Sohoni [15] suggested to study an orbit closure problem related to (*). Note that the group $\mathsf{GL}_{d^2} = \mathsf{GL}_{d^2}(\mathbb{C})$ acts on the space $\mathrm{Sym}^d(\mathbb{C}^{d\times d})^*$ of homogeneous polynomials of degree d in the variables x_{ij} by substitution. Instead of (*), we ask now whether

$$z^{d-m}\mathrm{per}_m \in \overline{\mathsf{GL}_{d^2} \cdot \det_d}. \qquad (**)$$

Mulmuley and Sohoni [15] conjectured that (**) is impossible for d polynomially bounded in m, which would imply Valiant's Hypothesis.

Moreover, in [15, 16] it was proposed to show that (**) is impossible for specific values m, d by exhibiting an irreducible representation of SL_{d^2} in the coordinate ring of the orbit closure of $z^{d-m}\mathrm{per}_m$, that does not occur in the coordinate ring $\mathbb{C}[\overline{\mathsf{GL}_{d^2} \cdot \det_d}]$ of $\overline{\mathsf{GL}_{d^2} \cdot \det_d}$. We call such a representation of SL_{d^2} an *obstruction for (**)* for the values m, d.

We can label the irreducible SL_{d^2}-representations by partitions λ into at most d^2-1 parts: For $\lambda \in \mathbb{N}^{d^2}$ such that $\lambda_1 \geq \ldots \geq \lambda_{d^2-1} \geq \lambda_{d^2} = 0$ we shall denote by $\mathscr{V}_\lambda(\mathsf{SL}_{d^2})$ the irreducible SL_{d^2}-representation obtained from the irreducible GL_{d^2}-representation $\mathscr{V}_\lambda$ with the highest weight λ by restriction.

If $\mathscr{V}_\lambda(\mathsf{SL}_{d^2})$ is an obstruction for m, d, then we must have $|\lambda| = \sum_i \lambda_i = \ell d$ for some ℓ, see [3, Prop. 5.6.2]. We call the representation $\mathscr{V}_\lambda(\mathsf{SL}_{d^2})$ a *candidate for an obstruction* iff $\mathscr{V}_\lambda(\mathsf{SL}_{d^2})$ does not occur in $\mathbb{C}[\overline{\mathsf{GL}_{d^2} \cdot \det_d}]$. The following proposition relates the search for obstructions to the symmetric Kronecker coefficient.

Proposition 1. *Suppose that $|\lambda| = \ell d$ and write $\square = (\ell, \ldots, \ell)$ with ℓ occurring d times. Then $\mathscr{V}_\lambda(\mathsf{SL}_{d^2})$ is a candidate for an obstruction iff the symmetric Kronecker coefficient $sg_{\square}^{\lambda}$ vanishes.*

Proof. This is an immediate consequence of Prop. 4.4.1 and Prop. 5.2.1 in [3]. □

We may thus interpret this paper's main results by saying that candidates for obstructions are rare.

5. Proofs

5.1. Proof of Theorem 1

We know that $\mathsf{Kron}(d, d, d^2)$ is a rational polytope, i.e., defined by finitely many affine linear inequalities with rational coefficients. This easily implies that a rational point in $\overline{\mathsf{Kron}(d, d, d^2)}$ actually lies in $\mathsf{Kron}(d, d, d^2)$. Hence the second part of Theorem 1 follows from the first part.

The first part of Theorem 1 follows from the spectral characterization of $\overline{\mathsf{Kron}(d, d, d^2)}$ described in Section 2.3 and the following result.

