

NO OCCURRENCE OBSTRUCTIONS IN GEOMETRIC COMPLEXITY THEORY*

PETER BÜRGISSER¹, CHRISTIAN IKENMEYER², GRETA PANOVA³

ABSTRACT. The permanent versus determinant conjecture is a major problem in complexity theory that is equivalent to the separation of the complexity classes VP_{ws} and VNP . Mulmuley and Sohoni [33] suggested to study a strengthened version of this conjecture over the complex numbers that amounts to separating the orbit closures of the determinant and padded permanent polynomials. In that paper it was also proposed to separate these orbit closures by exhibiting *occurrence obstructions*, which are irreducible representations of $\text{GL}_{n^2}(\mathbb{C})$, which occur in one coordinate ring of the orbit closure, but not in the other. We prove that this approach is impossible. However, we do not rule out the general approach to the permanent versus determinant problem via *multiplicity obstructions* as proposed in [33].

1. INTRODUCTION

A central problem in algebraic complexity theory is to prove that there is no efficient algorithm to evaluate the *permanent*

$$\text{per}_n := \sum_{\pi \in S_n} X_{1\pi(1)} \cdots X_{n\pi(n)}.$$

The natural model of computation to study this question is the one of straight-line programs (or arithmetic circuits), which perform arithmetic operations $+$, $-$, $*$ in the polynomial ring, starting with the variables X_{ij} and complex constants. Efficient means that the number of arithmetic operations is bounded by a polynomial in n . The permanent arises in combinatorics and physics as a generating function. Its relevance for complexity theory derives from Valiant's discovery [40, 41] that the evaluation of the permanent is a complete problem for the complexity class VNP (and also for the class $\#\text{P}$ in the model of Turing machines); see [3, 29] for more information.

The *determinant*

$$\det_n := \sum_{\pi \in S_n} \text{sgn}(\pi) X_{1\pi(1)} \cdots X_{n\pi(n)}$$

is known to have an efficient algorithm. Its evaluation is complete for the complexity class VP_{ws} ; cf. [40, 39]. From the definition it is clear that $\text{VP}_{\text{ws}} \subseteq \text{VNP}$ and proving the separation $\text{VP}_{\text{ws}} \neq \text{VNP}$ is the flagship problem in algebraic complexity theory. It can be seen as an “easier” version of the famous $\text{P} \neq \text{NP}$ problem; cf. [4].

The conjecture $\text{VP}_{\text{ws}} \neq \text{VNP}$ can be restated without any reference to complexity classes by directly comparing permanents and determinants. The *determinantal complexity* $\text{dc}(f)$ of a polynomial $f \in \mathbb{C}[X_1, \dots, X_N]$ is defined as the smallest nonnegative integer $n \in \mathbb{N}$ such that f can

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¹Technische Universität Berlin, pbuerg@math.tu-berlin.de. Partially supported by DFG grant BU 1371/3-2.

²Max Planck Institute for Informatics, Saarland Informatics Campus, cikenmey@mpi-inf.mpg.de.

³University of Pennsylvania, panova@math.upenn.edu. Partially supported by NSF grant DMS-1500834.

be written as a determinant of an n by n matrix whose entries are affine linear forms in the variables X_i . It is known [40] that $\text{dc}(f) \leq s + 1$ if f has a formula of size s . Valiant [40, 42] and Toda [39] proved that $\text{VP}_{\text{ws}} \neq \text{VNP}$ is equivalent to the following conjecture.

1.1. Conjecture (Valiant 1979). *The determinantal complexity $\text{dc}(\text{per}_n)$ grows superpolynomially in n .*

It is known [18] that $\text{dc}(\text{per}_n) \leq 2^n - 1$. Finding lower bounds on $\text{dc}(\text{per}_n)$ is an active area of research [31, 12, 26, 21, 2, 44], but the best known lower bounds are only $\Omega(n^2)$.

1(a). An attempt via algebraic geometry and representation theory. Towards answering Conjecture 1.1, Mulmuley and Sohoni [33, 34, 32] proposed an approach based on algebraic geometry and representation theory, for which they coined the name geometric complexity theory.

We consider $\text{Sym}^n \mathbb{C}^{n^2}$ as the space of homogeneous polynomials of degree n in n^2 variables. Clearly, $\det_n \in \text{Sym}^n \mathbb{C}^{n^2}$. The group GL_{n^2} acts on $\text{Sym}^n \mathbb{C}^{n^2}$ by linear substitution. The orbit $\text{GL}_{n^2} \cdot \det_n$ is obtained by applying all possible invertible linear transformations to \det_n . Consider the closure

$$(1.2) \quad \Omega_n := \overline{\text{GL}_{n^2} \cdot \det_n} \subseteq \text{Sym}^n \mathbb{C}^{n^2}$$

of this orbit with respect to the Euclidean topology. By a general principle, this is the same as the closure with respect to the Zariski topology; see [35, §2.C]. It is easy to see that $\Omega_2 = \text{Sym}^2 \mathbb{C}^4$. For $n = 3$, the boundary of Ω_n has been determined recently [19], but for $n = 4$ it is already unknown.

For $n > m$ we consider the *padded permanent* defined as $X_{11}^{n-m} \text{per}_m \in \text{Sym}^n \mathbb{C}^{m^2}$. (Sometimes the padding is achieved by using a variable not appearing in per_m , but this is irrelevant, cf. [23, Appendix].)

1.3. Conjecture (Mulmuley and Sohoni 2001). *For all $c \in \mathbb{N}_{\geq 1}$ we have $X_{11}^{m^c - m} \text{per}_m \notin \Omega_{m^c}$ for infinitely many m .*

This conjecture was stated in [33]. We refer to [11, Prop. 9.3.2] for an equivalent formulation in terms of complexity classes that goes back to [5]. (In particular see [5, Problem 4.3].)

Conjecture 1.3 implies Conjecture 1.1. Indeed, using that GL_{n^2} is dense in $\mathbb{C}^{n^2 \times n^2}$, one shows (e.g., see [6]) that $\text{dc}(\text{per}_m) \leq n$ implies $X_{11}^{n-m} \text{per}_m \in \Omega_n$. (The latter must be a point in the boundary of Ω_n if $n > m$.)

The following strategy towards Conjecture 1.3 was proposed in [33]. The action of the group $G := \text{GL}_{n^2}$ on Ω_n induces a corresponding action on the coordinate ring $\mathbb{C}[\Omega_n]$. It is well known [16] that the irreducible polynomial representations of G can be labeled by partitions λ into at most n^2 parts. The coordinate ring $\mathbb{C}[\Omega_n]$ is a direct sum of its irreducible submodules since G is reductive. We say that λ occurs in $\mathbb{C}[\Omega_n]$ if it contains the dual of an irreducible G -module of type λ . (It is useful to identify spaces with their duals to avoid negative weights.)

Let $Z_{n,m}$ denote the orbit closure of the padded permanent $X_{11}^{n-m} \text{per}_m \in \text{Sym}^n \mathbb{C}^{m^2}$. If the latter is contained in Ω_n , then $Z_{n,m} \subseteq \Omega_n$, and the restriction defines a surjective G -equivariant homomorphism $\mathbb{C}[\Omega_n] \rightarrow \mathbb{C}[Z_{n,m}]$ of the coordinate rings. Schur's lemma implies that if λ occurs in $\mathbb{C}[Z_{n,m}]$, then it must also occur in $\mathbb{C}[\Omega_n]$. A partition λ violating this condition is called an *occurrence obstruction*. Its existence thus proves that $Z_{n,m} \not\subseteq \Omega_n$ and hence $\text{dc}(\text{per}_m) > n$. It is known that occurrence obstructions λ must satisfy $|\lambda| = nd$ and $\ell(\lambda) \leq m^2$, cf. [33, 34, 11]. Here $|\lambda| := \sum_i \lambda_i$ denotes the *size* of λ and $\ell(\lambda)$ denotes the *length* of λ , which is defined as the number of nonzero parts of λ . We write $\lambda \vdash |\lambda|$, so in our case $\lambda \vdash nd$.

In [33, 34] it was suggested to prove Conjecture 1.3 by exhibiting occurrence obstructions. More specifically, the following conjecture was put forth.

1.4. Conjecture (Mulmuley and Sohoni 2001). *For all $c \in \mathbb{N}_{\geq 1}$, for infinitely many m , there exists a partition λ occurring in $\mathbb{C}[Z_{m^c, m}]$ but not in $\mathbb{C}[\Omega_{m^c}]$.*

This conjecture implies Conjecture 1.3 by the above reasoning.

Conjecture 1.4 on the existence of occurrence obstructions has stimulated a lot of research and has been the main focus of researchers in geometric complexity theory in the past years, see Section 1 (c). Unfortunately, this conjecture is false! This is the main result of this work. More specifically, we show the following.

1.5. Theorem. *Let n, d, m be positive integers with $n \geq m^{25}$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[Z_{n,m}]$, then λ also occurs in $\mathbb{C}[\Omega_n]$. In particular, Conjecture 1.4 is false.*

One can likely improve the bound on n by a more careful analysis.

1 (b). **Proof outline.** Besides combinatorics, the proofs only require some multilinear algebra. The rough outline of the proof of Theorem 1.5 closely follows the structure of the proof of the main theorem in the recent paper [23], which is concerned with the positivity of rectangular Kronecker coefficients. Recall that the *Kronecker coefficient* $k(\lambda, \mu, \nu)$ of three partitions λ, μ, ν of the same size d can be defined as the dimension of the space of \mathfrak{S}_d -invariants of $[\lambda] \otimes [\mu] \otimes [\nu]$, where $[\lambda]$ denotes the irreducible \mathfrak{S}_d -module of type λ and \mathfrak{S}_d is the symmetric group on d symbols; see [28, I§7, internal product]. We write $n \times d$ for the rectangular partition (d, \dots, d) of size nd and call $k_n(\lambda) := k(\lambda, n \times d, n \times d)$ the *rectangular Kronecker coefficient* of $\lambda \vdash nd$. In [33] it was realized that if $k_n(\lambda) = 0$, then λ does not occur in $\mathbb{C}[\Omega_n]$. This was proposed as a potential method of proving Conjecture 1.4, which has stimulated research in algebraic combinatorics on these quantities. The main result in [23] says that proving Conjecture 1.4 in this way is not possible:

1.6. Theorem ([23]). *Let n, d, m be positive integers with $n > 3m^4$ and $\lambda \vdash nd$. If λ occurs in $\mathbb{C}[Z_{n,m}]$, then $k_n(\lambda) > 0$.*

The *body* $\bar{\lambda}$ of a partition λ is obtained from λ by removing its first row. The proofs of Theorem 1.5 and Theorem 1.6 crucially use the following observation by Kadish and Landsberg [24] (see Proposition 3.13). If $\lambda \vdash nd$ appears in $\mathbb{C}[Z_{n,m}]$, then $|\bar{\lambda}| \leq md$. This is equivalent to $\lambda_1 \geq (n-m)d$, which implies that λ must have a very long first row if n is substantially larger than m . This fact can be used to prove a lower bound on all possible degrees d , although in the two proofs this lower bound is obtained using very different techniques. Then λ gets decomposed by cutting it vertically into several long rectangles of short height, treating the first row separately. For these rectangles it is shown independently that they appear in $\mathbb{C}[\Omega_n]$ (for the proof of Theorem 1.5) or that their rectangular Kronecker coefficient is positive (for the proof of Theorem 1.6). Besides taking care of several technicalities, the main result then follows from glueing λ back together via the so-called *semigroup property*. Although this rough outline is shared among both proofs, the details and techniques differ vastly.

A crucial technique in the present paper is the encoding of a generating system of highest weight vectors v_T in plethysms $\mathbf{Sym}^d \mathbf{Sym}^n V$ by tableaux with content $d \times n$, as well as the analysis of their evaluation at tensors of rank one in a combinatorial way. This is similar to [7, 20]. A further technique is the “lifting” of highest weight vectors of $\mathbf{Sym}^d \mathbf{Sym}^n V$, when increasing the inner degree n , as introduced by Kadish and Landsberg [24]. This is closely related to stability property of the plethysm coefficients [43, 13, 30]. A concrete understanding of the stability property in terms of lifting highest weight vectors is important for the proof of the main result. Remarkably, for the proof of Theorem 1.5, the only information we need about the orbit closures Ω_n is that they contain certain padded power sums (cf. Theorem 2.8), see also [10].

1 (c). **Related work.** Let $\tilde{k}_n(\lambda)$ denote the multiplicity by which the irreducible GL_{n^2} -module of type $\lambda \vdash nd$ occurs in $\mathbb{C}[\Omega_n]_d$. We call the numbers $\tilde{k}_n(\lambda)$ the *GCT-coefficients*. Note that an occurrence obstruction for $Z_{n,m} \not\subseteq \Omega_n$ is a partition λ for which $\tilde{k}_n(\lambda) = 0$ and such that λ occurs in $\mathbb{C}[Z_{n,m}]$.

In [33] it was realized that GCT-coefficients can be upper bounded by rectangular Kronecker coefficients: we have $\tilde{k}_n(\lambda) \leq k_n(\lambda)$ for $\lambda \vdash nd$. In fact, the multiplicity of λ in the coordinate ring of the orbit $\mathrm{GL}_{n^2} \cdot \det_n$ equals the so-called symmetric rectangular Kronecker coefficient [11], which is upper bounded by $k_n(\lambda)$.

Since Kronecker coefficients are fundamental quantities that have been the object of study in algebraic combinatorics for a long time [36], much of the research in geometric complexity has focused on these quantities, with the emphasis on understanding when they vanish for rectangular formats. A difficulty in the study of Kronecker coefficients is that there is no known counting interpretation of them [38].

A first attempt towards finding occurrence obstructions was by asymptotic considerations (moment polytopes): this was ruled out in [8], where it was proven that for all $\lambda \vdash nd$ there exists a stretching factor $\ell \geq 1$ such that $k_n(\ell\lambda) > 0$. In [22] it was shown that deciding positivity of Kronecker coefficients in general is NP-hard, but this proof fails for rectangular formats.

Clearly, $\tilde{k}_n(\lambda)$ is bounded from above by the multiplicity $a_\lambda(d[n])$ of the partition λ in the plethysm $\mathrm{Sym}^d \mathrm{Sym}^n \mathbb{C}^{n^2}$. Kumar [25] ruled out asymptotic considerations for the GCT-coefficients: assuming the Alon-Tarsi Conjecture [1], Kumar derived that $\tilde{k}_n(n\lambda) > 0$ for all $\lambda \vdash nd$ and even n . A similar conclusion, unconditional, although with less information on the stretching factor, was obtained in [9].

