Exotic quantifiers, complexity classes, and complete problems

(Extended Abstract)

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Abstract. We define new complexity classes in the Blum-Shub-Smale theory of computation over the reals, in the spirit of the polynomial hierarchy, with the help of infinitesimal and generic quantifiers. Basic topological properties of semialgebraic sets like boundedness, closedness, compactness, as well as the continuity of semialgebraic functions are shown to be complete in these new classes. All attempts to classify the complexity of these problems in terms of the previously studied complexity classes have failed. We also obtain completeness results in the Turing model for the corresponding discrete problems. In this setting, it turns out that infinitesimal and generic quantifiers can be eliminated, so that the relevant complexity classes can be described in terms of usual quantifiers only.

1 Introduction

The complexity theory over the real numbers introduced by L. Blum, M. Shub, and S. Smale developed quickly after their foundational paper [4]. Complexity classes other than $\mathrm{P}_{\mathbb{R}}$ and $\mathrm{NP}_{\mathbb{R}}$ were introduced (e.g., in [8, 17, 11]), completeness results were proven (e.g., in [8, 17, 22]), separations were obtained ([10, 16]), machine-independent characterizations of complexity classes were exhibited ([6, 14, 18]).

There are two points in this development which we would like to stress. Firstly, all the considered complexity classes were natural versions over the real numbers of existing complexity classes in the classical setting. Secondly, the catalogue of completeness results is disapointingly small. For a given semialgebraic set $S \subseteq \mathbb{R}^n$, deciding whether a point in $\mathbb{R}^n$ belongs to S is $\mathrm{P}_{\mathbb{R}}$-complete [17], deciding whether S is non-empty (or non-convex, or of dimension at least d for a given $d \in \mathbb{N}$) is $\mathrm{NP}_{\mathbb{R}}$-complete [4, 15, 22], and computing its Euler-Yao characteristic is $\mathrm{FP}_{\mathbb{R}}^{\#\mathrm{P}_{\mathbb{R}}}$-complete [8]. That is, essentially, all.

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Yet, there are plenty of natural problems involving semialgebraic sets: computing local dimensions, deciding denseness, closedness, unboundedness, etc. Consider, for instance, the latter. We can express that S is unbounded by

$$\forall K \in \mathbb{R}\, \exists x \in \mathbb{R}^n\, (x \in S \wedge \|x\| \geq K). \tag{1}$$

Properties describable with expressions like this one are common in classical complexity theory and in recursive function theory. Extending an idea by Kleene [19] for the latter, Stockmeyer introduced in [24] the polynomial time hierarchy which is built on top of NP and coNP in a natural way.[1] Recall, a set S is in NP when there is a polynomial time decidable relation R such that, for every $x \in \{0,1\}^*$,

$$x \in S \iff \exists y \in \{0,1\}^{\mathrm{size}(x)^{\mathcal{O}(1)}}\, R(x,y).$$

The class coNP is defined replacing $\exists$ by $\forall$. Classes in the polynomial hierarchy are then defined by allowing the quantifiers $\exists$ and $\forall$ to alternate (with a bounded number of alternations). If there are k alternations of quantifiers, we obtain the classes $\mathsf{\Sigma}^{k+1}$ (if the first quantifier is $\exists$) and $\mathsf{\Pi}^{k+1}$ (if the first quantifier is $\forall$). Note that $\mathsf{\Sigma}^1 = \mathsf{NP}$ and $\mathsf{\Pi}^1 = \mathsf{coNP}$. The definition of these classes over $\mathbb{R}$ is straightforward [3, Ch. 21].

It follows thus from (1) that deciding unboundedness is in $\Pi^2_{\mathbb{R}}$, the universal second level of the polynomial hierarchy over $\mathbb{R}$. On the other hand, it is easy to prove that this problem is $\mathrm{NP}_{\mathbb{R}}$-hard. But we do not have completeness for any of these two classes.

A similar situation appears for deciding denseness. We can express that $S \subseteq \mathbb{R}^n$ is Euclidean dense by

$$\forall x \in \mathbb{R}^n\, \forall \varepsilon > 0\, \exists y \in \mathbb{R}^n\, (y \in S \wedge \|x - y\| \leq \varepsilon)$$

thus showing that this problem is in $\Pi^2_{\mathbb{R}}$. But we can not prove hardness in this class. Actually, we can not even manage to prove $\mathrm{NP}_{\mathbb{R}}$-hardness or $\mathrm{coNP}_{\mathbb{R}}$-hardness. Yet a similar situation occurs with closedness, which is in $\Pi^3_{\mathbb{R}}$ since we express that S is closed by

$$\forall x \in \mathbb{R}^n\, \exists \varepsilon > 0\, \forall y \in \mathbb{R}^n\, (x \notin S\ \wedge\ \|x - y\| \leq \varepsilon \Rightarrow y \notin S)$$

but the best hardness result we can prove is $\mathrm{coNP}_{\mathbb{R}}$-hardness. It would seem that the landscape of complexity classes between $\mathrm{P}_{\mathbb{R}}$ and the third level of the polynomial hierarchy is not enough to capture the complexity of the problems above.