Proposition 2. *For any decreasing probability distribution r on $[d^2]$ there exists a density operator $\rho_{AB} \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with spectrum r such that $\mathrm{tr}_A(\rho_{AB}) = \mathrm{tr}_B(\rho_{AB}) = \mathbf{u}_d$, where $\mathcal{H}_A \simeq \mathcal{H}_B \simeq \mathbb{C}^d$.*

The proof of Proposition 2 proceeds by different lemmas. It will be convenient to use the bra and ket notation of quantum mechanics. Suppose that $\mathcal{H}_A$ and $\mathcal{H}_B$ are d-dimensional Hilbert spaces. We recall first the *Schmidt decomposition*: for any $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, there exist orthonormal bases $\{|u_i\rangle\}$ of $\mathcal{H}_A$ and $\{|v_i\rangle\}$ of $\mathcal{H}_B$ as well as nonnegative real numbers α_i, called *Schmidt coefficients*, such that $|\psi\rangle = \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle$. Indeed, the α_i are just the singular values of $|\psi\rangle$ when we interpret it as a linear operator in $\mathcal{L}(\mathcal{H}_A^*, \mathcal{H}_B) \simeq \mathcal{H}_A \otimes \mathcal{H}_B$.

Lemma 4. *Suppose that $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ has the Schmidt coefficients α_i and consider $\rho := |\psi\rangle\langle\psi| \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Then $\mathrm{tr}_B(\rho) \in \mathcal{L}(\mathcal{H}_A)$, obtained by tracing over the B-spaces, has eigenvalues α_i^2.*

Proof. We have $|\psi\rangle = \sum_i \alpha_i |u_i\rangle \otimes |v_i\rangle$ for some orthonormal bases $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$ of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. This implies

$$\rho = |\psi\rangle\langle\psi| = \sum_{i,j} \alpha_i \alpha_j |u_i\rangle\langle u_j| \otimes |v_i\rangle\langle v_j|$$

and tracing over the B-spaces yields $\mathrm{tr}_B(|\psi\rangle\langle\psi|) = \sum_i \alpha_i^2 |u_i\rangle\langle u_i|$. □

Let $|0\rangle, \ldots, |d-1\rangle$ denote the standard orthonormal basis of $\mathbb{C}^d$. We consider the discrete Weyl operators $X, Z \in \mathcal{L}(\mathbb{C}^d)$ defined by

$$X|i\rangle = |i+1\rangle, \quad Z|i\rangle = \omega^i\,|i\rangle,$$

where ω denotes a primitive dth root of unity and the addition is modulo d (see for instance [6]). We note that X and Z are unitary matrices and $X^{-1}ZX = \omega Z$.

We consider now two copies $\mathcal{H}_A$ and $\mathcal{H}_B$ of $\mathbb{C}^d$ and define the "maximal entangled state" $|\psi_{00}\rangle := \frac{1}{\sqrt{d}} \sum_\ell |\ell\rangle|\ell\rangle$ of $\mathcal{H}_A \otimes \mathcal{H}_B$. By definition, $|\psi_{00}\rangle$ has the Schmidt coefficients $\frac{1}{\sqrt{d}}$. Hence the vectors

$$|\psi_{ij}\rangle := (\mathrm{id} \otimes X^i Z^j)|\psi_{00}\rangle,$$

obtained from $|\psi_{00}\rangle$ by applying a tensor product of unitary matrices, have the Schmidt coefficients $\frac{1}{\sqrt{d}}$ as well.

Lemma 5. *The vectors $|\psi_{ij}\rangle$, for $0 \le i, j < d$, form an orthonormal bases of $\mathcal{H}_A \otimes \mathcal{H}_B$.*

Proof. We have, for some dth root of unity θ,

$$\begin{aligned}\langle\psi_{ij}|\psi_{k\ell}\rangle &= \langle\psi_{00}|(\mathrm{id}\otimes Z^{-j}X^{-i})(\mathrm{id}\otimes X^kZ^\ell)\,|\psi_{00}\rangle \\ &= \theta\,\langle\psi_{00}|\mathrm{id}\otimes X^{k-i}Z^{\ell-j}|\psi_{00}\rangle \\ &= \frac{\theta}{d}\sum_{m,m'}\langle mm|\mathrm{id}\otimes X^{k-i}Z^{\ell-j}|m'm'\rangle \\ &= \frac{\theta}{d}\sum_m\langle m|X^{k-i}Z^{\ell-j}|m\rangle = \frac{\theta}{d}\,\mathrm{tr}\big(X^{k-i}Z^{\ell-j}\big).\end{aligned}$$