1 (d). **Future directions.** While our main result, Theorem 1.5, rules out the possibility of proving the Conjecture 1.3 via occurrence obstructions, there still remains the possibility that one may succeed so by comparing multiplicities. If the orbit closure $Z_{n,m}$ of the padded permanent $X_{11}^{n-m} \mathrm{per}_m$ is contained in Ω_n , then the restriction defines a surjective G -equivariant homomorphism $\mathbb{C}[\Omega_n] \rightarrow \mathbb{C}[Z_{n,m}]$ of the coordinate rings, and hence the multiplicity of the type λ in $\mathbb{C}[Z_{n,m}]$ is bounded from above by the GCT-coefficient $\tilde{k}_n(\lambda)$. Thus, proving that $\tilde{k}_n(\lambda)$ is strictly smaller than the latter multiplicity implies that $Z_{n,m} \not\subseteq \Omega_n$. We note that the paper [14] rules out one natural asymptotic method for achieving this. Mulmuley pointed out to us a paper by Larsen and Pink [27] that is of potential interest in this connection.

2. PRELIMINARIES

2 (a). **Some multilinear algebra.** We denote by \mathbb{N} the set of nonnegative integers. Let V be a finite dimensional complex vector space and $d \in \mathbb{N}$. We assume that V is endowed with an inner product, i.e., a nondegenerate symmetric bilinear form. E.g., $V = \mathbb{C}^N$ and $\langle v, w \rangle := \sum_i v_i w_i$ for $v, w \in \mathbb{C}^N$. With $v \in V$ we associate the linear form $v^* \in V^*$ given by $v^*(w) = \langle v, w \rangle$. This gives a linear isomorphism $V \rightarrow V^*$ that allows to identify V with V^* . The inner product on V extends to an inner product on the d th tensor power $V^{\otimes d}$ characterized by $\langle v_1 \otimes \cdots \otimes v_d, w_1 \otimes \cdots \otimes w_d \rangle = \langle v_1, w_1 \rangle \cdots \langle v_d, w_d \rangle$. We associate with $f \in V^{\otimes d}$ the linear form $f^* \in (V^{\otimes d})^*$ defined by $f^*(t) = \langle f, t \rangle$. Note that $(V^{\otimes d})^* \simeq (V^*)^{\otimes d}$ canonically. Suppose that X_1, \dots, X_N is an orthonormal basis of V . For a list $I = (i_1, \dots, i_m) \in [N]^m$ we define

$$X_I := X_{i_1} \otimes X_{i_2} \otimes \cdots \otimes X_{i_m} \in \otimes^m V$$

and note that the X_I form an orthonormal basis of $\otimes^m V$.

The symmetric group \mathfrak{S}_d on d symbols acts on $V^{\otimes d}$ by permuting the factors. This action preserves the inner product on $V^{\otimes d}$. The *symmetric power* $\mathrm{Sym}^d V$ is the subspace of $V^{\otimes d}$ consisting of the \mathfrak{S}_d -invariant tensors. It can be obtained as the image of the projection $\Pi_d: \otimes^d V \rightarrow \mathrm{Sym}^d V$ defined by $\Pi_d := \frac{1}{d!} \sum_{\pi \in \mathfrak{S}_d} \pi$. Let $C_N(d)$ denote the set of all $\alpha \in \mathbb{N}^N$ with $|\alpha| = d$. For $I \in [N]^m$ we define the *type* $\zeta(I) \in C_N(d)$, letting $\zeta(I)_i$ denote the number of appearances of i in I . We

associate with $\alpha \in C_N(d)$ the *monomial*

$$(2.1) \quad X^\alpha := \frac{1}{\binom{m}{\alpha}} \sum_{\zeta(I)=\alpha} X_I,$$

where the sum is over all $\alpha \in C_N(d)$ such that $\zeta(I) = \alpha$ and $\binom{m}{\alpha}$ is the multinomial coefficient. The elements of $\text{Sym}^d V$ are degree d homogeneous polynomial expressions in the variables X_i , but their interpretation as symmetric tensors is a crucial technical ingredient in our work. The multiplication of polynomials corresponds to the *symmetric product* ($d_1, d_2 \in \mathbb{N}$)

$$(2.2) \quad \text{Sym}^{d_1} V \times \text{Sym}^{d_2} V \rightarrow \text{Sym}^{d_1+d_2} V, (p, q) \mapsto p \cdot q := \Pi_{d_1+d_2}(p \otimes q),$$

which is defined by symmetrizing $p \otimes q$.

Let W be a finite dimensional \mathbb{C} -vector space. The degree d part of its coordinate ring $\mathbb{C}[W]_d$ consists the homogeneous functions $F: W \rightarrow \mathbb{C}$ of degree d . For any such F , there is a unique \mathfrak{S}_d -invariant linear form $f^*: W^{\otimes d} \rightarrow \mathbb{C}$ such that $F(w) = f^*(w^{\otimes d})$ for $w \in W$. This is a consequence of the *polarization formula*, cf. [15, p. 5], which states that for $w_1, \dots, w_d \in W$,

$$(2.3) \quad f^*(w_1 \otimes \dots \otimes w_d) = \frac{1}{d!} \sum_{I \subseteq [d]} (-1)^{d-|I|} F\left(\sum_{i \in I} w_i\right).$$

We choose an inner product on W . This defines a linear isomorphism $W^{\otimes d} \rightarrow (W^{\otimes d})^*$, which induces a linear isomorphism

$$(2.4) \quad \text{Sym}^d W \rightarrow \text{Sym}^d W^*, f \mapsto f^*,$$

where we have identified $\text{Sym}^d W^*$ with the space of \mathfrak{S}_d -invariants of $(W^{\otimes d})^*$ via the canonical isomorphism. This way, f^* corresponds to a symmetric tensor $f \in \text{Sym}^d W$, that is characterized by

$$(2.5) \quad \forall w \in W \quad F(w) = f^*(w^{\otimes d}) = \langle f, w^{\otimes d} \rangle.$$

We refer to [37] for background on multilinear algebra.

Via the above isomorphisms, we will write $\text{Sym}^d W^*$ for the space of degree d homogeneous functions.

We present now a useful lemma on the evaluation of polynomials at ‘‘points of low rank’’.

2.6. Lemma. *Suppose $F \in \text{Sym}^d W^*$ is such that $F(\sum_{j=1}^r w_j) \neq 0$ for some $w_1, \dots, w_r \in W$. Then there exists $J \subseteq [r]$ with $|J| \leq d$ and $F(\sum_{j \in J} w_j) \neq 0$.*

Proof. By multilinearity and (2.5) we have

$$0 \neq F\left(\sum_{j=1}^r w_j\right) = \left\langle f, \left(\sum_{j=1}^r w_j\right)^{\otimes d} \right\rangle = \sum_{j_1, \dots, j_d} \left\langle f, w_{j_1} \otimes \dots \otimes w_{j_d} \right\rangle,$$

hence there exist $j_1, \dots, j_d \in [r]$ such that $\langle f, w_{j_1} \otimes \dots \otimes w_{j_d} \rangle \neq 0$. The polarization formula (2.3) implies the existence of a subset $I \subseteq [d]$ such that $F(\sum_{i \in I} w_{j_i}) \neq 0$, which completes the proof with $J := \{j_i \mid i \in I\}$. \square

We will study homogeneous polynomial functions on a space $W = \text{Sym}^n V$, where V is a finite dimensional \mathbb{C} -vector space. These functions can be seen as elements of the *plethysm* $\text{Sym}^d W^* = \text{Sym}^d(\text{Sym}^n V)^* \simeq \text{Sym}^d \text{Sym}^n V^*$. Plethysms play a central role in our work.

We show now that if $F \in \text{Sym}^d(\text{Sym}^n V)^*$ is nonzero, then F does not vanish on a some power sum $v_1^n + \dots + v_d^n \in \text{Sym}^n V$ with at most d terms, where $v_i \in V$. More specifically:

2.7. Corollary. *Let V be a finite dimensional \mathbb{C} -vector space and $d, n \geq 1$. If $F \in \text{Sym}^d(\text{Sym}^n V)^*$ is nonzero, then F does not vanish on $v_1^n + \dots + v_d^n$, for Zariski almost all $(v_1, \dots, v_d) \in V^d$.*

Proof. Let $w \in \text{Sym}^n V$ be such that $F(w) \neq 0$. It is well known that there exist $r \in \mathbb{N}$ and $v_1, \dots, v_r \in V$ such that $w = v_1^n + \dots + v_r^n$. By Lemma 2.6, there are $1 \leq j_1, \dots, j_d \leq r$ such that $F(v_{j_1}^n + \dots + v_{j_d}^n)$ is nonzero. Therefore, the open set $\{(v_1, \dots, v_d) \in V^d \mid F(v_1^n + \dots + v_d^n) \neq 0\}$ is nonempty and hence Zariski dense. \square

2(b). **Padded power sum embedding.** Let $V = \mathbb{C}^{n^2}$ and $\Omega_n \subseteq \text{Sym}^n V$ denote the orbit closure of \det_n ; cf. (1.2). The following result goes back to Valiant [40]. Remarkably, it is the only property of the orbit closure Ω_n that we will exploit in our paper!

2.8. Theorem. *Let n, s, d be positive integers such that $n \geq sd$ and $\ell, v_1, \dots, v_d \in V$. Then we have $\ell^{n-s}(v_1^s + \dots + v_d^s) \in \Omega_n$.*

Proof. Let X_1, \dots, X_{n^2} denote the standard basis of V . Writing the power sum $X_1^s + \dots + X_d^s$ as a formula requires at most $(s-1)d + d - 1 = sd - 1$ many additions and multiplications. Valiant's construction [40] implies that $X_1^s + \dots + X_d^s$ has the determinantal complexity at most $sd \leq n$, i.e., it can be written as the determinant of an $n \times n$ -matrix with affine linear entries in X_1, \dots, X_d . The determinantal complexity is invariant under invertible linear transformations. Hence the determinantal complexity of $v_1^s + \dots + v_d^s$ is at most n , for any linearly independent system v_1, \dots, v_d of linear combinations of X_1, \dots, X_{n^2} . By homogenizing with respect to a new variable Y and then substituting Y by $\ell \in V$, we see that $\ell^{n-s}(v_1^s + \dots + v_d^s) \in \Omega_n$. Since Ω_n is closed, we can go over to the limit and drop the assumption that the v_i are linearly independent. \square

2(c). **Basics on highest weight vectors.** We refer to [17] for more details and proofs for the following basic notions and facts.

A *partition* λ is a nondecreasing finite sequence of nonnegative integers $(\lambda_1, \dots, \lambda_N)$. It can be visualized as a *Young diagram*, which is a finite collection of boxes, arranged in left-justified rows, with λ_i boxes in the i th row. Depending on the context, we also write λ for the set of boxes of the diagram. We shall use the notation $|\lambda| := \sum_i \lambda_i$ for the *size* of λ , which is the number of the boxes of the Young diagram. Moreover, we write $\ell(\lambda)$ for the *length* of λ , which is defined as the number of its nonzero parts λ_i , i.e., the number of rows of the diagram. We also say that λ_i is the *length* of the i th row of λ . By the symbol $k \times \ell$ we denote the rectangular diagram with k rows of length ℓ , so with a total of $k\ell$ boxes. In partition notation, we have $k \times \ell = (\ell, \dots, \ell)$ with ℓ appearing k times. We shall also use the convenient notation $\lambda \vdash D$ to express that λ is a partition of size D . The *body* $\bar{\lambda}$ of λ is obtained from λ by removing its first row.

Recall that \mathfrak{S}_D denotes the symmetric group on D symbols. It is well known that the irreducible \mathfrak{S}_D -modules can be encoded by partitions $\lambda \vdash D$ of size D . They are called *Specht modules* and we shall denote them by $[\lambda]$.

Let $G := \text{GL}_N(\mathbb{C})$ and \mathcal{V} be a finite dimensional, rational G -module. We denote by $U_N \subseteq G$ the subgroup of upper triangular matrices with ones on the main diagonal. Moreover, let $\text{diag}(\alpha_1, \dots, \alpha_N)$ denote the diagonal matrix with entries α_i on the diagonal. A vector $f \in \mathcal{V}$ is called a *highest weight vector* of weight $\lambda \in \mathbb{Z}^N$ iff f is U_N -invariant, i.e., $u \cdot f = f$ for all $u \in U_N$, and f is a weight vector of weight λ , i.e., $\text{diag}(\alpha_1, \dots, \alpha_N) \cdot f = \alpha_1^{\lambda_1} \dots \alpha_N^{\lambda_N} f$ for all $\alpha_i \in \mathbb{C}^\times$. We remark that necessarily $\lambda_1 \geq \dots \geq \lambda_N$, so that λ is a partition if its entries are nonnegative. We denote by $\text{HWV}_\lambda(\mathcal{V})$ the vector space of highest weight vectors of weight λ . An irreducible G -module \mathcal{V} is called a *Schur-Weyl module*. It is known that there is a unique λ such that $\text{HWV}_\lambda(\mathcal{V})$ is one-dimensional. Moreover, $\text{HWV}_\mu(\mathcal{V}) = 0$ for all $\mu \neq \lambda$. We call λ the *type* of the Schur-Weyl module \mathcal{V} and shall abbreviate \mathcal{V} by the symbol $\{\lambda\}$.

We assume $V := \mathbb{C}^N$. The group $G := \text{GL}(V)$ acts on the D th tensor power $\otimes^D V$ by $g(v_1 \otimes \dots \otimes v_D) = (gv_1) \otimes \dots \otimes (gv_D)$ and the group \mathfrak{S}_D acts by permuting the factors. Since these actions commute, we have an action of $\text{GL}(V) \times \mathfrak{S}_D$ on $\otimes^D V$. We next explain how to construct highest weight vectors in $\otimes^D V$. We denote by X_1, \dots, X_N the standard basis vectors of \mathbb{C}^N . Let $\lambda \vdash D$ and μ denote the transpose of λ , so μ_i denotes the number of boxes in the i -th column of λ . For

$j \leq N$ we note that $v_{j \times 1} := X_1 \wedge X_2 \wedge \cdots \wedge X_j$ is a highest weight vector of weight $j \times 1$. We define now:

$$(2.9) \quad v_\lambda := v_{\mu_1 \times 1} \otimes \cdots \otimes v_{\mu_{\lambda_1} \times 1} \in \bigotimes^D V.$$

It is easy to check that v_λ is a nonzero highest weight vector of weight λ .