A main goal of this paper is to show that the two features we pointed out earlier namely, a theory uniquely based upon real versions of classical complexity classes, and a certain scarsity of completeness results, are not unrelated. With the help of infinitesimal and generic quantifiers we shall define complexity classes

[1] All along this paper we use a subscript $\mathbb{R}$ to differentiate complexity classes over $\mathbb{R}$ from discrete complexity classes. To further emphasize this difference, we use sans serif to denote the latter.

lying in between the different levels of the polynomial hierarchy. These new classes will allow us to determine the complexity of some of the problems we mentioned (and of others we didn't mention) or, in some cases, to decrease the gap between their lower and upper complexity bounds as we know them today.

A remarkable feature of these classes is that, as with the classes in the polynomial hierarchy, they are defined using quantifiers which act as operators on complexity classes. The properties of these operators naturally become an object of study for us. Thus, another goal of this paper is to provide some structural results for these operators.

We remark that a similar classification has already been achieved in the so called additive BSS model, without the need to introduce exotic quantifiers [7, 9].

2 Preliminaries

We assume some basic knowledge on real machines and complexity as presented, for instance, in [3, 4].

An *algebraic circuit* $\mathscr{C}$ over $\mathbb{R}$ is an acyclic directed graph where each node has indegree 0, 1 or 2. Nodes with indegree 0 are either labeled as *input nodes* or with elements of $\mathbb{R}$ (we shall call them *constant nodes*). Nodes with indegree 2 are labeled with the binary operators of $\mathbb{R}$, i.e., one of $\{+, \times, -, /\}$. They are called *arithmetic nodes*. Nodes with indegree 1 are either *sign nodes* or *output nodes*. All the output nodes have outdegree 0. Otherwise, there is no upper bound for the outdegree of the other kinds of nodes. For an algebraic circuit $\mathscr{C}$, the *size* of $\mathscr{C}$, is the number of nodes in $\mathscr{C}$. The *depth* of $\mathscr{C}$, is the length of the longest path from some input node to some output node.

An arithmetic node computes a function of its input values in an obvious manner. Sign nodes compute the function sgn defined by $\text{sgn}(x) = 1$ if $x \geq 0$ and $\text{sgn}(x) = 0$ otherwise. To a circuit $\mathscr{C}$ with n input gates and m output gates is associated a function $f_{\mathscr{C}} : \mathbb{R}^n \to \mathbb{R}^m$. This function may not be total since divisions by zero may occur (in which case, by convention, $f_{\mathscr{C}}$ is not defined on its input). We say that an algebraic circuit is a *decision* circuit if it has only one output gate whose parent is a sign gate. Thus, a decision circuit $\mathscr{C}$ with n input gates computes a function $f_{\mathscr{C}} : \mathbb{R}^n \to \{0, 1\}$. The set *decided* by the circuit is

$$S_{\mathscr{C}} = \{x \in \mathbb{R}^n \mid f_{\mathscr{C}}(x) = 1\}.$$

Subsets of $\mathbb{R}^n$ decidable by algebraic circuits are known as *semialgebraic sets*. They are defined as those sets which can be written as a Boolean combination of solution sets of polynomial inequalities $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$.

Semialgebraic sets will be inputs to problems considered in this paper. They will be either given by a Boolean combination of polynomial equalities and inequalities or by a decision circuit. If not otherwise specified, we mean the first variant. In this case, polynomials are encoded with the so called *dense encoding*, i.e., they are represented by the complete list of their coefficients (including zero coefficients).

We close this section by recalling a completeness result, which will play an important role in our developments. For $d \in \mathbb{N}$ let $\textsc{Dim}_{\mathbb{R}}(d)$ be the problem of, given a semialgebraic set S, deciding whether $\dim S \geq d$. In [22] Koiran proved that $\textsc{Dim}_{\mathbb{R}}$ is $\mathrm{NP}_{\mathbb{R}}$-complete.