It is easy to check that $\frac{\theta}{d}\,\mathrm{tr}\big(X^{k-i}Z^{\ell-j}\big) = 0$ if $\ell\neq j$ or $k\neq i$. □

Proof of Proposition 2. Let r_{ij} be the given probability distribution assuming some bijection $[d^2]\simeq[d]^2$. According to Lemma 5, the density operator $\rho_{AB} := \sum_{ij} r_{ij}|\psi_{ij}\rangle\langle\psi_{ij}|$ has the eigenvalues r_{ij}. Lemma 4 tells us that $\mathrm{tr}_B(|\psi_{ij}\rangle\langle\psi_{ij}|)$ has the eigenvalues $1/d$, hence $\mathrm{tr}_B(|\psi_{ij}\rangle\langle\psi_{ij}|) = \mathbf{u}_d$. It follows that $\mathrm{tr}_B(\rho_{AB}) = \mathbf{u}_d$. Analogously, we get $\mathrm{tr}_A(\rho_{AB}) = \mathbf{u}_d$. □

5.2. Proof of Theorem 2

The proof is essentially the one of Theorem 2 in [5] carried out in the special case at hand. Suppose that $\lambda\vdash_{d^2}\ell d$. By Proposition 2 there is a density operator ρ_{AB} having the spectrum $\overline{\lambda}$ such that $\mathrm{tr}_A(\rho_{AB}) = \mathbf{u}_d$, $\mathrm{tr}_B(\rho_{AB}) = \mathbf{u}_d$. Let P_X denote the orthogonal projection of $(\mathcal{H}_A)^{\otimes k}$ onto the sum of its isotypical components $\mathscr{S}_\mu\otimes\mathscr{V}_\mu$ satisfying $\|\overline{\mu}-\mathbf{u}_d\|_1\leq\epsilon$. Then $P_{\overline{X}} := \mathrm{id} - P_X$ is the orthogonal projection of $(\mathcal{H}_A)^{\otimes k}$ onto the sum of its isotypical components $\mathscr{S}_\mu\otimes\mathscr{V}_\mu$ satisfying $\|\overline{\mu}-\mathbf{u}_d\|_1 > \epsilon$. The estimation theorem (3) implies that

$$\mathrm{tr}(P_{\overline{X}}\,(\rho_A)^{\otimes k}) \leq (k+1)^d(k+1)^{d(d-1)/2}\,e^{-\frac{k}{2}\epsilon^2} \leq (k+1)^{d(d+1)/2}\,e^{-\frac{k}{2}\epsilon^2},$$

since there are at most $(k+1)^d$ partitions of k of length at most d.

Let P_Y denote the orthogonal projection of $(\mathcal{H}_B)^{\otimes k}$ onto the sum of its isotypical components $\mathscr{S}_\nu\otimes\mathscr{V}_\nu$ satisfying $\|\overline{\nu}-u\|_1\leq\epsilon$, and let P_Z denote the orthogonal projection of $(\mathcal{H}_A\otimes\mathcal{H}_B)^{\otimes k}$ onto the sum of its isotypical components $\mathscr{S}_\Lambda\otimes\mathscr{V}_\Lambda$ satisfying $\|\overline{\Lambda}-\overline{\lambda}\|_1\leq\epsilon$. We set $P_{\overline{Y}} := \mathrm{id} - P_Y$ and $P_{\overline{Z}} := \mathrm{id} - P_Z$. Then we have, similarly as for P_X,

$$\begin{aligned}\mathrm{tr}(P_{\overline{Y}}\,(\rho_B)^{\otimes k}) &\leq (k+1)^{d(d+1)/2}\,e^{-\frac{k}{2}\epsilon^2}, \\ \mathrm{tr}(P_{\overline{Z}}\,(\rho_{AB})^{\otimes k}) &\leq (k+1)^{d^2(d^2+1)/2}\,e^{-\frac{k}{2}\epsilon^2}.\end{aligned}$$