2.10. Proposition. *Let $\lambda \vdash D$. Then the vector space $\text{HWV}_\lambda(\bigotimes^D V)$ is spanned by the \mathfrak{S}_D -orbit of v_λ .*

Proof. Schur-Weyl duality provides a $\text{GL}(V) \times \mathfrak{S}_D$ -isomorphism

$$\bigotimes^D V \simeq \bigoplus_{\lambda \vdash D} \{\lambda\} \otimes [\lambda].$$

Recalling that $\text{HWV}_\lambda(\{\lambda\})$ is one-dimensional, we see that $\text{HWV}_\lambda(\bigotimes^D V)$ is isomorphic to $[\lambda]$ as an \mathfrak{S}_D -module. It follows that $\text{HWV}_\lambda(\bigotimes^D V)$ is spanned by the \mathfrak{S}_D -orbit of any of its nonzero elements. \square

We analyze now the v_λ in more detail. A *Young tableau* of shape $\lambda \vdash D$ is a filling of the boxes of the diagram λ with numbers. We shall assume that each of the numbers $1, \dots, D$ occurs exactly once, so that we obtain an enumeration of the boxes. The *column-standard Young tableau* T_λ^{std} of shape λ is the Young tableau of shape λ that contains the numbers $1, \dots, D$ ordered columnwise, from top to bottom and left to right. For example,

$$T_{(4,2)}^{\text{std}} = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 6 \\ \hline 2 & 4 & & \\ \hline \end{array}$$

is column-standard. The symmetric group \mathfrak{S}_D acts on the set of Young tableaux of the diagram λ by replacing each entry i with $\pi(i)$. For example, for $\pi = (2453)$, we obtain

$$(2.11) \quad \pi T_{(4,2)}^{\text{std}} = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 6 \\ \hline 4 & 5 & & \\ \hline \end{array}.$$

Describing πv_λ in terms of the permutation π has redundancies that can be read off the tableau $\pi T_\lambda^{\text{std}}$. Namely, the following is an easy consequence of the definition of v_λ in (2.9). For a transposition $\tau = (i \ j)$ we have:

$$(2.12) \quad \tau \pi v_\lambda = -\pi v_\lambda \quad \text{if } i \text{ and } j \text{ are in the same column in } \pi T_\lambda^{\text{std}}.$$

Moreover, if τ is a permutation that switches two columns of the same length in $\pi T_\lambda^{\text{std}}$, then

$$(2.13) \quad \tau \pi v_\lambda = \pi v_\lambda.$$

3. PLETHYSMS

Again assume $V = \mathbb{C}^N$. We partition the position set $[dn] := \{1, \dots, dn\}$ into the *blocks* B_1, \dots, B_d , where $B_u := \{(u-1)n + v \mid 1 \leq v \leq n\}$. The subgroup of \mathfrak{S}_{dn} of permutations that preserve the partition into blocks is called the *wreath product* $\mathfrak{S}_d \wr \mathfrak{S}_n$. It is generated by the permutations leaving the blocks invariant, and the permutations of the form $(u-1)n + v \mapsto (\tau(u)-1)n + v$ with $\tau \in \mathfrak{S}_d$, which simultaneously permute the blocks. Structurally, the wreath product is a semidirect product $\mathfrak{S}_d \wr \mathfrak{S}_n \simeq (\mathfrak{S}_n)^d \rtimes \mathfrak{S}_d$. Note that its order equals $d! n!^d$. Symmetrizing over $\mathfrak{S}_d \wr \mathfrak{S}_n$, we obtain

$$(3.1) \quad \Sigma_{d,n} := \frac{1}{d! n!^d} \sum_{\sigma \in \mathfrak{S}_d \wr \mathfrak{S}_n} \sigma$$

in the group algebra $\mathbb{C}[\mathfrak{S}_{dn}]$. We define the plethysm $\text{Sym}^d \text{Sym}^n V$ as the space of $\mathfrak{S}_d \wr \mathfrak{S}_n$ -invariants in $\bigotimes^{dn} V$. This space is the image of the projection $\bigotimes^{dn} V \rightarrow \text{Sym}^d \text{Sym}^n V, w \mapsto \Sigma_{d,n} w$. Moreover, for $\lambda \vdash dn$, we define the *plethysm coefficient*

$$a_\lambda(d[n]) := \dim \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V),$$

as the multiplicity of $\{\lambda\}$ in $\text{Sym}^d \text{Sym}^n V$.

3(a). A system of generators encoded by tableaux. As a consequence of Proposition 2.10, the highest weight vector space of $\text{Sym}^d \text{Sym}^n V$ of weight $\lambda \vdash dn$ is spanned by the projections $\Sigma_{d,n} \pi v_\lambda$, where π runs over all permutations in \mathfrak{S}_{dn} . However, this description is highly redundant: we have $\Sigma_{d,n} \pi = \Sigma_{d,n} \pi'$ iff $(\mathfrak{S}_d \wr \mathfrak{S}_n) \pi = (\mathfrak{S}_d \wr \mathfrak{S}_n) \pi'$, for $\pi, \pi' \in \mathfrak{S}_{dn}$. We next give an intuitive description of the cosets of $\mathfrak{S}_d \wr \mathfrak{S}_n$ in terms of certain tableaux.

3.2. Definition. A tableau T of shape $\lambda \vdash dn$ with content $d \times n$ is a partition of the set of boxes of the Young diagram of λ into d classes C_1, \dots, C_d , each of size n .

Intuitively, we think of a such a tableau T as a filling of the Young diagram of λ with d different letters, where all boxes in the class C_u have the same letter (and the order of the letters is irrelevant).

We assign to a permutation $\pi \in \mathfrak{S}_{dn}$ and $\lambda \vdash dn$ a tableau $T_\lambda(\pi)$ of shape λ with content $d \times n$ as follows: take d different letters and replace in $\pi T_\lambda^{\text{std}}$ the numbers in the block B_u by the same letter. It is clear that any tableau of shape $\lambda \vdash dn$ with content $d \times n$ can be obtained this way. For example, for the tableau in (2.11) we get for $n = 3, d = 2$, using the letters a, b ,

$$\begin{array}{|c|c|c|c|} \hline a & a & a & b \\ \hline b & b & & \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline b & b & b & a \\ \hline a & a & & \\ \hline \end{array}.$$

It should be clear that $T_\lambda(\pi) = T_\lambda(\pi')$ iff $(\mathfrak{S}_n \wr \mathfrak{S}_d) \pi = (\mathfrak{S}_n \wr \mathfrak{S}_d) \pi'$, for $\pi, \pi' \in \mathfrak{S}_{dn}$.

By this observation, the following is well-defined.

3.3. Definition. Let T be a tableau of shape λ with content $d \times n$. We define $v_T := \Sigma_{d,n} \pi v_\lambda$ where $\pi \in \mathfrak{S}_{dn}$ is such that $T = T_\lambda(\pi)$.

By Proposition 2.10, v_T is a highest weight vector in $\text{Sym}^d \text{Sym}^n V$ of weight λ . We can restrict our attention to certain T because of the following.

3.4. Lemma. (1) Let T be a tableau of shape λ with content $d \times n$. If the same letter appears in a column of T more than once, then $v_T = 0$.

(2) Let T and T' be two tableaux of shape λ with content $d \times n$ that can be obtained from each other by switching two columns that have the same length. Then $v_T = v_{T'}$.

Proof. For the first assertion let (r, c) and (r', c) be different positions in T in the same column that have the same letter. Assume $T = T_\lambda(\pi)$ and let i and j denote the entries of $\pi T_\lambda^{\text{std}}$ at the positions (r, c) and (r', c) , respectively. Then i and j lie in the same block, since they are mapped to the same letter. Hence the transposition $\tau := (i \ j)$ is an element of $\mathfrak{S}_d \wr \mathfrak{S}_n$. Using (2.12), we see that symmetrizing πv_λ over the 2-element subgroup $\{\text{id}, \tau\} \subseteq \mathfrak{S}_d \wr \mathfrak{S}_n$ maps πv_λ to zero. Hence symmetrizing over the full group $\mathfrak{S}_d \wr \mathfrak{S}_n$ maps πv_λ to zero as well.

The second assertion is shown analogously, but using (2.13) instead of (2.12). \square

By a *singleton column* we understand a column of length one. According to Lemma 3.4(2), singleton columns of T can be permuted without changing the value of v_T . Moreover, according to Lemma 3.4(1), we can restrict attention to tableaux, where no letter appears more than once in a column. This leads to the following definition.

3.5. Definition. Let T and T' be tableaux of shape λ with content $d \times n$, where no letter appears more than once in a column. We call T and T' equivalent if they differ only by a reordering of their singleton columns. In this case we write $T \simeq T'$.

By Lemma 3.4(2), v_T depends only on the equivalence class of T . For example, the following two tableaux with content 2×3 are equivalent:

$$\begin{array}{|c|c|c|c|} \hline a & a & a & b \\ \hline b & b & & \\ \hline \end{array} \simeq \begin{array}{|c|c|c|c|} \hline a & a & b & a \\ \hline b & b & & \\ \hline \end{array}.$$

Summarizing, we arrived at the following result.

3.6. Proposition. *The vector space $\text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V)$ is spanned by the highest weight vectors v_T , where T ranges over all equivalence classes of tableaux of shape λ with content $d \times n$, such that no letter appears more than once in a column of T .*

3(b). **Dual spaces.** We are actually dealing with the duals of the spaces $\text{Sym}^d \text{Sym}^n V$ since we investigate degree d homogenous polynomial functions F on $W = \text{Sym}^n V$, encoded by symmetric tensors $f^* \in \text{Sym}^d W^* = \text{Sym}^d(\text{Sym}^n V)^*$. Since we have chosen an inner product on W , the tensor f^* corresponds to a tensor $f \in \text{Sym}^d W = \text{Sym}^d \text{Sym}^n V$, see (2.5). Note that if $f \in \text{Sym}^d \text{Sym}^n V$ is a highest weight vector of weight $\lambda \vdash dn$, then $f^* \in \text{Sym}^d W^*$ is a highest weight vector of weight $-\lambda$.

Let $Z \subseteq W$ be a Zariski closed G -invariant subset, where $G = \text{GL}(V)$. The following two situations are of interest to us: the orbit closure Ω_n of the determinant of \det_n , cf. (1.2), and the orbit closure $Z_{n,m}$ of the padded permanent $X_{11}^{n-m} \text{per}_m$ (here $V = \mathbb{C}^{n^2}$ and $n > m$). The coordinate ring $\mathbb{C}[Z]$ consists of the polynomial functions on W . Its degree d part $\mathbb{C}[Z]_d$ contains an irreducible G -module of highest weight $-\lambda$ iff there exists a highest weight vector $F: W \rightarrow \mathbb{C}$ of weight $-\lambda$ such that $F(p) \neq 0$ for some $p \in Z$. If F corresponds to $f \in \text{Sym}^d W$, this means that $\langle f, p^{\otimes d} \rangle \neq 0$, cf. Section 2(a). If this is the case, we say that λ occurs in $\mathbb{C}[Z]_d$. This useful convention allows more elegant statements by avoiding negative weights. We will also say that λ occurs in $\text{Sym}^d(\text{Sym}^n V)^*$ if the latter contains an irreducible G -module of highest weight $-\lambda$.

The following *semigroup property* is a crucial ingredient of our proofs.

3.7. Lemma. *If λ and μ occur in $\mathbb{C}[Z]_d$, then $\lambda + \mu$ occurs in $\mathbb{C}[Z]_d$.*

Proof. If $F, G: W \rightarrow \mathbb{C}$ are highest weight vectors with the weights $-\lambda$ and $-\mu$, respectively, then the product $F \cdot G$ is a highest weight vector of weight $-\lambda - \mu$. \square

3(c). **Contracting highest weight vectors in plethysms with rank one tensors.** Again assume $V = \mathbb{C}^N$ and let X_1, \dots, X_N denote the standard basis of $V = \mathbb{C}^N$. The standard basis vectors $X_{s(1)} \otimes \dots \otimes X_{s(dn)}$ of $V^{\otimes d}$ are encoded by maps $s: [dn] \rightarrow [N]$. Our goal is to provide a combinatorial description of the contraction $\langle v_T, X_{s(1)} \otimes \dots \otimes X_{s(dn)} \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product.

We view $\lambda \vdash dn$ as a Young diagram and, for convenience, denote by λ also the set of boxes of the diagram. Recall that a tableau T of shape λ with content $d \times n$ is given by a partition $\lambda = C_1 \cup \dots \cup C_d$ of the set of boxes of λ into classes C_u of size n . Also, recall from the beginning of Section 3 the decomposition $[dn] = B_1 \cup \dots \cup B_d$ into blocks of size n . We consider now bijective assignments $\vartheta: \lambda \rightarrow [dn]$ that map boxes in the same class to numbers in the same block. More formally:

3.8. Definition. *Let T be a tableau of shape λ with content $d \times n$ and $\vartheta: \lambda \rightarrow [dn]$ be a bijection. We say that ϑ respects T iff there exists a permutation $\tau \in \mathfrak{S}_d$ such that $\vartheta(C_i) = B_{\tau(i)}$ for all i .*

This notion is closely related to the wreath product $\mathfrak{S}_d \wr \mathfrak{S}_n$ as follows. Let us call *standard enumeration* $\vartheta_0: \lambda \rightarrow [dn]$ the labeling of the boxes in T_λ^{std} and let $\pi \in \mathfrak{S}_{dn}$. Then the assignments ϑ respecting $T_\lambda(\pi)$ are given by $\vartheta = \sigma \circ \pi \circ \vartheta_0$, where $\sigma \in \mathfrak{S}_d \wr \mathfrak{S}_n$.

We need to introduce some further notations. Let $j = (j_1, \dots, j_k)$ be a list of integers. If $\{j_1, \dots, j_k\} = \{1, 2, \dots, k\}$, then $\text{sgn}(j)$ denotes the *sign* of the permutation j ; otherwise, we define $\text{sgn}(j) := 0$.

Suppose $\vartheta: \lambda \rightarrow [dn]$ respects the tableau T of shape λ with content $d \times n$, and take a map $s: [dn] \rightarrow [N]$. We define the *value* $\text{val}_\vartheta(s)$ of ϑ at $s: [dn] \rightarrow [N]$ by

$$(3.9) \quad \text{val}_\vartheta(s) := \prod_{\text{column } c \text{ of } \lambda} \text{sgn}(s \circ \vartheta)|_c.$$

Here, $\text{sgn}(s \circ \vartheta)|_c$ denotes the sign of the list of integers $(s(\vartheta(1, c)), \dots, s(\vartheta(\mu_c, c)))$ corresponding to the c th column of the diagram λ . It is important to note that if $\text{val}_\vartheta(s) \neq 0$, then $s(\vartheta(\square)) = 1$ for all singleton columns \square of λ .

We shall use the following important rule throughout the paper.

3.10. Theorem. *Let T be a tableau of shape $\lambda \vdash dn$ with content $d \times n$ and $s: [dn] \rightarrow [N]$ be a map. Then*

$$\langle v_T, X_{s(1)} \otimes \dots \otimes X_{s(dn)} \rangle = \frac{1}{d! n!^d} \sum_{\vartheta} \text{val}_\vartheta(s),$$

where the sum is over all bijections $\vartheta: \lambda \rightarrow [dn]$ respecting T .