2.1 Infinitesimal and generic quantifiers

We are going to define three logical quantifiers in the theory of the reals. Suppose $\varphi(\varepsilon)$ is a formula with one free variable ε. The expression $\mathsf{H}\varepsilon\varphi(\varepsilon)$ shall express that $\varphi(\varepsilon)$ holds *for sufficiently small real* $\varepsilon > 0$, that is,

$$\mathsf{H}\varepsilon\,\varphi(\varepsilon) \stackrel{\text{def}}{\equiv} \exists\mu > 0\;\forall\varepsilon \in (0,\mu)\;\varphi(\varepsilon). \tag{2}$$

Suppose that $\psi(x)$ is a formula with n free variables $x_1, \ldots, x_n$. We shall write $\forall^* x\,\psi(x)$ in order to express that *almost all* $x \in \mathbb{R}^n$ (with respect to the *Euclidean topology*) satisfy $\psi(x)$. Explicitly,

$$\forall^* x\,\psi(x) \stackrel{\text{def}}{\equiv} \forall x_0\,\forall\varepsilon > 0\,\exists x\,(\|x - x_0\| < \varepsilon \,\wedge\, \psi(x)). \tag{3}$$

If we put $S_\psi = \{x \in \mathbb{R}^n \mid \psi(x) \text{ holds}\}$ this is equivalent to $\dim(\mathbb{R}^n - S_\psi) < n$, as S_ψ is semialgebraic, cf. [5]. Furthermore, we shall write $\exists^* x\,\psi(x)$ to express that *almost all* $x \in \mathbb{R}^n$ (with respect to the *Zariski topology*) satisfy $\psi(x)$. This is the case iff $\dim S_\psi = n$, which is in turn equivalent to

$$\exists^* x\,\psi(x) \stackrel{\text{def}}{\equiv} \exists x_0\,\exists\varepsilon > 0\,\forall x\,(\|x - x_0\| < \varepsilon \Rightarrow \psi(x)), \tag{4}$$

which expresses that S_ψ contains an open ball. (For a proof of this equivalence see [5].) The generic quantifiers $\forall^*$ and $\exists^*$ were previously introduced by Koiran [22], while the infinitesimal quantifier H so far hasn't been studied in a complexity framework.

By definition, $\exists^*\psi(x)$ is equivalent to $\neg(\forall^*\neg\psi(x))$. By contrast, it is easy to see that the quantifier H allows to pull in negations: $\neg\mathsf{H}\varepsilon\varphi(\varepsilon)$ is equivalent to $\mathsf{H}\varepsilon\neg\varphi(\varepsilon)$.

We are next going to interpret the new quantifiers as operators acting on complexity classes. We denote by $\mathbb{R}^\infty$ the disjoint union $\sqcup_{n\geq 0}\mathbb{R}^n$. If $x \in \mathbb{R}^n \subset \mathbb{R}^\infty$ we define its *size* to be $|x| = n$.

Definition 2.1. Let $\mathcal{C}$ be a complexity class of decision problems.

1. The class $\mathsf{H}\mathcal{C}$ consists of the $A \subseteq \mathbb{R}^\infty$ such that there exists $B \subseteq \mathbb{R} \times \mathbb{R}^\infty$, $B \in \mathcal{C}$, such that, for all $x \in \mathbb{R}^\infty$,
$$x \in A \iff \mathsf{H}\varepsilon\,(\varepsilon, x) \in B.$$
2. Let Q be one of the quantifiers $\forall, \forall^*, \exists, \exists^*$. The class $Q\,\mathcal{C}$ consists of the $A \subseteq \mathbb{R}^\infty$ such that there exists a polynomial p and $B \subseteq \mathbb{R}^\infty \times \mathbb{R}^\infty$, $B \in \mathcal{C}$, such that, for all $x \in \mathbb{R}^\infty$,
$$x \in A \iff Qz \in \mathbb{R}^{p(|x|)}\,(z, x) \in B.$$

By repeatedly applying these operators to $\mathrm{P}_{\mathbb{R}}$ we may define many new complexity classes, which can be seen as a refinement of the polynomial hierarchy over the reals. These classes somehow take into account the *topology* of $\mathbb{R}$, an aspect completely absent in the discrete setting.

In order to simplify notation we will omit $\mathrm{P}_{\mathbb{R}}$ and write simply $\mathrm{NP}_{\mathbb{R}} = \exists\,\mathrm{P}_{\mathbb{R}} = \exists$, $\mathrm{coNP}_{\mathbb{R}} = \forall\,\mathrm{P}_{\mathbb{R}} = \forall$ etc. We call the classes defined this way *polynomial classes*. It is easy to see that they are closed under many-one reductions. Completeness shall always refer to such reductions.