By choosing $k = O(\frac{d^4}{\epsilon^2}\log\frac{d}{\epsilon})$ we can achieve that

$$\mathrm{tr}(P_{\overline{X}}(\rho_A)^{\otimes k}) < \frac{1}{3}, \quad \mathrm{tr}(P_{\overline{Y}}(\rho_B)^{\otimes k}) < \frac{1}{3}, \quad \mathrm{tr}(P_{\overline{Z}}(\rho_{AB})^{\otimes k}) < \frac{1}{3}.$$

We put $\sigma := (\rho_{AB})^{\otimes k}$ in order to simplify notation and claim that

$$\mathrm{tr}((P_X \otimes P_Y)\sigma P_Z) > 0. \tag{4}$$

In order to see this, we decompose $\mathrm{id} = P_X \otimes P_Y + P_{\overline{X}} \otimes \mathrm{id} + P_X \otimes P_{\overline{Y}}$. From the definition of the partial trace we have

$$\mathrm{tr}\big((P_{\overline{X}} \otimes \mathrm{id})\sigma\big) = \mathrm{tr}\big(P_{\overline{X}}(\rho_A)^{\otimes k}\big) < \frac{1}{3}.$$

Similarly,

$$\mathrm{tr}\big((P_X \otimes P_{\overline{Y}})\sigma\big) \le \mathrm{tr}\big((\mathrm{id} \otimes P_{\overline{Y}})\sigma\big) = \mathrm{tr}\big(P_{\overline{Y}}(\rho_B)^{\otimes k}\big) < \frac{1}{3}.$$

Hence $\mathrm{tr}\big((P_X \otimes P_Y)\sigma\big) > \frac{1}{3}$. Using $\mathrm{tr}\big((P_X \otimes P_Y)\sigma P_{\overline{Z}}\big) \le \mathrm{tr}(\sigma P_{\overline{Z}}) < \frac{1}{3}$, we get

$$\mathrm{tr}\big((P_X \otimes P_Y)\sigma P_Z\big) = \mathrm{tr}\big((P_X \otimes P_Y)\sigma\big) - \mathrm{tr}\big((P_X \otimes P_Y)\sigma P_{\overline{Z}}\big) > \frac{1}{3} - \frac{1}{3} = 0,$$

which proves Claim (4).

Claim (4) implies that there exist partitions μ, ν, Λ with normalizations ϵ-close to $\mathbf{u}_d, \mathbf{u}_d, r$, respectively, such that $(P_\mu \otimes P_\nu)P_\Lambda \neq 0$. Recalling the isotypical decomposition (2), we infer that

$$(\mathscr{S}_\Lambda \otimes \mathscr{V}_\Lambda) \cap (\mathscr{S}_\mu \otimes \mathscr{V}_\mu) \otimes (\mathscr{S}_\nu \otimes \mathscr{V}_\nu) \neq 0.$$

Statement (1) implies that $g_{\mu,\nu,\Lambda} \neq 0$ and hence the assertion follows for $R_1 = \mu, R_2 = \nu$. □

5.3. Proof of Lemma 3

We assume that $\lambda, \mu \vdash_d n$. The group $\mathsf{GL}_d \times \mathsf{GL}_d \times \mathsf{GL}_d$ operates on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ by tensor product, which induces an action on the polynomial ring A on $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. Schur-Weyl duality implies that the submodule A_n of homogeneous polynomials of degree n splits as follows (cf. [13]):

$$A_n = \bigoplus_{\lambda,\mu,\nu \vdash_d n} \big(\mathscr{S}_\lambda \otimes \mathscr{S}_\mu \otimes \mathscr{S}_\nu\big)^{S_n} \otimes \mathscr{V}_\lambda^* \otimes \mathscr{V}_\mu^* \otimes \mathscr{V}_\nu^*.$$

We assume now that $g_{\lambda,\mu,\mu} = \dim \left(\mathscr{S}_\lambda \otimes \mathscr{S}_\mu \otimes \mathscr{S}_\mu\right)^{S_n} \neq 0$ for some $\lambda, \mu \vdash_d n$. Hence there exists a highest weight vector $F \in A_n$ of weight (λ, μ, μ). We may assume that the coefficients of F are real (cf. [1]).