Proof. We begin with a general reasoning. Let μ denote the transpose of λ and put $r := \lambda_1$. Take a rank one tensor $t = t_1 \otimes \dots \otimes t_{dn} \in \bigotimes^{dn} V$, where $t_i \in V$. From the definition (2.9) of the highest weight vector $v_\lambda \in \bigotimes^{dn} V$, we see that contracting v_λ with t yields the following product of determinants

$$\langle v_\lambda, t \rangle = \widetilde{\det}(t_1, t_2, \dots, t_{\mu_1}) \cdot \widetilde{\det}(t_{\mu_1+1}, t_{\mu_1+2}, \dots, t_{\mu_1+\mu_2}) \cdot \dots \cdot \widetilde{\det}(t_{dn-\mu_r+1}, t_{dn-\mu_r+2}, \dots, t_{dn}),$$

where $\widetilde{\det}(t_{i+1}, \dots, t_{i+j})$ denotes the determinant of the top $j \times j$ minor of the $N \times j$ matrix whose columns are t_{i+1}, \dots, t_{i+j} . Applying the permutation $\pi \in \mathfrak{S}_{dn}$ to v_λ and using $\langle \pi v_\lambda, t \rangle = \langle v_\lambda, \pi^{-1} t \rangle$, we conclude that

$$(3.11) \quad \langle \pi v_\lambda, t \rangle = \widetilde{\det}(t_{\pi(1)}, t_{\pi(2)}, \dots, t_{\pi(\mu_1)}) \cdot \dots \cdot \widetilde{\det}(t_{\pi(dn-\mu_r+1)}, t_{\pi(dn-\mu_r+2)}, \dots, t_{\pi(dn)}).$$

Pictorially, we place the vector t_k into the box of $\pi T_\lambda^{\text{std}}$ with entry k and then take the product over the determinants given by the columns.

Suppose that $T = T_\lambda(\pi)$ for $\pi \in \mathfrak{S}_{dn}$. From the definition of $\Sigma_{d,n}$ in (3.1) and $v_T = \Sigma_{d,n} \pi v_\lambda$, we obtain that

$$\langle v_T, t \rangle = \langle \Sigma_{d,n} \pi v_\lambda, t \rangle = \frac{1}{d! n!^d} \sum_{\sigma \in \mathfrak{S}_d \wr \mathfrak{S}_n} \langle \sigma \pi v_\lambda, t \rangle = \frac{1}{d! n!^d} \sum_{\sigma \in \mathfrak{S}_d \wr \mathfrak{S}_n} \langle \pi v_\lambda, \sigma^{-1} t \rangle.$$

Assume now $t_i := X_{s(i)}$ for a map $s: [dn] \rightarrow [N]$. Applying (3.11) to $\sigma^{-1} t = X_{\sigma(s(1))} \otimes \dots \otimes X_{\sigma(s(dn))}$, we obtain that

$$\langle \pi v_\lambda, \sigma^{-1} t \rangle = \widetilde{\det}(t_{\sigma(\pi(1))}, t_{\sigma(\pi(2))}, \dots, t_{\sigma(\pi(\mu_1))}) \cdot \dots \cdot \widetilde{\det}(t_{\sigma(\pi(dn-\mu_r+1))}, t_{\sigma(\pi(dn-\mu_r+2))}, \dots, t_{\sigma(\pi(dn))}).$$

We are dealing here with determinants of matrices whose columns are standard basis vectors, thus the determinants are either zero or equal the sign of the corresponding permutation. More specifically, the first determinant can be written as

$$\begin{aligned} \widetilde{\det}(t_{\sigma(\pi(1))}, t_{\sigma(\pi(2))}, \dots, t_{\sigma(\pi(\mu_1))}) &= \widetilde{\det}(X_{s(\sigma(\pi(1)))}, X_{s(\sigma(\pi(2)))}, \dots, X_{s(\sigma(\pi(\mu_1)))}) \\ &= \text{sgn}(s(\sigma(\pi(1))), \dots, s(\sigma(\pi(\mu_1)))). \end{aligned}$$

Put $\vartheta := \sigma \circ \pi \circ \vartheta_0$. We observe that the list $(s(\sigma(\pi(1))), \dots, s(\sigma(\pi(\mu_1))))$ is obtained by restricting $s \circ \vartheta$ to the first column of λ . The remaining determinants correspond to the remaining columns of λ in a similar way. From this we see that

$$\langle \pi v_\lambda, \sigma^{-1} t \rangle = \text{val}_\vartheta(s).$$

The assertion follows now by noting that if σ runs through the wreath group $\mathfrak{S}_d \wr \mathfrak{S}_n$, then $\vartheta = \sigma \circ \pi \circ \vartheta_0$ runs through all assignments respecting T . \square

Here is a first application.

3.12. Corollary. *Let n be even and T be the tableau of shape $d \times n$ with content $d \times n$, in which in each row all boxes have the same letter. Then $\langle v_T, (c_1 X_1^n + \dots + c_n X_d^n)^{\otimes d} \rangle = d! c_1 \dots c_d$ for $c_1, \dots, c_d \in \mathbb{C}$.*

Proof. Put $w := c_1 X_1^n + \dots + c_n X_d^n$. Since $X_i^n = X_i^{\otimes n}$ by (2.1), we have $w^{\otimes d} = \sum_{i_1, \dots, i_d} c_{i_1} \dots c_{i_d} X_{i_1}^{\otimes n} \otimes \dots \otimes X_{i_d}^{\otimes n}$, where the sum is over all $1 \leq i_1, \dots, i_d \leq d$.

Assume first $(i_1, \dots, i_d) = (1, \dots, d)$. We apply Theorem 3.10 to compute $\langle v_T, X_1^{\otimes n} \otimes \dots \otimes X_d^{\otimes n} \rangle$. Using a rowwise enumeration of the boxes of λ , the bijections $\vartheta: \lambda \rightarrow [dn]$ respecting T are in one to one correspondence with the elements of the wreath product $\mathfrak{S}_d \wr \mathfrak{S}_n$. They are given by permutations $\tau \in \mathfrak{S}_d$ of the rows, and permutations of the numbers within the rows. Such a bijection ϑ contributes $\text{val}_\vartheta(s) = \text{sgn}(\tau)^n$, where $s = (1, \dots, 1, \dots, d, \dots, d)$ (each index occurring n times). Hence we obtain, by the assumption that n is even,

$$\langle v_T, c_1 \dots c_d X_1^{\otimes n} \otimes \dots \otimes X_d^{\otimes n} \rangle = \frac{c_1 \dots c_d}{d!} \sum_{\tau \in \mathfrak{S}_d} \text{sgn}(\tau)^n = c_1 \dots c_d.$$

We turn now to the contributions of an arbitrary sequence (i_1, \dots, i_d) . If it is a permutation of $1, \dots, d$, then, by the same argument as before, we see that we get the same contribution as for $(1, \dots, d)$. On the other hand, if the sequence is not a permutation of $1, \dots, d$, we get zero. Altogether, we obtain $\langle v_T, X_1^n + \dots + X_d^n \rangle = d! c_1 \dots c_d$ as claimed. \square

3(d). The restriction on highest weights resulting from padding. We give another application of Theorem 3.10. Let $n > m$ and consider the orbit closure $Z_{n,m}$ of the padded permanent $X_1^{n-m} \text{per}_m \in \text{Sym}^n \mathbb{C}^{n^2}$ (X_1 standing for X_{11}). Recall that the body $\bar{\lambda}$ of a partition λ is obtained from λ by removing its first row. The following insight is due to Kadish and Landsberg [24].

3.13. Proposition ([24]). *If $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]_d$, then $\ell(\lambda) \leq m^2$ and $|\bar{\lambda}| \leq md$.*

Proof. Suppose that $\lambda \vdash nd$ satisfies $\ell(\lambda) > m^2$ or $|\bar{\lambda}| > md$. We need to show that λ does not occur in $\mathbb{C}[Z_{n,m}]_d$. Due to Proposition 3.6 and (2.5), this amounts to show that $\langle v_T, p^{\otimes d} \rangle = 0$ for any tableau T of shape λ with content $d \times n$ and any $p \in Z_{n,m}$. So fix a tableau T of shape λ with content $d \times n$.

Assume first that $\ell := \ell(\lambda) > m^2$. We prove that $\langle v_T, p^{\otimes d} \rangle = 0$ for all polynomials $p \in \text{Sym}^m \mathbb{C}^{m^2}$ in at most m^2 variables. For this, it is enough to show that $\langle v_T, t \rangle = 0$ for all tensors $t = X_{s(1)} \otimes \dots \otimes X_{s(dn)}$, where $s: [dn] \rightarrow [m^2]$. The first column c of λ has ℓ boxes. We compose s with a bijection $\vartheta: \lambda \rightarrow [dn]$. Then the restriction of $s \circ \vartheta$ to the first column c is not injective since $\ell > m^2$. Hence $\text{sgn}(s \circ \vartheta)|_c = 0$, and Theorem 3.10 implies that indeed $\langle v_T, t \rangle = 0$.

Assume now $|\bar{\lambda}| > md$, so that $\lambda_1 < (n - m)d$. It is sufficient to prove that $\langle v_T, q^{\otimes d} \rangle = 0$ for all $q = X_1^{n-m} p$, where $p \in \text{Sym}^m \mathbb{C}^{n^2}$. We can express $q^{\otimes d}$ as a linear combination of tensors $t = X_{s(1)} \otimes \dots \otimes X_{s(dn)}$, where $s: [dn] \rightarrow [n^2]$ maps at least $(n - m)d$ elements to 1. Fix such a tensor t and consider a map $\vartheta: \lambda \rightarrow [dn]$. Since $\lambda_1 < (n - m)d$, λ has less than $(n - m)d$ columns. By the pigeonhole principle, there is a column c in which $s \circ \vartheta$ puts a 1 in at least two boxes. Therefore, $\text{sgn}(s \circ \vartheta)|_c = 0$ and Theorem 3.10 implies that indeed $\langle v_T, t \rangle = 0$. \square

4. LIFTING HIGHEST WEIGHT VECTORS IN PLETHYSMS

In the following we analyze two ways of ‘‘lifting’’ highest weight vectors in $\text{Sym}^d \text{Sym}^m V$ by raising either the *inner degree* m or the *outer degree* d . If d and m are sufficiently large in comparison with $\mu \vdash dm$, then these liftings provide isomorphisms of the spaces of highest weight vectors. In particular, the multiplicity $a_\mu(d[m])$ does not increase, which is known as the stability property of the plethysm coefficients [43, 13, 30]. A detailed understanding of the lifting in terms of highest

weight vectors is crucial for the proof of our main result and this is not considered in [43, 13, 30]. As a side result, we also obtain new proofs for the stability properties.

4(a). **Symmetric embeddings of tensor spaces.** Assume $V = \mathbb{C}^N$ and denote by X_1 the first standard basis vector of V . We consider the following linear map

$$M_m: \otimes^m V \rightarrow \otimes^{m+1} V, \quad v_1 \otimes \cdots \otimes v_m \mapsto \frac{1}{m+1} \sum_{i=0}^m v_1 \otimes \cdots \otimes v_{i-1} \otimes X_1 \otimes v_{i+1} \otimes \cdots \otimes v_m.$$

Using the cycles $\tau_i := (i+1 \ i+2 \ \cdots \ m+1)$, we can express M_m in the following way:

$$(4.1) \quad M_m(w) = \frac{1}{m+1} \sum_{i=0}^m \tau_i(w \otimes X_1), \quad \text{for } w \in \otimes^m V.$$

For $n \geq m$ we define the linear map $M_{m,n} := M_{n-1} \circ \cdots \circ M_{m+1} \circ M_m: \otimes^m V \rightarrow \otimes^n V$ by composition. Even though the maps $M_{m,n}$ are not equivariant, we will see that they have the nice property of mapping highest weight vectors to highest weight vectors.

Recall the projection $\Pi_m: \otimes^m \rightarrow \mathbf{Sym}^m V$. In the sequel, we write $\alpha + (k) := (\alpha_1 + k, \alpha_2, \dots, \alpha_N)$ for $\alpha \in \mathbb{Z}^N$ and $k \in \mathbb{Z}$. If μ is a partition and $k \geq 0$, this amounts to adding k boxes to the first row of the diagram of μ .

4.2. **Lemma.** (1) *The map $M_{m,n}$ is injective.*

(2) *A highest weight vector of weight $\mu \vdash m$ is mapped by $M_{m,n}$ to a highest weight vector of weight $\mu + ((n-m))$.*

(3) *We have $\Pi_n(w \otimes X_1^{\otimes(n-m)}) = M_{m,n}(\Pi_m(w))$ for $w \in \otimes^m V$. In particular, $M_{m,n}$ maps symmetric tensors to symmetric tensors. When interpreting the elements of $\mathbf{Sym}^m V$ as homogeneous polynomials, $M_{m,n}$ is the multiplication with the power X_1^{n-m} .*

Proof. It suffices to prove the assertion in the case $n = m+1$, i.e., for the map $M_m = M_{m,m+1}$.

1. We order the basis elements $X_{i_1} \otimes \cdots \otimes X_{i_m}$ of $\otimes^m V$ lexicographically, with $X_1^{\otimes m}$ being the smallest element. Then $M_m(X_{i_1} \otimes \cdots \otimes X_{i_m})$ equals a positive multiple of $X_1 \otimes X_{i_1} \otimes \cdots \otimes X_{i_m}$ plus a linear combination of strictly larger basis elements. Therefore, the matrix of M_m is lower triangular with nonzero diagonal elements, and hence of full rank.

2. The basis element $X_{i_1} \otimes \cdots \otimes X_{i_m}$ is a weight vector of weight $\alpha \in \mathbb{N}^N$, where α_k counts the number of occurrences of k in (i_1, \dots, i_m) . Clearly, $M_m(X_{i_1} \otimes \cdots \otimes X_{i_m})$ is a weight vector of weight $\alpha + (1)$.

Now we claim that $u(M_m(w)) = M_m(u(w))$ for $u \in U_N$ and $w \in \otimes^m V$. Indeed, it suffices to check this for $w = v_1 \otimes \cdots \otimes v_m$ of rank one. By (4.1), since $u\tau_i = \tau_i u$ and $u(X_1) = X_1$,

$$uM_m(w) = \frac{1}{m+1} \sum_i \tau_i(u(w) \otimes uX_1) = M_m(u(w)).$$

It follows that M_m maps U_N -invariant vectors to U_N -invariant vectors and the assertion follows.