2.2 Standard complete problems

Let $\textsc{Standard}(\mathsf{H}\exists)$ be the problem of deciding, given a polynomial f in $n+1$ variables (in dense encoding), whether

$$\mathsf{H}\varepsilon\,\exists x \in \mathbb{R}^n\, f(\varepsilon, x) = 0.$$

The problem $\textsc{Standard}(\mathsf{H}\forall)$ is analogously defined by requiring $f(\varepsilon, x) \neq 0$ instead. The usual proof of $\mathrm{NP}_{\mathbb{R}}$-completeness of the real feasibility problem [3, 4] yields:

Proposition 2.2. $\textsc{Standard}(\mathsf{H}\exists)$ *is* $\mathsf{H}\exists$*-complete and* $\textsc{Standard}(\mathsf{H}\forall)$ *is* $\mathsf{H}\forall$*-complete.*

We remark that any polynomial class can be shown to have a standard complete problem.

3 Natural problems complete for $\mathsf{H}\exists$ and $\mathsf{H}\forall$

Consider the following problems

$\textsc{Unbounded}_{\mathbb{R}}$ (*Unboundedness*) Given a semialgebraic set S, is it unbounded?

$\textsc{EAdh}_{\mathbb{R}}$ (*Euclidean Adherence*) Given a semialgebraic set S and a point x, decide whether x belongs to the Euclidean closure $\overline{S}$ of S.

$\textsc{LocDim}_{\mathbb{R}}$ (*Local Dimension*) Given a semialgebraic set $S \subseteq \mathbb{R}^n$, a point $x \in S$, and $d \in \mathbb{N}$, is $\dim_x S \geq d$?

Proposition 3.1. $\textsc{Unbounded}_{\mathbb{R}}$*,* $\textsc{EAdh}_{\mathbb{R}}$*, and* $\textsc{LocDim}_{\mathbb{R}}$ *are* $\mathsf{H}\exists$*-complete.*

Proof. A set S is unbounded if and only if

$$\mathsf{H}\varepsilon\,\exists x \in \mathbb{R}^n\,(\varepsilon\|x\| \geq 1 \;\wedge\; x \in S).$$

This shows $\textsc{Unbounded}_{\mathbb{R}} \in \mathsf{H}\exists$. In a similar way one sees that $\textsc{EAdh}_{\mathbb{R}} \in \mathsf{H}\exists$.

Let $B(x, \varepsilon)$ denote the open ε-ball centered at x. From the equivalence

$$\dim_x S \geq d \iff \mathsf{H}\varepsilon\,\dim(S \cap B(x,\varepsilon)) \geq d$$

and the fact [22] that $\text{DIM}_{\mathbb{R}} \in \text{NP}_{\mathbb{R}}$ we conclude $\text{LOCDIM}_{\mathbb{R}} \in \mathsf{H}\exists$.

For showing hardness, consider the auxiliary problem $\mathcal{L} \subseteq \mathbb{R}^\infty$ consisting of, given $g \in \mathbb{R}[\varepsilon, X_1, \dots, X_n]$, deciding whether

$$\mathsf{H}\varepsilon\, \exists t \in (-1,1)^n \; g(\varepsilon, t_1, \dots, t_n) = 0.$$

We first reduce $\text{STANDARD}(\mathsf{H}\exists)$ to $\mathcal{L}$, which will show that $\mathcal{L}$ is $\mathsf{H}\exists$-complete, cf. Proposition 2.2. To do so, note that the existence of a root in $\mathbb{R}^n$ of a polynomial f is equivalent to the existence of a root in the open unit cube $(-1,1)^n$ for a suitable other polynomial. This is so since the mapping $\psi(\lambda) = \frac{\lambda}{1-\lambda^2}$ bijects $(-1,1)$ with $\mathbb{R}$. Therefore, for $f \in \mathbb{R}[Y, X_1, \dots, X_n]$,

$$\mathsf{H}\varepsilon\, \exists x \in \mathbb{R}^n \; f(\varepsilon, x_1, \dots, x_n) = 0 \iff \mathsf{H}\varepsilon\, \exists t \in (-1,1)^n \; g(\varepsilon, t_1, \dots, t_n) = 0,$$

where $d_i = \deg_{x_i} f$ and $g \in \mathbb{R}[Y, T_1, \dots, T_n]$ is given by

$$g(\varepsilon, t_1, \dots, t_n) := (1-t_1^2)^{d_1}(1-t_2^2)^{d_2} \cdots (1-t_n^2)^{d_n} f(\varepsilon, \psi(t_1), \dots, \psi(t_n)).$$

Note that we can construct g in time polynomial in the size of f. (As we are representing f and g in the dense encoding, the divisions can be eliminated in polynomial time.) So the mapping $f \mapsto g$ indeed reduces $\text{STANDARD}(\mathsf{H}\exists)$ to $\mathcal{L}$.