Consider the linear automorphism that exchanges the last two factors of $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$. This induces an automorphism σ of the algebra A. It is easy to see that $F' := \sigma(F)$ is a highest weight vector of weight (λ, μ, μ). Therefore, both squares F^2 and $(F')^2$ are highest weight vectors of weight $(2\lambda, 2\mu, 2\mu)$. Since $F^2 + (F')^2$ is nonzero and invariant under σ, we see that $\left(\mathscr{S}_{2\lambda} \otimes \mathscr{S}_{2\mu} \otimes \mathscr{S}_{2\mu}\right)^{S_n}$ has a nonzero invariant with respect to σ. Hence

$$\left(\mathscr{S}_{2\lambda} \otimes \mathrm{Sym}^2(\mathscr{S}_{2\mu})\right)^{S_n} \neq 0,$$

which means that $sg_{2\mu}^{2\lambda} \neq 0$. □

References

[1] Bürgisser, P., Christandl, M., Ikenmeyer, C., 2011. Even partitions in plethysms. Journal of Algebra 328, 322–329.

[2] Bürgisser, P., Clausen, M., Shokrollahi, M.A., 1997. Algebraic Complexity Theory. Volume 315 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin. With the collaboration of Thomas Lickteig.

[3] Bürgisser, P., Landsberg, J., Manivel, L., Weyman, J., 2009. An overview of mathematical issues arising in the geometric complexity theory approach to VP≠VNP. arXiv:0907.2850v2.

[4] Christandl, M., Harrow, A., Mitchison, G., 2007. On nonzero Kronecker coefficients and what they tell us about spectra. Comm. Math. Phys. 270, 575–585.

[5] Christandl, M., Mitchison, G., 2006. The spectra of density operators and the Kronecker coefficients of the symmetric group. Comm. Math. Phys. 261, 789–797.

[6] Christandl, M., Winter, A., 2005. Uncertainty, monogamy, and locking of quantum correlations. IEEE Trans.Inf. Theory 51, 3159–3165.

[7] Franz, M., 2002. Moment polytopes of projective G-varieties and tensor products of symmetric group representations. J. Lie Theory 12, 539–549.

[8] Fulton, W., Harris, J., 1991. Representation theory. Volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[9] Keyl, M., Werner, R., 2001. Estimating the spectrum of a density operator. Phys. Rev. A 64, 052311.

[10] Klyachko, A.A., 1998. Stable bundles, representation theory and Hermitian operators. Selecta Math. (N.S.) 4, 419–445.

[11] Klyachko, A., 2004. Quantum marginal problem and representations of the symmetric group. arXiv:quant-ph/0409113v1.

[12] Knutson, A., Tao, T., 1999. The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture. J. Amer. Math. Soc. 12, 1055–1090.

[13] Landsberg, J.M., Manivel, L., 2004. On the ideals of secant varieties of Segre varieties. Found. Comput. Math. 4, 397–422.

[14] Manivel, L., 1997. Applications de Gauss et pléthysme. Ann. Inst. Fourier (Grenoble) 47, 715–773.

[15] Mulmuley, K.D., Sohoni, M., 2001. Geometric complexity theory. I. An approach to the P vs. NP and related problems. SIAM J. Comput. 31, 496–526.

[16] Mulmuley, K.D., Sohoni, M., 2008. Geometric complexity theory. II. Towards explicit obstructions for embeddings among class varieties. SIAM J. Comput. 38, 1175–1206.

[17] Stanley, R.P., 1999. Enumerative combinatorics. Vol. 2. Volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge.

[18] Stanley, R.P., 2000. Positivity problems and conjectures in algebraic combinatorics, in: Mathematics: frontiers and perspectives. Amer. Math. Soc., Providence, RI, pp. 295–319.

[19] Valiant, L.G., 1979. Completeness classes in algebra, in: Conference Record of the Eleventh Annual ACM Symposium on Theory of Computing (Atlanta, Ga., 1979). ACM, New York, pp. 249–261.

[20] Valiant, L.G., 1982. Reducibility by algebraic projections. Enseign. Math. (2) 28, 253–268.