3. Using (4.1) and $\cup_{i=0}^m \tau_i \mathfrak{S}_m = \mathfrak{S}_{m+1}$, we verify that $M_m(\Pi_m(w)) = \Pi_{m+1}(w \otimes X_1)$. \square

4(b). **Inner degree lifting.** For $n \geq m$ and $d \geq 1$ we consider the d -fold tensor power

$$\otimes^d M_{m,n}: \otimes^d \otimes^m V \rightarrow \otimes^d \otimes^n V$$

of the linear map $M_{m,n}$. Lemma 4.2 immediately implies the following.

4.3. **Lemma.** (1) *The linear map $\otimes^d M_{m,n}$ is injective.*

(2) *A highest weight vector of weight $\mu \vdash dm$ is mapped by $\otimes^d M_{m,n}$ to a highest weight vector of weight $\mu + (d(n-m))$.*

The restriction of $\bigotimes^d M_{m,n}$ yields an injective linear map

$$(4.4) \quad \kappa_{m,n}^d : \text{Sym}^d \text{Sym}^m V \rightarrow \text{Sym}^d \text{Sym}^n V$$

that maps highest weight vectors to highest weight vectors as in Lemma 4.3(2). We call $\kappa_{m,n}^d$ the *inner degree lifting* by $n - m$ and write $\kappa_m^d := \kappa_{m,m+1}^d$ for the lifting by 1.

Recall the generators v_T of $\text{Sym}^d \text{Sym}^m V$ labeled by tableaux T (Definition 3.3). We show now that $\kappa_{m,n}^d$ maps v_T to a generator $v_{T'}$, whose tableau T' arises from T in a simple way.

4.5. Theorem. *Let T be a tableau of shape μ with content $d \times m$ and let the tableau T' of shape $\mu' := \mu + (d(n - m))$ with content $d \times n$ be obtained from T by adding $n - m$ copies of each of the d letters in the first row (in some order). Then $\kappa_{m,n}^d(v_T) = v_{T'}$.*

Proof. Since $\kappa_{m,n}^d = \kappa_{n-1}^d \circ \cdots \circ \kappa_m^d$, it suffices to prove the assertion for a lifting by 1, i.e., $n = m + 1$ and $\mu' := \mu + (d)$.

We can write the symmetrizer $\Sigma_{d,m}$ of the wreath group $\mathfrak{S}_d \wr \mathfrak{S}_m$ (cf. (3.1)) as the composition $\Sigma_{d,m} = \Pi_{d,m} \circ \bigotimes^d \Pi_m$, where $\Pi_{d,m}$ denotes the symmetrizer of the subgroup isomorphic to \mathfrak{S}_d of the permutations in \mathfrak{S}_{dn} of the form $(u - 1)n + v \mapsto (\tau(u) - 1)n + v$ with $\tau \in \mathfrak{S}_d$, (simultaneous permutations of the blocks).

We denote by $\rho \in \mathfrak{S}_{d(n+1)}$ the permutation that merges the last d entries into the first d blocks, each at the end of the block, respectively. More specifically,

$$\rho(v_1 \otimes \cdots \otimes v_{dn} \otimes w_1 \otimes \cdots \otimes w_d) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes v_{n+1} \cdots \otimes v_{2n} \otimes w_2 \otimes \cdots \otimes v_{(d-1)n+1} \otimes \cdots \otimes v_{dn} \otimes w_d.$$

For example, for $n = 3, d = 2$, we have $\rho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 7 & 4 & 5 & 6 & 8 \end{pmatrix}$ and hence $\rho = (4567)$.

For $\pi \in \mathfrak{S}_{dn}$ we define $\pi' := \rho \tilde{\pi} \in \mathfrak{S}_{d(n+1)}$, where $\tilde{\pi}$ acts like π on the first dn factors. In the above example, for $\pi = (2453)$, we obtain $\pi' = \rho \tilde{\pi} = (253)(467)$.

We have $v_{\mu'} = v_{\mu} \otimes X_1^{\otimes d}$ and hence

$$(4.6) \quad \pi' v_{\mu'} = \rho \tilde{\pi} v_{\mu'} = \rho \tilde{\pi} (v_{\mu} \otimes X_1^{\otimes d}) = \rho((\pi v_{\mu}) \otimes X_1^{\otimes d}).$$

Suppose now that $T = T_{\mu}(\pi)$ for $\pi \in \mathfrak{S}_{dn}$ so that $v_T = \Sigma_{d,m}(\pi v_{\mu})$ (see Definition 3.3). The reader should verify that, by the construction of π' , we have $T' \simeq T_{\mu'}(\pi')$ (see Definition 3.5). Therefore, $v_{T'} = \Sigma_{d,m+1}(\bigotimes^d \Pi_{m+1})(v_{\mu'})$. We define

$$w_T := (\bigotimes^d \Pi_m)(\pi v_{\mu}), \quad w_{T'} := (\bigotimes^d \Pi_{m+1})(\pi' v_{\mu'})$$

and note that $v_T = \Pi_{d,m}(w_T)$ and $v_{T'} = \Pi_{d,m+1}(w_{T'})$. By Lemma 4.9 below, it is sufficient to prove that

$$(4.7) \quad (\bigotimes^d M_m)(w_T) = w_{T'}.$$

For showing this, we first claim that for $w \in \bigotimes^d \bigotimes^m V$, we have

$$(4.8) \quad (\bigotimes^d \Pi_{m+1})(\rho(w \otimes X_1^{\otimes d})) = (\bigotimes^d (M_m \circ \Pi_m))(w).$$

For verifying this, we may assume that $w = w_1 \otimes \cdots \otimes w_d$ with $w_i \in \bigotimes^m V$. By the definition of ρ , we have $\rho(w \otimes X_1^{\otimes d}) = w_1 \otimes X_1 \otimes \cdots \otimes w_d \otimes X_1$, and hence, using Lemma 4.2(3) for the second equality,

$$(\bigotimes^d \Pi_{m+1})(\rho(w \otimes X_1^{\otimes d})) = \bigotimes_{i=1}^d \Pi_{m+1}(w_i \otimes X_1) = \bigotimes_{i=1}^d M_m(\Pi_m(w_i)) = (\bigotimes_{i=1}^d (M_m \circ \Pi_m))(w),$$

and (4.8) follows.

Using (4.6) and (4.8) with $w = \pi v_{\mu}$, we argue now as follows:

$$w_{T'} = (\bigotimes^d \Pi_{m+1})(\pi' v_{\mu'}) = (\bigotimes^d \Pi_{m+1})\rho((\pi v_{\mu}) \otimes X_1^{\otimes d}) = (\bigotimes^d (M_m \circ \Pi_m))(\pi v_{\mu}) = (\bigotimes^d M_m)(w_T),$$

which shows (4.7) and completes the proof. \square

4.9. **Lemma.** *The following diagram is commutative:*

$$\begin{array}{ccc} \bigotimes^d \text{Sym}^m V & \xrightarrow{\bigotimes^d M_m} & \bigotimes^d \text{Sym}^{m+1} V \\ \Pi_{d,m} \downarrow & & \downarrow \Pi_{d,m+1} \\ \text{Sym}^d \text{Sym}^m V & \xrightarrow{\kappa_m^d} & \text{Sym}^d \text{Sym}^{m+1} V. \end{array}$$

Proof. We need to show that $\Pi_{d,m+1}(\bigotimes^d M_m(w)) = (\bigotimes^d M_m)(\Pi_{d,m}(w))$ for $w \in \bigotimes^d \text{Sym}^m V$. For this, we can assume w.l.o.g. that $w = w_1 \otimes \cdots \otimes w_d$ with $w_i \in \text{Sym}^m V$. Then we have

$$\frac{1}{d!} \sum_{\tau \in \mathfrak{S}_d} M_m(w_{\tau(1)}) \otimes \cdots \otimes M_m(w_{\tau(d)}) = (\bigotimes^d M_m) \left(\frac{1}{d!} \sum_{\tau \in \mathfrak{S}_d} w_{\tau(1)} \otimes \cdots \otimes w_{\tau(d)} \right)$$

and the assertion follows. \square

4.10. **Proposition.** (1) *Suppose $\mu \vdash md$ is such that $\mu_2 \leq m$ and let $n \geq m$. Then the lifting (4.4) defines an isomorphism*

$$(4.11) \quad \text{HWV}_\mu(\text{Sym}^d \text{Sym}^m V) \rightarrow \text{HWV}_{\mu+(d(n-m))}(\text{Sym}^d \text{Sym}^n V), f \mapsto \kappa_{m,n}^d(f).$$

(2) *Suppose that $\lambda \vdash nd$ satisfies $\lambda_2 \leq m$ and $\lambda_2 + |\bar{\lambda}| \leq md$. Then every highest weight vector of weight λ in $\text{Sym}^d \text{Sym}^n V$ is obtained by lifting a highest weight vector in $\text{Sym}^d \text{Sym}^m V$ of weight μ , where $\mu \vdash md$ such that $\bar{\mu} = \bar{\lambda}$.*

Proof. (1) Let T' be a tableau of shape $\mu + (d(n-m))$ with content $d \times n$ such that no letter appears more than once in a column. Then each of the d letters appears at least $n - \mu_2 \geq n - m$ times in singleton columns. Hence T' is obtained from a tableau T of shape μ with content $d \times m$ as in Theorem 4.5 (the order of the letters in the singleton columns is irrelevant, see Definition 3.5). Since $\kappa_{m,n}^d(v_T) = v_{T'}$ by Theorem 4.5, the assertion follows with Proposition 3.6.

(2) Note that $\lambda_2 + |\bar{\lambda}| \leq md$ is the number of boxes of λ that appear in columns that are not singleton columns. We can therefore shorten the given λ to a partition $\mu \vdash md$ by removing singleton columns. Then $\bar{\mu} = \bar{\lambda}$ and $\lambda = \mu + (d(n-m))$ and we conclude with part one. \square

We remark that the stability property of plethysms expressed in Proposition 4.10(1) was first shown in [43, Cor. 1.8] and [13] by different methods (not relying on the construction of highest weight vectors).

4(c). **A fundamental property of the inner degree lifting.** We will see that the inner degree lifting of symmetric tensors preserves the inner products in a certain way.

Recall that X_1, \dots, X_N denotes the standard basis of $V = \mathbb{C}^N$. The monomials X^α , for $\alpha \in C_N(m)$, form a basis of $\text{Sym}^m V$. For $n \geq m$ we define the following linear automorphism of $\text{Sym}^m V$ (called *scaling by $n-m$*):

$$(4.12) \quad S_{m,n}: \text{Sym}^m V \rightarrow \text{Sym}^m V, X^\alpha \mapsto \frac{\binom{n}{m}}{\binom{\alpha_1+n-m}{n-m}} X^\alpha.$$

In order to analyze this, let us consider the following linear map

$$\mathcal{M}_m: \text{Sym}^m V \rightarrow \text{Sym}^{m+1} V, p \mapsto X_1 \cdot S_{m,m+1}(p),$$

which consists of a scaling by 1, followed by a multiplication with X_1 . The result of composing k of these maps is $S_{m,m+k}$, followed by a multiplication with X_1^k . More specifically,

4.13. **Lemma.** *We have $X_1^k \cdot S_{m,m+k}(p) = \mathcal{M}_{m+k-1}(\cdots \mathcal{M}_{m+1}(\mathcal{M}_m(p)))$ for $k \geq 1$, $p \in \text{Sym}^m V$.*

Proof. This follows from

$$\frac{\binom{n}{n-m}}{\binom{\alpha_1+n-m}{n-m}} = \frac{n \cdot (n-1) \cdots (m+1)}{(\alpha_1+n-m) \cdot (\alpha_1+n-m-1) \cdots (\alpha_1+1)},$$

and $\mathcal{M}_m(X^\alpha) = \frac{m+1}{\alpha_1+1} X^{\alpha+(1)}$. \square

The degree $\deg_{X_1} q$ of $q \in \text{Sym}^n V$ is defined by interpreting q as a homogeneous polynomial in the variables X_1, \dots, X_N (put $\deg 0 := -\infty$). We can now state a fundamental property of the inner degree lifting.

4.14. Theorem. *Let $n \geq m$, $p \in \text{Sym}^m V$, $r \in \text{Sym}^n V$ such that $\deg_{X_1} r < n - m$. For all $f \in \text{Sym}^d \text{Sym}^m V$ we have*

$$\langle \kappa_{m,n}^d(f), q^{\otimes d} \rangle = \langle f, p^{\otimes d} \rangle, \quad \text{where } q = X_1^{n-m} \cdot S_{m,n}(p) + r.$$

Proof. Taking into account Lemma 4.13 and $\kappa_{m,n}^d = \kappa_{n-1}^d \circ \cdots \circ \kappa_m^d$, we easily see that it is sufficient to prove the assertion in the case $n = m + 1$. So it suffices to prove the following for $p \in \text{Sym}^m V$ and $r \in \text{Sym}^n V$ with $\deg_{X_0} \leq 0$:

$$(4.15) \quad \langle v_{T'}, q^{\otimes d} \rangle = \langle v_T, p^{\otimes d} \rangle, \quad \text{where } q = \mathcal{M}_m(p) + r,$$

since the v_T span $\text{Sym}^d \text{Sym}^m V$ (Proposition 3.6) and $\kappa_m^d(v_T) = v_{T'}$ by Theorem 4.5.

Recall the notation from Section 2 (a). Expanding p and using (2.1), we get ($c_\alpha \in \mathbb{C}$)

$$p = \sum_{\alpha \in C_N(m)} \binom{m}{\alpha} c_\alpha X^\alpha = \sum_{\alpha \in C_N(m)} c_\alpha \sum_{\zeta(I)=\alpha} X_I.$$

We take the d th tensor power and expand, obtaining

$$p^{\otimes d} = \sum_{A:[d] \rightarrow C_N(m)} C_A \sum_{\zeta(I^k)=A(k)} X_{I^1} \otimes \cdots \otimes X_{I^d},$$

where $C_A := \prod_{k=1}^d c_{A(k)}$ and the right-hand sum is over all d -tuples (I^1, \dots, I^d) of lists in $[N]^m$ such that $\zeta(I^k) = A(k)$ for all k . Therefore, using Theorem 3.10, we get

$$(4.16) \quad \begin{aligned} \langle v_T, p^{\otimes d} \rangle &= \sum_{A:[d] \rightarrow C_N(m)} C_A \sum_{\zeta(I^k)=A(k)} \langle v_T, X_{I^1} \otimes \cdots \otimes X_{I^d} \rangle \\ &= \sum_{A:[d] \rightarrow C_N(m)} C_A \frac{1}{d! m!^d} \sum_{\zeta(I^k)=A(k)} \sum_{\vartheta} \text{val}_\vartheta(I^1, \dots, I^d), \end{aligned}$$

where the last sum is over all bijections $\vartheta: \mu \rightarrow [dm]$ respecting T and (I^1, \dots, I^d) is interpreted as a single list of dm elements.