In order to reduce $\mathcal{L}$ to $\text{UNBOUNDED}_{\mathbb{R}}$ we associate to $g \in \mathbb{R}[Y, T_1, \dots, T_n]$ the semialgebraic set $S := \{(y,t) \in \mathbb{R} \times (-1,1)^n \mid h(y,t) = 0\}$, where h is the polynomial defined by $h(Y,T) = Y^{2\deg_Y g} g(1/Y^2, T)$. Then $g \in \mathcal{L}$ if and only if S is unbounded. This proves that $\text{UNBOUNDED}_{\mathbb{R}}$ is $\mathsf{H}\exists$-complete.

We reduce now $\text{UNBOUNDED}_{\mathbb{R}}$ to $\text{EADH}_{\mathbb{R}}$. To a polynomial f of degree d in n variables we assign $f' := \|X\|^{2d} f(\|X\|^{-2}X)$. Let $S \subseteq \mathbb{R}^n$ be a semialgebraic set given by a Boolean combination of inequalities of the form $f(x) > 0$. Without loss of generality, $0 \notin S$. The set defined by the same Boolean combination of the inequalities $f'(x) > 0$ and the condition $x \neq 0$ is the image of S under the inversion map $i\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}, x \mapsto \|x\|^{-2}x$. Hence S is unbounded if and only if 0 belongs to the closure of $i(S) \setminus \{0\}$.

Finally, it is easy to reduce $\text{EADH}_{\mathbb{R}}$ to $\text{LOCDIM}_{\mathbb{R}}$. For given $S \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$ put take $S' = \mathbb{R}^n$ if $x \in S$. Else, put $S' = S \cup \{x\}$. Then $x \in \overline{S}$ iff $\dim_x S' \geq 1$. □

A *basic semialgebraic set* is the solution set $S \subseteq \mathbb{R}^n$ of a system of polynomial equalities and inequalities of the form

$$f = 0, h_1 \geq 0, \dots, h_p \geq 0, g_1 > 0, \dots, g_q > 0. \tag{5}$$

Consider the following problems:

$\text{BASICCLOSED}_{\mathbb{R}}$ (*Closedness for basic semialgebraic sets*) Given a basic semialgebraic set S, is it closed?

$\text{BASICCOMPACT}_{\mathbb{R}}$ (*Compactness for basic semialgebraic sets*) Given a basic semialgebraic set S, is it compact?

Theorem 3.2. $\text{BASICCLOSED}_{\mathbb{R}}$ *and* $\text{BASICCOMPACT}_{\mathbb{R}}$ *are* $\mathsf{H}\forall$*-complete.*

The proof needs some preparation. For a basic semialgebraic set S given as in (5) define for $\varepsilon > 0$

$$S_\varepsilon = \{f = 0, h_1 \geq 0, \ldots, h_p \geq 0, g_1 \geq \varepsilon, \ldots, g_q \geq \varepsilon\}.$$

Note that $S_\varepsilon \subseteq S_{\varepsilon'} \subseteq S$ for $0 < \varepsilon' < \varepsilon$ and that $S = \cup_{\varepsilon > 0} S_\varepsilon$.

Lemma 3.3. *Suppose that $K^S := \{f = 0, h_1 \geq 0, \ldots, h_p \geq 0\}$ is bounded. Then S is closed iff $S_\varepsilon = S$ for sufficiently small $\varepsilon > 0$.*

The condition $\mathsf{H}\varepsilon\,(S_\varepsilon = S)$ is testable in $\mathsf{H}\forall$. For showing membership of $\text{BasicClosed}_{\mathbb{R}}$ to $\mathsf{H}\forall$ it is therefore sufficient to reduce the general situation to the one with bounded K^S.

Let $\mathbb{S}^n$ denote the n-dimensional unit sphere and $\mathcal{N} = (0, \ldots, 0, 1)$. The *stereographic projection* $\pi : \mathbb{S}^n - \{\mathcal{N}\} \to \mathbb{R}^n, (x, t) \mapsto \frac{1}{1-t} x$ is a homeomorphism. Consider $\tilde{S} := \pi^{-1}(S) \cup \{\mathcal{N}\}$ Then, $\tilde{S}$ is a basic semialgebraic set such that $K^{\tilde{S}}$ is bounded. Moreover, S is closed in $\mathbb{R}^n$ iff $\tilde{S}$ is closed in $\mathbb{R}^{n+1}$. This shows membership of $\text{BasicClosed}_{\mathbb{R}}$ to $\mathsf{H}\forall$. The claimed membership of $\text{BasicCompact}_{\mathbb{R}}$ follows now by using $\text{Unbounded}_{\mathbb{R}} \in \mathsf{H}\exists$.

The proof of $\mathsf{H}\forall$-hardness is based on the following lemma.