We first assume that $r = 0$ so that $q = \mathcal{M}_m(p)$. Then

$$q = \sum_{\alpha \in C_N(m)} \frac{m+1}{\alpha_1+1} \binom{m}{\alpha} c_\alpha X^{\alpha+(1)} = \sum_{\alpha \in C_N(m)} \binom{m+1}{\alpha+(1)} c_\alpha X^{\alpha+(1)} = \sum_{\alpha \in C_N(m)} c_\alpha \sum_{\zeta(J)=\alpha+(1)} X_J.$$

Taking the d th tensor power and expanding yields

$$q^{\otimes d} = \sum_{A:[d] \rightarrow C_N(m)} C_A \sum_{\zeta(J^k)=A(k)+(1)} X_{J^1} \otimes \cdots \otimes X_{J^d},$$

where the right-hand sum is over all d -tuples (J^1, \dots, J^d) of lists in $[N]^{m+1}$ such that $\zeta(J^k) = A(k) + 1$ for all k . Using Theorem 3.10, we get

$$(4.17) \quad \begin{aligned} \langle v_{T'}, q^{\otimes d} \rangle &= \sum_{A: [d] \rightarrow C_N(m)} C_A \sum_{\zeta(J^k) = A(k) + 1} \langle v_{T'}, X_{J^1} \otimes \cdots \otimes X_{J^d} \rangle \\ &= \sum_{A: [d] \rightarrow C_N(m)} C_A \frac{1}{d! (m+1)!^d} \sum_{\zeta(J^k) = A(k) + 1} \sum_{\vartheta'} \text{val}_{\vartheta'}(J^1, \dots, J^d), \end{aligned}$$

where the last sum is over all bijections $\vartheta': \mu' \rightarrow [d(m+1)]$ respecting T' , where we have set $\mu' := \mu + (d)$.

The following claim, together with (4.16) and (4.17), implies the asserted equality (4.15) in the special case where $r = 0$.

Claim 1. We have for all $A: [d] \rightarrow C_N(m)$

$$\sum_{\zeta(J^k) = A(k) + 1} \sum_{\vartheta'} \text{val}_{\vartheta'}(J^1, \dots, J^d) = (m+1)^d \sum_{\zeta(I^k) = A(k)} \sum_{\vartheta} \text{val}_{\vartheta}(I^1, \dots, I^d).$$

We prove now this claim. Suppose we have a bijection $\vartheta': \mu' \rightarrow [d(m+1)]$ respecting T' , and a tuple (J^1, \dots, J^d) of lists in $[N]^{m+1}$ such that $\text{val}_{\vartheta'}(J^1, \dots, J^d) \neq 0$. The d singleton boxes of T' that have been added to T are mapped to different blocks by ϑ' . Moreover, those boxes are mapped to positions that are assigned the value 1 by (J^1, \dots, J^d) . Now we remove these positions to obtain a new tuple (I^1, \dots, I^d) of lists. This process is best explained by an example. For instance, suppose that $d = 2, m = 3$, and $J^1 = (2, 1, 1, 2)$, $J^2 = (1, 1, 2, 2)$. Let \square_1 and \square_2 denote the singleton boxes added to T . Assume that $\vartheta'(\square_1) = 2$ and $\vartheta'(\square_2) = 5$. This means that \square_1 is mapped to the second position of the block $B_1 = \{1, 2, 3, 4\}$ and \square_2 is mapped to the first position of the block $B_2 = \{5, 6, 7, 8\}$. In both positions, J^1 and J^2 provide the value 1. After removing the two positions we get the lists $I^1 = (2, 1, 2)$, $I^2 = (1, 2, 2)$.

By adjusting for the positions shifts, we obtain from ϑ' an assignment $\vartheta: \mu' \rightarrow [dm]$ that clearly respects T . A moment's thought shows that the values are preserved: $\text{val}_{\vartheta'}(J^1, \dots, J^d) = \text{val}_{\vartheta}(I^1, \dots, I^d)$. In addition, if $\zeta(J^k) = A(k) + 1$, then $\zeta(I^k) = A(k)$.

Summarizing, we have set up a map from the set of pairs $(\vartheta', (J^1, \dots, J^d))$, such that ϑ' respects T' , $\text{val}_{\vartheta'}(J^1, \dots, J^d) \neq 0$, and $\zeta(J^k) = A(k) + 1$ for all k , to the set of pairs $(\vartheta, (I^1, \dots, I^d))$, such that ϑ respects T , $\text{val}_{\vartheta}(I^1, \dots, I^d) \neq 0$, and $\zeta(I^k) = A(k)$ for all k .

It is easily checked that that all the fibers of this map have the cardinality $(m+1)^d$. The reason for this is that in each of the d blocks, we have $m+1$ possibilities to insert a new position, which is then assigned the value 1 under the J^k to be defined. This completes the proof of Claim 1.

In order to provide the proof of (4.15) in the general case $r \neq 0$, we have to make some small modifications of the above reasonings. Let us call a map $A': [d] \rightarrow C_N(m+1)$ *relevant* iff all the $A'(k)$ encode a monomial divisible by X_1 , i.e., $A'(k)_1 \geq 1$. Otherwise, we call A' *irrelevant*. Put $q := \mathcal{M}_m(p) + r$. In analogy with (4.17), we see that there are coefficients $D_{A'} \in \mathbb{C}$ such that

$$\langle v_{T'}, q^{\otimes d} \rangle = \sum_{A': [d] \rightarrow C_N(m+1)} D_{A'} \sum_{\zeta(J^k) = A'(k)} \langle v_{T'}, X_{J^1} \otimes \cdots \otimes X_{J^d} \rangle,$$

where for the relevant A' , the $D_{A'}$ can be expressed in terms of the coefficients studied before: we have $D_{A'} = C_A$ where $A(k) = A'(k) - (1, 0, \dots, 0)$. In the special case $r = 0$, we even have $D_{A'} = 0$ for all irrelevant A' . While this is not true in the general case, the following claim holds. Note that this claim implies the assertion (4.15) and thus finishes the proof of the theorem.

Claim 2. Suppose that A' is irrelevant and $J^1, \dots, J^d \in [N]^{m+1}$ are such that $\zeta(J^k) = A'(k)$ for all k . Then $\langle v_{T'}, X_{J^1} \otimes \cdots \otimes X_{J^d} \rangle = 0$.

We show now Claim 2.

Since A' is irrelevant, we have $A'(k_0)_1 = 0$ for some k_0 . We expand $\langle v_{T'}, X_{J_1} \otimes \cdots \otimes X_{J_d} \rangle$ according to Theorem 3.10 and show that each of the resulting summands vanishes. Recall from Section 3 the decomposition of $[d(m+1)] = B_1 \cup B_2 \cup \cdots \cup B_d$ into blocks of size $m+1$. Take a bijection $\vartheta: \mu' \rightarrow [d(m+1)]$ respecting T' and suppose that a is the letter such that ϑ maps the boxes labelled with a to entries in the block B_{k_0} . By construction of T' , there exists a singleton column in T' with label a ; let \square denote the single box in this column. If $s: [d(m+1)] \rightarrow [N]$ denotes the list $(J_1^1, \dots, J_{m+1}^1, J_1^2, \dots, J_{m+1}^2, \dots, J_1^d, \dots, J_{m+1}^d)$, then $s(\vartheta(\square)) \neq 1$ since $\zeta(J^{k_0})_1 = A'(k_0)_1 = 0$. Hence the summand in Theorem 3.10 corresponding to ϑ vanishes (see the comment after (3.9)). \square

4.18. Remark. In the conference version of our paper at FOCS 2016, there is an error regarding the definition of the lifting map and the statement of Theorem 4.14. In fact, this theorem is unexpectedly subtle.

4(d). **Outer degree lifting.** We keep the notation $V = \mathbb{C}^N$ with the standard basis X_1, \dots, X_N from the previous section. Note that $X_1^m \in \text{Sym}^m V$ is a highest weight vector of weight (m) . Let $k \leq d$. By multiplying with the $(d-k)$ th power of X_1^m , we obtain the injective linear map

$$(4.19) \quad \text{Sym}^k \text{Sym}^m V \rightarrow \text{Sym}^d \text{Sym}^m V, f \mapsto (X_1^m)^{d-k} \cdot f.$$

(The multiplication is defined via the symmetric product, which corresponds to the multiplication of polynomials, cf. (2.2) with V replaced by $\text{Sym}^m V$.) It is clear that (4.19) maps a highest weight vector of weight $\nu \vdash mk$ to a highest weight vector of weight $\nu + ((d-k)m)$.

4.20. Lemma. *Let T be a tableau of shape ν with content $k \times m$ and let the tableau T'' of shape $\nu + ((d-k)m)$ be obtained from T by adding m copies of $d-k$ new letters to the first row (in some order). Then $v_{T''}$ is obtained as the image of v_T under the map (4.19)*

Proof. Let $\nu'' := \nu + ((d-k)m)$. We note that the lifting from degree k to d can be obtained as a composition of liftings that increase the degree by one only. Hence we may assume without loss of generality that $d = k+1$. So we assume that T'' is obtained from T by adding m copies of a new letter to the first row. In order to show that $v_{T''} = X_1^m \cdot v_T$, it suffices to prove that

$$(4.21) \quad \langle v_{T''}, t \rangle = \langle X_1^m \cdot v_T, t \rangle$$

for all rank one tensors $t = t_1 \otimes \cdots \otimes t_{k+1}$, where $t_i \in \otimes^m V$. We shall evaluate the inner products with Theorem 3.10. For this, consider the decomposition $[(k+1)m] = B_1 \cup \cdots \cup B_{k+1}$ into the blocks B_i of size m as in Section 3. For $1 \leq i \leq k+1$, let $\tau_i \in \mathfrak{S}_{(k+1)m}$ be the permutation that exchanges the blocks B_i and B_{k+1} and preserves the order within the blocks (note $\tau_{k+1} = \text{id}$).

Suppose that $\vartheta: \nu \rightarrow [km]$ respects T . We extend ϑ to a map $\tilde{\vartheta}: \nu'' \rightarrow [km+m]$ by sending the m boxes of $T'' \setminus T$ bijectively to the numbers in the block B_{k+1} . Clearly $\tilde{\vartheta}$ respects T'' . Let $1 \leq i \leq k+1$ and π be a permutation of B_{k+1} . Composing $\tilde{\vartheta}$ with $\tau_i \pi$, we obtain a map $\vartheta': \nu'' \rightarrow [km+m]$ respecting T'' . Moreover, any assignment ϑ'' respecting T'' arises this way from uniquely determined ϑ, i , and π .

Taking this observation into account, we deduce from Theorem 3.10 after some thought that:

$$\langle v_{T''}, t \rangle = \frac{1}{k+1} \left(\langle v_T, t_1 \otimes \cdots \otimes t_k \rangle \cdot \langle X_1^m, t_{k+1} \rangle + \sum_{i=1}^k \langle v_T, t_1 \otimes \cdots \otimes t_{i-1} \otimes t_{k+1} \otimes t_{i+1} \otimes \cdots \otimes t_k \rangle \cdot \langle X_1^m, t_i \rangle \right).$$

(Note that the first summand corresponds to $i = k+1$.) This equals $\langle X_1^m \cdot v_T, t \rangle$ by the definition of the symmetric product. \square

The outer degree lifting (4.19) behaves nicely with respect to highest weight vectors.

4.22. Proposition. (1) *Suppose $\nu \vdash mk$ such that $\nu_2 + |\bar{\nu}| \leq k$ and let $d \geq k$. The lifting (4.19) defines an isomorphism*

$$\text{HWV}_\nu(\text{Sym}^k \text{Sym}^m V) \rightarrow \text{HWV}_{\nu + ((d-k)m)}(\text{Sym}^d \text{Sym}^m V).$$

(2) *Suppose that f is a nonzero highest weight vector in $\text{Sym}^d \text{Sym}^m V$ of weight $\mu \vdash dm$ and assume that $\mu_2 + |\bar{\mu}| \leq k \leq d$ for some k . Then $\mu = \nu + ((d-k)m)$ for some $\nu \vdash mk$ and $f = (X_1^m)^{d-k} \cdot g$ for some $g \in \text{HWV}_\nu(\text{Sym}^k \text{Sym}^m V)$.*

Proof. (1) Let T'' be a tableau of shape $\mu := \nu + ((d-k)m)$ with content $d \times m$. Note that $\nu_2 + |\bar{\nu}|$ is the number of boxes in μ that are not singleton columns. Hence there are least $d - (\nu_2 + |\bar{\nu}|) \geq d - k$ many letters appearing in singleton columns of T'' only. Removing the $(d-k)m$ many boxes with these letters from the first row of μ leads to a tableau T of shape ν with content $k \times m$. We conclude with Lemma 4.20 and Proposition 3.6.

(2) There exists a tableau T'' of shape μ with content $d \times m$ by assumption and Proposition 3.6. As before, we see that there are at least $d - k$ many letters appearing in singleton columns of T'' only. In particular, we have $\mu = \nu + ((d-k)m)$ for some $\nu \vdash mk$. Now we apply part one. \square

We remark that the stability of plethysm in Proposition 4.22(1), for the slightly weaker condition $|\bar{\nu}| \leq k \leq d$, was first shown in [30] with a geometric method.

5. PROOF OF THEOREM 1.5

5(a). **Small degrees or extremely long first rows.** In the following we put $V := \mathbb{C}^{n^2}$. To warm up, we first show that (n) occurs in $\mathbb{C}[\Omega_n]_1$. Consider the linear map $\text{lc}: \text{Sym}^m V \rightarrow \mathbb{C}$ that assigns to $p = cX_1^m + \dots \in \text{Sym}^m V$ the coefficient c of p (where X_1 is the first variable). Then lc is a highest weight vector in $(\text{Sym}^n V)^*$ of weight (n) and $\text{lc}(X_1^n) = 1$. Since $X_1^n \in \Omega_n$, we see that (n) indeed occurs in $\mathbb{C}[\Omega_n]_1$.

The following result deals with the case of partitions $\lambda \vdash nd$ where d is small.

5.1. Proposition. *Let $\lambda \vdash nd$ be such that there exists a positive integer m satisfying $|\bar{\lambda}| \leq md$ and $md^2 \leq n$. Then every nonzero highest weight vector H of weight $-\lambda$ in $\text{Sym}^d(\text{Sym}^n V)^*$ does not vanish on Ω_n . In particular, if λ occurs in $\text{Sym}^d(\text{Sym}^n V)^*$, then λ occurs in $\mathbb{C}[\Omega_n]_d$.*

Proof. The case $d = 1$ is trivial as (n) occurs in $\mathbb{C}[\Omega_n]_1$, as noted before. So suppose $d \geq 2$.

Let $H \in \text{Sym}^d(\text{Sym}^n V)^*$ be a highest weight vector of weight $-\lambda$ and $h \in \text{Sym}^d \text{Sym}^n V$ be such that $H(p) = \langle h, p^{\otimes d} \rangle$ for all $p \in \text{Sym}^n V$, cf. (2.5). Then h is a highest weight vector of weight λ . We have $\lambda_2 \leq |\bar{\lambda}| \leq md$ and $\lambda_2 + |\bar{\lambda}| \leq 2|\bar{\lambda}| \leq 2md \leq md \cdot d$. Therefore, we are in the setting of Proposition 4.10(2) with respect to the lifting $\text{Sym}^d \text{Sym}^{md} V \rightarrow \text{Sym}^d \text{Sym}^n V$ (note that the inner degree on the left-hand side is md and not d). We conclude that h arises by an inner degree lifting from a highest weight vector $f \in \text{Sym}^d \text{Sym}^{md} V$ of weight λ ; so we have $h = \kappa_{md,n}^d(f)$. Suppose that $F \in \text{Sym}^d(\text{Sym}^{md} V)^*$ corresponds to f via (2.5). Recall the scaling map $S_{md,n}: \text{Sym}^{md} V \rightarrow \text{Sym}^{md} V$ from (4.12), which is a linear isomorphism.

By Corollary 2.7, there are $v_1, \dots, v_d \in V$ such that F does not vanish on

$$p := S_{md,n}^{-1}(v_1^{md} + \dots + v_d^{md}).$$

We apply now Theorem 4.14 with $q := X_1^{n-md} S_{md,n}(p) = X_1^{n-md}(v_1^{md} + \dots + v_d^{md})$, and obtain

$$H(q) = \langle h, q^{\otimes d} \rangle = \langle f, p^{\otimes d} \rangle = F(p) \neq 0.$$

By Theorem 2.8 we have $q \in \Omega_n$ since $n \geq md \cdot d$. Therefore, H does not vanish on Ω_n . \square

The next result deals with the extreme situation, where the body $\bar{\lambda}$ is very small compared with the size of λ . (In this situation, the splitting strategy in the proof of Proposition 5.5 below would fail.)

5.2. Proposition. *Let $\lambda \vdash nd$ and assume there exist positive integers s, m such that $\ell(\lambda) \leq m^2$, $\lambda_2 \leq s$, $m^2 s^2 \leq n$, and $m^2 s \leq d$. Then every nonzero $H \in \text{HWV}_{-\lambda}(\text{Sym}^d(\text{Sym}^n V)^*)$ of weight $-\lambda$ does not vanish on Ω_n .*

Proof. Suppose H corresponds to $h \in \text{HWV}_\lambda(\text{Sym}^d \text{Sym}^n V)$ via (2.5). We first consider the inner degree lifting $\text{Sym}^d \text{Sym}^s V \rightarrow \text{Sym}^d \text{Sym}^n V$, cf. (4.4). Since

$$\lambda_2 \leq s, \quad \lambda_2 + |\bar{\lambda}| \leq \lambda_2 + (\ell(\lambda) - 1)\lambda_2 = \ell(\lambda)\lambda_2 \leq m^2 s \leq d \leq ds,$$

the assumptions of Proposition 4.10(2) are satisfied and we conclude that h arises by lifting some $f \in \text{HWV}_\mu(\text{Sym}^d \text{Sym}^s V)$ with $\mu \vdash ds$ and $\bar{\mu} = \bar{\lambda}$.

By assumption, $k := m^2 s \leq d$. We continue with the outer degree lifting map $\text{Sym}^k \text{Sym}^s V \rightarrow \text{Sym}^d \text{Sym}^s V$, see (4.19). We have, using the above,

$$\mu_2 + |\bar{\mu}| = \lambda_2 + |\bar{\lambda}| \leq m^2 s = k,$$

hence the assumptions of Proposition 4.22(2) are satisfied and we have $f = (X_1^s)^{d-k} \cdot g$ for some highest weight vector $g \in \text{Sym}^k \text{Sym}^s V$ of weight $\nu \vdash ks$ such that $\bar{\nu} = \bar{\mu}$. Suppose g corresponds to $G \in \text{Sym}^k(\text{Sym}^s V)^*$. By Corollary 2.7, there are $v_1, \dots, v_k \in V$ such that G does not vanish on

$$p := S_{s,n}^{-1}(v_1^s + \dots + v_k^s)$$

and $\langle X_1^s, p \rangle \neq 0$. Applying Theorem 4.14, we obtain with $q := X_1^{n-s} S_{s,n}(p) = X_1^{n-s}(v_1^s + \dots + v_k^s)$ that

$$H(q) = \langle h, q^{\otimes d} \rangle = \langle f, p^{\otimes d} \rangle = \langle X_1^s, p \rangle^{d-k} \langle g, p^{\otimes k} \rangle \neq 0.$$

On the other hand, by Theorem 2.8, the padded polynomial $X_1^{n-s}(v_1^s + v_2^s + \dots + v_k^s)$ is contained in Ω_n , as $n \geq sk$ by assumption. Therefore, H does not vanish on Ω_n . \square

5(b). Building blocks and splitting technique. We construct as ‘‘building blocks’’ certain partitions that occur in $\mathbb{C}[\Omega_n]$. Using Theorem 2.8, this will boil down to prove that certain plethysms coefficients are nonzero. We achieve this by providing explicit tableau constructions and showing that the corresponding highest weight vectors do not vanish on a certain tensor.

In a first step we consider the case of even row length. Let $\lambda^{\sharp M}$ denote the partition $\lambda + (M - |\lambda|)$, which is λ with a prolonged first row so that $\lambda^{\sharp M}$ has M boxes.

5.3. Proposition. *Let N be even and $n \geq Nd$. Then $(d \times N)^{\sharp nd}$ occurs in $\mathbb{C}[\Omega_n]_d$.*

Proof. We use the construction from [7]. Let T denote the tableau of shape $d \times N$ with content $d \times N$ from Corollary 3.12. Let $h := \kappa_{N,n}^d(v_T) \in \text{Sym}^d \text{Sym}^n V$ denote the lifting of v_T , which is a highest weight vector of weight $(d \times N)^{\sharp nd}$. Note that by the definition (4.12) of the scaling map, we have for some $c_1 > 0$

$$p := S_{N,n}^{-1}(X_1^N + \dots + X_d^N) = c_1 X_1^N + X_2^N \dots + X_d^N.$$

Corollary 3.12 implies that $\langle v_T, (c_1 X_1^N + X_2^N + \dots + X_d^N)^{\otimes d} \rangle \neq 0$. Applying Theorem 4.14, we obtain with $q := X_1^{n-N} S_{N,n}(p) = X_1^{n-N}(c_1 X_1^N + X_2^N + \dots + X_d^N)$ that $\langle h, q^{\otimes d} \rangle = \langle v_T, p^{\otimes d} \rangle \neq 0$. By Theorem 2.8, we have $X_1^n \in \Omega_n$ since $n \geq dN$. Therefore, the polynomial $H \in \text{Sym}^d(\text{Sym}^n V)^*$ corresponding to h does not vanish on Ω_n and the assertion follows. \square

In order to handle partitions with odd parts, we use as further building blocks partitions obtained from rectangles by adding a single row and a single column.

We postpone the proof of the following technical result to Section 6. (It is based on an explicit construction of a highest weight vector.)

5.4. **Theorem.** *Let $2 \leq b, c \leq m^2$ and let $n \geq 24m^6$. Then there exists an even $i \leq 2m^4$, such that*

$$\lambda = b \times 1 + c \times i + 1 \times j$$

occurs in $\mathbb{C}[\Omega_n]_{3m^4}$ for $j = 3m^4n - b - ic$.

The splitting strategy in the following proof is a refinement of the one in [23]. The proof relies on the semigroup property stated in Lemma 3.7.

5.5. **Proposition.** *Given a partition λ with $|\lambda| = nd$ such that there exists $m \geq 2$ with $\ell(\lambda) \leq m^2$, $m^{10} \leq |\bar{\lambda}| \leq md$, $n \geq 24m^6$, and $d > 4m^6$. Then λ occurs in $\mathbb{C}[\Omega_n]_d$.*

Proof. Let $L := \ell(\lambda)$ and c_k denote the number of columns of length k in λ for $1 \leq k \leq L$. Let K be the index $k \geq 2$, for which c_k is maximal, i.e., $c_K = \max(c_k; k = 2, \dots, L)$. By assumption, we have $2 \leq K \leq m^2$ and

$$m^{10} \leq |\bar{\lambda}| = \sum_{k=2}^L (k-1)c_k \leq c_K \sum_{k=2}^L (k-1) \leq c_K \frac{L^2}{2} \leq c_K \frac{m^4}{2},$$

hence $c_K \geq 2m^6$.

The columns of odd length of λ need a special treatment: let S denote the set of integers $k \in \{2, \dots, L\}$ for which c_k is odd. For $k \in S$ we define the partition

$$\omega_k := k \times 1 + K \times i_k,$$

where the even integer $i_k \leq 2m^4$ is taken from Theorem 5.4, so that $\omega_k^{\#3nm^4}$ occurs in $\mathbb{C}[\Omega_n]_{3m^4}$. (Here we have used the assumption $n \geq 24m^6$.)

Assume first that $K \notin S$, that is, c_K is even. Then we can split λ vertically in rectangles as follows:

$$\begin{aligned} \lambda &= 1 \times c_1 + \sum_{\substack{k=2 \\ k \notin S \cup \{K\}}}^L k \times c_k + \sum_{\substack{k=2 \\ k \in S}}^L k \times c_k + K \times c_K \\ &= 1 \times c_1 + \sum_{\substack{k=2 \\ k \notin S \cup \{K\}}}^L k \times c_k + \sum_{\substack{k=2 \\ k \in S}}^L k \times (c_k - 1) + \sum_{k \in S} \omega_k + K \times \left(c_K - \sum_{k \in S} i_k \right). \end{aligned}$$

If, for $k \leq L$, we set $d_k := c_k$ if $k \notin S \cup \{K\}$ and $d_k := c_k - 1$ if $k \in S$, and define $d_K := c_K - \sum_{k \in S} i_k$, then the above can be briefly written as

$$(5.6) \quad \lambda = 1 \times c_1 + \sum_{k=2}^L k \times d_k + \sum_{k \in S} \omega_k.$$

By construction, all d_k are even. It is crucial to note that, using $i_k \leq 2m^4$,

$$d_K = c_K - \sum_{k \in S} i_k \geq c_K - (m^2 - 1) \cdot 2m^4 \geq c_K - 2m^6 \geq 0.$$

The last inequality is due to our observation at the beginning of the proof.

In the case where $K \in S$, we achieve the same decomposition as in (5.6) with the modified definition $d_K := c_K - 1 - \sum_{k \in S} i_k$. Here, as well $d_K \geq 0$ and all d_k are even.

We need to round down rational numbers to the next even number, so for $a \in \mathbb{Q}$ we define $\llbracket a \rrbracket := 2\lfloor a/2 \rfloor$. Note that $\llbracket a \rrbracket \geq a - 2$ for all $a \in \mathbb{Q}$. Hence $\llbracket n/k \rrbracket \geq n/k - 2 \geq 2$ for all $2 \leq k \leq m^2$, since $n \geq 4m^2$.

Using division with remainder, let us write $d_k = q_k \lfloor \frac{n}{k} \rfloor + r_k$ with $0 \leq r_k < \lfloor \frac{n}{k} \rfloor$. Then we split $k \times d_k = q_k(k \times \lfloor \frac{n}{k} \rfloor) + k \times r_k$. Since d_k is even and $\lfloor n/k \rfloor$ is even, r_k is even as well. From (5.6) we obtain that the partition

$$(5.7) \quad \mu := \sum_{k=2}^L q_k ((k \times \lfloor n/k \rfloor)^{\sharp nk}) + \sum_{k=2}^L (k \times r_k)^{\sharp nk} + \sum_{k \in S} \omega_k^{\sharp 3nm^4}$$

coincides with λ in all but possibly the first row.

Since $\lfloor n/k \rfloor \leq n/k$, $r_k \leq n/k$, and both $\lfloor n/k \rfloor$ and r_k are even, Proposition 5.3 implies that $(k \times \lfloor n/k \rfloor)^{\sharp nk}$ and $(k \times r_k)^{\sharp nk}$ occur as highest weights in $\mathbb{C}[\Omega_n]_k$. Moreover, Theorem 5.4 tells us that $\omega_k^{\sharp 3nm^4}$ occurs as a highest weight in $\mathbb{C}[\Omega_n]_{3m^4}$. The semigroup property implies that μ occurs in $\mathbb{C}[\Omega_n]$.

Claim. $|\mu| \leq dn$.

Let us finish the proof assuming the claim. If $|\mu| \leq dn$, we can obtain λ from μ by adding boxes to the first row of μ . Note that $|\lambda| - |\mu|$ is a multiple of n . Since $(n) \in \mathbb{C}[\Omega_n]$, the semigroup property implies that λ occurs in $\mathbb{C}[\Omega_n]_d$.

It remains to verify the claim. From (5.7) we get

$$|\mu| \leq \sum_{k=2}^L (q_k nk + nk + 3nm^4).$$

We have, using $\lfloor a \rfloor \geq a - 2$,

$$q_k \leq \frac{d_k}{\lfloor n/k \rfloor} \leq \frac{kd_k}{n - 2k}.$$

This implies

$$|\mu| \leq n \sum_{k=2}^L \left(\frac{k^2 d_k}{n - 2k} + k + 3m^4 \right).$$

Using $d_k \leq c_k$ and $L \leq m^2$, we get

$$|\mu| \leq n \sum_{k=2}^L \frac{m^2}{n - 2m^2} k c_k + n \sum_{k=2}^{m^2} k + 3nm^4(m^2 - 1).$$

Noting that $\sum_{k=2}^L k c_k = |\bar{\lambda}| + \lambda_2 \leq 2|\bar{\lambda}|$, we continue with

$$\begin{aligned} |\mu| &\leq \frac{nm^2}{n - 2m^2} \cdot 2|\bar{\lambda}| + n \left(\frac{m^2(m^2 + 1)}{2} + 3m^4(m^2 - 1) \right) \\ &\leq \frac{nm^2}{12m^6 - m^2} \cdot |\bar{\lambda}| + n \left(3m^6 - \frac{5}{2}m^4 + \frac{1}{2}m^2 \right), \end{aligned}$$

where we have used $n > 24m^6$ for the second inequality. Plugging in the assumptions $|\bar{\lambda}| \leq dm$ and $d > 4m^6$, we obtain

$$|\mu| \leq \frac{dnm^3}{11m^6} + 3nm^6 \leq \frac{dn}{11} + 3nm^6 \leq \frac{dn}{11} + \frac{3dn}{4} < dn,$$

which shows the claim and completes the proof. \square

We can now complete the proof of our main result.