Lemma 3.4. *There exists a constant $c > 0$ with the following property. To $f \in \mathbb{R}[\varepsilon, X_1, \ldots, X_n]$ of degree d and $N = (nd)^{cn}$ we assign the semialgebraic set*

$$S := \left\{ (\varepsilon, x, y) \in (0, \infty) \times (-1, 1)^n \times \mathbb{R} \mid f(\varepsilon, x) = 0 \ \wedge \ y \prod_{k=1}^{n} (1 - x_k^2) = \varepsilon^N \right\}.$$

Then for all f we have

$$\mathsf{H}\varepsilon\, \forall x \in (-1, 1)^n \ f(\varepsilon, x) \neq 0 \Longleftrightarrow S \textit{ is closed in } \mathbb{R}^{n+2}.$$

The proof of this lemma uses efficient quantifier elimination over $\mathbb{R}$, cf. [23, Part III], and the following auxiliary result, whose proof is based on the description of the half-branches of real algebraic curves by means of Puiseux series, cf. [2, §13].

Lemma 3.5. *Let $T \subseteq (0, \infty) \times (0, \infty)$ be a semialgebraic set given by a Boolean combination of inequalities of polynomials of degree strictly less than d and let $(0, 0) \in \overline{T}$. Then there exists a sequence of points (t_ν, ε_ν) in T such that*

$$\lim_{\nu \to \infty} \frac{\varepsilon_\nu^d}{t_\nu} = 0.$$

Proof of Theorem 3.2. It suffices to prove $\mathsf{H}\forall$-hardness. Lemma 3.4 (plus the reduction in the proof of Proposition 3.1 to allow the variables x_i to vary in $\mathbb{R}$) allows us to reduce $\text{Standard}(\mathsf{H}\forall)$ to $\text{BasicClosed}_{\mathbb{R}}$. Indeed, a description of the set S in its statement can be obtained in polynomial time from a description of f. However, the exponent N is exponential in the size of f. In order to reduce the degree N we introduce the variables $z_1, \ldots, z_{\log N}$ (assuming N is a power of 2) and replace $y \prod_{k=1}^{n}(1-x_k^2) = \varepsilon^N$ by the equalities

$$z_1 = \varepsilon^2, \quad z_j = z_{j-1}^2 \ (j = 2, \ldots, \log N), \quad y \prod_{k=1}^{n}(1-x_k^2) = z_{\log N}.$$

This defines a basic semialgebraic set S' homeomorphic to S whose size in dense encoding is polynomial in the size of f. This completes the proof for $\textsc{BasicClosed}_{\mathbb{R}}$. Hardness of $\textsc{BasicCompact}_{\mathbb{R}}$ follows as before by means of the stereographic projection. □

Problem 3.6. Can Theorem 3.2 be extended to arbitrary semialgebraic sets? We note that the three problems of deciding, for an arbitrary semialgebraic set S, whether S is compact, whether it is open, or whether it is closed are polynomial time equivalent.

Complexity results for problems involving functions instead of sets are also of interest. Consider the following problems:

$\textsc{Cont}_{\mathbb{R}}$ (*Continuity*) Given a circuit $\mathscr{C}$, decide whether $f_{\mathscr{C}}$ is total and continuous.

$\textsc{Cont}_{\mathbb{R}}^{\mathrm{DF}}$ (*Continuity for Division-Free Circuits*) Given a division-free circuit $\mathscr{C}$, decide whether $f_{\mathscr{C}}$ is continuous.

$\textsc{ContPoint}_{\mathbb{R}}^{\mathrm{DF}}$ (*Continuity at a Point for Division-Free Circuits*) Given a division-free circuit $\mathscr{C}$ with n input gates and a point $x \in \mathbb{R}^n$, decide whether $f_{\mathscr{C}}$ is continuous at x.

Theorem 3.7. $\textsc{ContPoint}_{\mathbb{R}}^{\mathrm{DF}}$ *is* $\mathsf{H}\forall$*-complete. Moreover,* $\textsc{Cont}_{\mathbb{R}}^{\mathrm{DF}} \in \mathsf{H}^2\forall$ *and* $\textsc{Cont}_{\mathbb{R}} \in \mathsf{H}^3\forall$ *and both problems are* $\forall$*-hard.*

4 Quantifying genericity

It is customary to express denseness in terms of adherence. For instance, a subset $S \subseteq \mathbb{R}^n$ is Euclidean dense in $\mathbb{R}^n$ iff $\forall x \in \mathbb{R}^n\ (x, S) \in \textsc{EAdh}_{\mathbb{R}}$. We formally define $\textsc{EDense}_{\mathbb{R}}$ as follows:

$\textsc{EDense}_{\mathbb{R}}$ (*Euclidean Denseness*) Given a decision circuit $\mathscr{C}$ with n input gates, decide whether $\overline{S_{\mathscr{C}}} = \mathbb{R}^n$.

Therefore, one would expect at least $\mathrm{NP}_{\mathbb{R}}$-hardness (if not $\Pi_{\mathbb{R}}^2$-completeness) for $\textsc{EDense}_{\mathbb{R}}$. The situation is quite different, however. Let the problem $\textsc{ZDense}_{\mathbb{R}}$ be the counterpart of $\textsc{EDense}_{\mathbb{R}}$ for the Zariski topology.

Proposition 4.1. $\textsc{EDense}_{\mathbb{R}}$ *is* $\forall^*$*-complete and* $\textsc{ZDense}_{\mathbb{R}}$ *is* $\exists^*$*-complete.*

The following result locates $\exists^*$ and $\forall^*$ with respect to the previously studied complexity classes.

Proposition 4.2. *We have* $\exists^* \subseteq \exists \subseteq \mathsf{H}^2\exists^*$ *and* $\forall^* \subseteq \forall \subseteq \mathsf{H}^2\forall^*$.

PROOF. The proof of the inclusion $\exists^* \subseteq \exists$ relies on a technique by Koiran [22] developed for showing that $\mathrm{DIM}(d)$ is in $\mathrm{NP}_{\mathbb{R}}$. Using this technique, one may in fact show the following general inclusion for any polynomial complexity class $\mathcal{C}$

$$\exists^*\mathcal{C} \subseteq \exists\mathcal{C} \text{ and } \forall^*\mathcal{C} \subseteq \forall\mathcal{C}. \tag{6}$$

In order to show that $\exists \subseteq \mathsf{H}^2\exists^*$ note that for $f \in \mathbb{R}[X_1, \ldots .X_n]$ we have

$$\begin{aligned}
\exists x\, f(x) = 0 &\iff \mathsf{H}\delta\, \exists x\, \big(\|x\|^2 \le \delta^{-1} \ \wedge\ f(x) = 0\big) \\
&\iff \mathsf{H}\delta\, \mathsf{H}\varepsilon\, \exists x\, \big(\|x\|^2 \le \delta^{-1} \ \wedge\ f(x)^2 < \varepsilon\big) \\
&\iff \mathsf{H}\delta\, \mathsf{H}\varepsilon\, \exists^* x\, \big(\|x\|^2 < \delta^{-1} \ \wedge\ f(x)^2 < \varepsilon\big)
\end{aligned}$$

the second equivalence by the compactness of closed balls. □

5 Discrete setting

We discuss here the relationship between polynomial classes and classical complexity theory. Thus we restrict the input polynomials in the problems considered so far to polynomials with integer coefficients (represented in binary), or to constant-free circuits (i.e., circuits which use only 0 and 1 as values associated to their constant nodes). The resulting problems can be encoded in a finite alphabet and studied in the classical Turing setting. In general, if L denotes a problem defined over $\mathbb{R}$ or $\mathbb{C}$, we denote its restriction to integer inputs by $L^{\mathbb{Z}}$. This way, the discrete problems $\text{UNBOUNDED}_{\mathbb{R}}^{\mathbb{Z}}$, $\text{EADH}_{\mathbb{R}}^{\mathbb{Z}}$, $\text{BASICCLOSED}_{\mathbb{R}}^{\mathbb{Z}}$, etc. are well defined.

Another natural restriction (considered e.g. in [13, 20, 21]), now for real machines, is the requirement that no constants other than 0 and 1 appear in the machine program. Complexity classes arising by considering such constant-free machines are indicated by a superscript 0 as in $\mathrm{P}^0_{\mathbb{R}}$, $\mathrm{NP}^0_{\mathbb{R}}$, etc.

The simultaneous consideration of both these restrictions leads to the notion of constant-free Boolean part.

Definition 5.1. Let $\mathcal{C}$ be a complexity class over $\mathbb{R}$. The *Boolean part* of $\mathcal{C}$ is the discrete complexity class

$$\mathrm{BP}(\mathcal{C}) = \{S \cap \{0,1\}^\infty \mid S \in \mathcal{C}\}.$$

We denote by $\mathcal{C}^0$ the subclass of $\mathcal{C}$ obtained by requiring all the considered machines over $\mathbb{R}$ to be constant-free. The *constant-free Boolean part* of $\mathcal{C}$ is defined as $\mathrm{BP}^0(\mathcal{C}) := \mathrm{BP}(\mathcal{C}^0)$.

Some of the classes $\mathrm{BP}^0(\mathcal{C})$ do contain natural complete problems. This raises the issue of characterizing these classes in terms of already known discrete complexity classes. Unfortunately, there are not many real complexity classes $\mathcal{C}$ for which $\mathrm{BP}^0(\mathcal{C})$ is completely characterized in such terms. The only such result we know is $\mathrm{BP}^0(\mathrm{PAR}_{\mathbb{R}}) = \mathsf{PSPACE}$, proved in [12]. An obvious solution (which may be the only one) is to define new discrete complexity classes in terms of Boolean parts. In this way we define the classes $\mathsf{PR} := \mathrm{BP}^0(\mathrm{P}_{\mathbb{R}})$, $\mathsf{NPR} := \mathrm{BP}^0(\mathrm{NP}_{\mathbb{R}})$ and $\mathsf{coNPR} = \mathsf{co}\mathrm{BP}^0(\mathrm{NP}_{\mathbb{R}}) = \mathrm{BP}^0(\mathrm{coNP}_{\mathbb{R}})$.

While never explicited as a complexity class, the computational resources behind PR have been around for quite a while. A constant-free machine over $\mathbb{R}$ restricted to binary inputs is, in essence, a unit-cost Random Access Machine (RAM). Therefore, PR is the class of subsets of $\{0,1\}^*$ decidable by a RAM in polynomial time. In [1] it was shown that PR is contained in the counting hierarchy and some empirical evidence pointing towards $\mathsf{P} \neq \mathsf{PR}$ was collected. We also note that the existential theory of the reals over the language $\{\{0,1\}, +, -, \times, \leq\}$ is an NPR-complete problem.

Theorem 5.2. *For any polynomial class $\mathcal{C}$ we have $\mathrm{BP}^0(\mathsf{H}\mathcal{C}) = \mathrm{BP}^0(\mathcal{C})$.*

The proof is based on the old idea of simulating the infinitesimal ε by a doubly exponentially small number $2^{2^{N^c}}$, which can be computed by a straight-line program in time polynomial in N by repeated squaring. A second ingredient is the theorem on efficient quantifier elimination [23, Part III].

Combining Theorem 5.2 with Proposition 4.2 we obtain:

Corollary 5.3. *We have $\mathrm{BP}^0(\exists^*) = \mathrm{BP}^0(\exists) = \mathrm{BP}^0(\mathsf{H}\exists) = \mathsf{NPR}$ and $\mathrm{BP}^0(\forall^*) = \mathrm{BP}^0(\forall) = \mathrm{BP}^0(\mathsf{H}\forall) = \mathrm{co}\mathsf{NPR}$.*

All our completeness results induce completeness results in the classical setting.

Corollary 5.4. *(a) The discrete versions of* $\mathrm{U{\scriptsize NBOUNDED}}_{\mathbb{R}}$, $\mathrm{EA{\scriptsize DH}}_{\mathbb{R}}$, $\mathrm{L{\scriptsize OC}D{\scriptsize IM}}_{\mathbb{R}}$, *and* $\mathrm{ZD{\scriptsize ENSE}}_{\mathbb{R}}$ *are* NPR*-complete.*

(b) The discrete versions of the following problems are $\mathrm{co}\mathsf{NPR}$*-complete:* $\mathrm{B{\scriptsize ASIC}C{\scriptsize LOSED}}_{\mathbb{R}}$, $\mathrm{B{\scriptsize ASIC}C{\scriptsize OMPACT}}_{\mathbb{R}}$, $\mathrm{ED{\scriptsize ENSE}}_{\mathbb{R}}$, $\mathrm{C{\scriptsize ONT}P{\scriptsize OINT}}^{\mathrm{DF}}_{\mathbb{R}}$, $\mathrm{C{\scriptsize ONT}}_{\mathbb{R}}$.

Proof. The claimed memberships follow from the definition of BP^0, Corollary 5.3, and a cursory look at the membership proofs for their real versions which show that the involved algorithms are constant-free.

For proving hardness we first note that $\mathrm{S{\scriptsize TANDARD}}(\mathsf{H}\exists)^{\mathbb{Z}}$ is hard for $\mathrm{BP}^0(\mathsf{H}\exists)$ (and similarly for $\mathrm{S{\scriptsize TANDARD}}(\exists^*)$). Indeed, when restricted to binary inputs, the reduction in the proof of Proposition2.2 can be performed by a Turing machine in polynomial time. We next note that the reductions shown in this paper for all the problems above also can be performed by a Turing machine in polynomial time when restricted to binary inputs. □

Thus, based on Theorem 5.2, we obtain in Corollary 5.4 the completeness for the discrete problems $\textsc{ContPoint}_{\mathbb{R}}^{\mathrm{DF}}$ and $\textsc{Cont}_{\mathbb{R}}^{\mathbb{Z}}$ even though we do not have completeness results for the corresponding real problems. This suggests that we are not far away from completeness.

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