Proof of Theorem 1.5. We may assume that $m \geq 2$, as the case $m = 1$ is trivial. Suppose that $\lambda \vdash nd$ occurs in $\mathbb{C}[Z_{n,m}]$ and $n \geq m^{25}$. Proposition 3.13 implies that $|\bar{\lambda}| \leq md$ and $\ell(\lambda) \leq m^2$.

In the case of ‘‘small degree’’, where $n \geq md^2$, Proposition 5.1 implies that λ occurs in $\mathbb{C}[\Omega_n]$.

So we may assume that $d > \sqrt{n/m}$. In this case we have $d \geq \sqrt{m^{25}/m} = m^{12}$. We conclude by two further case distinctions.

If $|\bar{\lambda}| < m^{10}$, we can apply Proposition 5.2 with $s := m^{10}$ since $\lambda_2 \leq |\bar{\lambda}| \leq s$, $m^2 s^2 = m^{22} \leq n$, and $m^2 s = m^{12} \leq d$. Thus λ occurs in $\mathbb{C}[\Omega_n]_d$.

Finally, if $|\bar{\lambda}| \geq m^{10}$, then the above Proposition 5.5 tells us that λ occurs in $\mathbb{C}[\Omega_n]_d$. \square

6. EXPLICIT CONSTRUCTIONS OF TABLEAUX AND POSITIVITY OF PLETHYSMS

The goal of this last section is to provide the proof of Theorem 5.4.

In order to motivate the construction, we begin with a general reasoning. Let T be a tableau of shape λ with content $d \times n$. For the sake of readability, we will use the natural numbers $1, \dots, d$ as letters. The set of boxes of T is partitioned as $C_1 \cup \dots \cup C_d$, where C_u denotes the set of boxes with the letter u . Note that $|C_u| = n$ for all u . We denote by C_u^1 the subset of C_u consisting of the boxes in singleton columns. On the other hand, the position set $[dn] = B_1 \cup \dots \cup B_d$ is partitioned into the blocks $B_u := \{(u-1)n + v \mid 1 \leq v \leq n\}$, where $|B_u| = n$ for all u .

We fix a map $s: [dn] \rightarrow [N]$, which defines the rank one tensor $t = X_{s(1)} \otimes \dots \otimes X_{s(dn)} \in \bigotimes^{dn} \mathbb{C}^N$. Depending on s , we denote by $B_u^1 := B_u \cap s^{-1}(1)$ the set of positions in block B_u that are mapped to 1 under the map s . (Hence the positions in B_u^1 are the ones mapped to the basis vector X_1 .)

Recall from Theorem 3.10 that $\langle v_T, t \rangle = (n!^d d!)^{-1} \sum_{\vartheta} \text{val}_{\vartheta}(s)$, where the sum is over all bijections $\vartheta: \lambda \rightarrow [dn]$ respecting T . The bijection ϑ respects the tableau T if there exists $\tau_{\vartheta} \in \mathfrak{S}_d$ such that $\vartheta(C_u) = B_{\tau_{\vartheta}(u)}$ for all u ; cf. Definition 3.8.

If $\text{val}_{\vartheta}(s) \neq 0$, then $s \circ \vartheta$ must map all boxes in singleton columns to 1. This means

$$(6.1) \quad \forall u \quad \vartheta(C_u^1) \subseteq B_{\tau_{\vartheta}(u)}^1.$$

For proving that $a_{\lambda}(d[n]) > 0$, we shall design T and s in such a way that $\text{val}_{\vartheta}(s) \geq 0$ for all ϑ and there are only few ϑ with $\text{val}_{\vartheta}(s) > 0$ (there must be at least one).

Part of the strategy for realizing this can be described as follows.

6.2. Claim. *Let T be tableau of shape λ and content $d \times n$, where C_u^1 denotes the set of boxes with letter u in the singleton columns of T . Let $s: [dn] \rightarrow [N]$ and recall $B_u^1 := B_u \cap s^{-1}(1)$. Assume there is an integer D with $\ell(\lambda) \leq D \leq d$ such that $|C_1^1| = |B_1^1|, \dots, |C_D^1| = |B_D^1| \leq n - 2$ are pairwise distinct numbers and $|C_u^1| > n - 2$ for $D < u \leq d$. Then for any $\vartheta: \lambda \rightarrow [dn]$ respecting T with $\text{val}_{\vartheta}(s) \neq 0$, we have $\tau_{\vartheta}(u) = u$ for all $1 \leq u \leq D$.*

Proof. Assume $\text{val}_{\vartheta}(s) \neq 0$ and write $\tau := \tau_{\vartheta}$. Then (6.1) implies $|C_u^1| \leq |B_{\tau(u)}^1|$ for all u . For $u > D$ we have $|C_u^1| > n - 2$, hence $\tau(u) > D$, since $|B_{u'}^1| \leq n - 2$ for $u' \leq D$. We conclude that τ permutes the set $[D]$. For $u \leq D$, by assumption, the cardinalities $w(u) := |C_u^1| = |B_u^1|$ are pairwise distinct and (6.1) gives $w(u) \leq w(\tau(u))$. This implies that $\tau(u) = u$ for $1 \leq u \leq D$. \square

By a concrete choice of a tableau T and map s we prove now the following.

6.3. Proposition. *Let $t \geq r$, $i \geq 2t + 3$ be positive integers and let $n \geq i$ and $d \geq 2t + i + 1$. Let $\nu = (t + 1) \times i + (r + 1) \times 1 + (j)$, where $j = dn - (t + 1)i - (r + 1)$. Then $a_{\nu}(d[n]) > 0$.*

Proof. We may assume that $n = i$ and $d = 2t + i + 1$ (see Lemma 4.3 and (4.19)).

Let T be a tableau of shape ν labeled with the integers $1, 2, 3, \dots, d$, each appearing n times, as explained in Figure 1 for the case $t = 5$, $r = 3$ and $i = 13$. Formally, if $1 \leq k \leq r$, the row $k + 1$ of T has $i + 1$ boxes: $k + 1$ boxes are labeled k , and the remaining $i - k$ boxes are labeled $2t + 1 - k$. If $r < k \leq t$, then the row $k + 1$ of T has i boxes: $k + 1$ boxes are labeled k and the remaining $i - k - 1$ boxes labeled $2t + 1 - k$. The first row of T starts with the first $i + 1$ boxes labeled with $2t + 1, \dots, d = 2t + i + 1$, respectively, and all the remaining j labels are put in the singleton columns of T such that each integer in $1, \dots, d$ appears exactly n times. Note that each integer $1, \dots, d$ appears in at least one singleton column, since $n \geq i \geq 2t + 3$.

														i							
														}							
														r							
														t							
11	12	13	14	15	16	17	18	19	20	21	22	23	24	1	1	1	...				
1	1	10	10	10	10	10	10	10	10	10	10	10	10								
2	2	2	9	9	9	9	9	9	9	9	9	9	9								
3	3	3	3	8	8	8	8	8	8	8	8	8	8								
4	4	4	4	4	7	7	7	7	7	7	7	7	7								
5	5	5	5	5	5	6	6	6	6	6	6	6	6								

FIGURE 1. Prop. 6.5: $t = 5$, $r = 3$, $i = 13$, $d = 24$, $n = 13$, $D = 10$, $dn = 312$, $j = 230$.

Put $D := 2t$. By construction, for any $1 \leq u \leq D$ in T , u appears in row 1 and in a unique row $k_u + 1$ for some $1 \leq k_u \leq t$. Let $\beta(u)$ denote the number of occurrences of u in row $k_u + 1$. Note that $2 \leq \beta(1) < \beta(2) < \dots < \beta(D)$ by construction. Using the notation introduced before the proof, we have by construction

$$(6.4) \quad |C_u^1| = n - \beta(u) \leq n - 2 \text{ for } 1 \leq u \leq D \quad \text{and} \quad |C_u^1| = n - 1 \text{ for } D < u \leq d.$$

We consider now the tensor

$$t = \bigotimes_{u=1}^D (X_{k_u+1}^{\otimes \beta(u)} \otimes X_1^{\otimes (n-\beta(u))}) \otimes \bigotimes_{u=D+1}^d X_1^n,$$

which, more precisely, is defined by the map, $s: [dn] \rightarrow [N]$,

$$s_{(u-1)n+v} = \begin{cases} k_u + 1 & \text{if } 1 \leq u \leq D \text{ and } 1 \leq v \leq \beta(u) \\ 1 & \text{otherwise.} \end{cases}$$

Using the notation introduced before the proof, we have $|C_u^1| = |B_u^1| = n - \beta(u)$ for $1 \leq u \leq D$. Hence, by (6.4), the assumptions of Claim 6.2 are satisfied. So if $\vartheta: \lambda \rightarrow [dn]$ respects T and gives a nonzero contribution to $\langle v_T, t \rangle$, then ϑ bijectively maps C_u^1 to B_u^1 for all $1 \leq u \leq D$ since $\tau_{\vartheta}(u) = u$, cf. (6.1). Hence a box \square with the label u , which is not in the first row of T , is mapped to a position in the u th block B_u . By the definition of s , this implies that $s(\vartheta(\square))$ equals the row number of \square . This also holds true for boxes in singleton columns of T . We conclude from Theorem 3.10 that ϑ contributes the value 1 to $\langle v_T, t \rangle$.

To complete the proof, it is sufficient to show the existence of some map $\vartheta: \lambda \rightarrow [dn]$ that respects T , which is now obvious. In fact, there are $(n!)^{n-D} (n-D)! \prod_{u=1}^D \beta(u)! \cdot (n - \beta(u))!$ many maps ϑ , that all contribute the value 1 to $\langle v_T, t \rangle$, since ϑ can permute within every label the X_{k_u+1} terms and the X_1 terms, and ϑ can as well permute the labels $D + 1, \dots, n$. \square

By generalizing this construction in the proof, we can show the following.

6.5. Proposition. *Let t, r be positive integers, $i \in [\frac{(r+2t)^2}{2t}, \frac{(r+2t)^2}{2t} + r + t + 1]$, and let $n > 6t + 2r$ and $d > r + 2t + i$. Let $\nu = (t + 1) \times i + (r + 1) \times 1 + (j)$, where $j = dn - (r + 1) - (t + 1)i$. Then $a_{\nu}(d[n]) > 0$.*

Proof. If $r < t$ then we can directly apply Proposition 6.3, noticing that

$$2t + 2 < \frac{(1 + 2t)^2}{2t} \leq \frac{(r + 2t)^2}{2t} \leq i \leq \frac{(t + 2t)^2}{2t} + r + t + 1 \leq \frac{11}{2}t + r + 1 \leq 6t + r \leq n.$$

Let now $r \geq t$. The proof is similar to the proof of Proposition 6.3, so we describe a more general construction which applies in the case $r < t$ as well. Define $e := 2(\lfloor (r - 1)/(2t) \rfloor + 1)$, so that

$r \leq te \leq r + 2t - 1$ and e is even. Put

$$i' := (te + 1)\frac{e}{2} \leq (r + 2t)\frac{e}{2} \leq (r + 2t)(\lfloor \frac{r-1}{2t} \rfloor + 1) \leq (r + 2t)\frac{(r + 2t - 1)}{2t} \leq i.$$

We will prove the statement for $i = i'$. When $i > i'$, the tableau construction below can be modified by increasing the number of appearances of the t largest labels by $i - i' \leq r + t$ in the subtableau T' as defined below. By assumption, $n > 6t + 2r \geq te + 2$ and $d > r + 2t + i \geq te + i + 1$. Indeed, we will prove the statement for the more general case in which we do not require $n > 6t + 2r$ and $d > r + 2t + i$, but only $n \geq te + 2$ and $d \geq te + i + 1$. It suffices to prove the statement with $n = te + 2$ and $d = te + i + 1$ (see Lemma 4.3 and (4.19)).

Let T be a tableau of shape ν filled with the labels $1, 2, 3, \dots, d = te + i + 1$, each number appearing $n = te + 2$ times, as in Figure 2 for the case $t = 2, r = 8, e = 4, i = 18, n = 10, d = 27$.

	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	1	1	1	...
r	8	1	4	4	4	4	5	5	5	5	5	8	8	8	8	8	8	8	8	t			
	7	2	2	3	3	3	6	6	6	6	6	6	7	7	7	7	7	7	7				
	6	i																					
	5																						
4																							
3																							
2																							
1																							

FIGURE 2. Prop. 6.5: $t = 2, r = 8, e = 4, i = 18, n = 10, d = 27$.

In the first row and in the first $i + 1$ columns we have the labels $te + 1, \dots, te + i + 1$. In the first column and in the rows 2 to $r + 1$ we have the labels $te, te - 1, \dots, te - r + 1$. The remaining rectangular $t \times i$ subshape of T , denoted T' , consisting of the columns 2 to $i + 1$ and the rows 2 to $t + 1$, is filled with the remaining labels $1, \dots, te$, so that each label appears a different number of times. More precisely, for each $1 \leq s \leq te$, let the label s appear in T' exactly s times and only in row $\min(\ell, 2t - \ell + 1)$, where $s \equiv \ell \pmod{2t}$, $1 \leq \ell \leq 2t$. (Note that the first row in T' , which we are referring to, is actually the second row in T .) So the row k of T' contains the e different labels $k, 2t + 1 - k, 2t + k, 4t + 1 - k, \dots, t(e - 2) + k, te + 1 - k$, each appearing that many times, adding up to the row length of

$$\sum_{\alpha=1}^{e/2} \left((2(\alpha - 1)t + k) + (2\alpha t + 1 - k) \right) = (te + 1)\frac{e}{2} = i.$$

The remaining labels of each kind are then put in the singleton boxes of T .

As in Proposition 6.3, we show that the corresponding highest-weight vector v_T in $\text{HWV}_\nu(\text{Sym}^d \text{Sym}^n V)$ is nonzero by taking the inner product with a particular monomial tensor t . For each label u , $1 \leq u \leq d$, let the associated monomial be

$$m_u = \bigotimes_{\square \in T, \text{label}(\square)=u} X_{\text{row}(\square)},$$

where the product goes over all boxes of T labeled u and for each such box we take the variable X whose index is the row the box is in. Again, let $t := \bigotimes_{u=1}^d m_u$ be the tensor. We evaluate $\langle v_T, t \rangle$ with Theorem 3.10. The crucial observation is again that the labels of T below row 1 all appear a different number of times, so each X_k appears in distinct nonzero degrees in the monomials m_u .

The rest of the proof of Proposition 6.3 applies almost verbatim, and we see that $\langle v_T, t \rangle \neq 0$: Again, as in the case of $r \leq t$, we see that all nonzero summands in Proposition 3.10 have the value 1 and there exists a nonzero summand that is trivial to construct. \square

Finally we can complete the proof of the promised technical result.

Proof of Theorem 5.4. We apply Proposition 6.5 with $r = b - 1 \leq m^2 - 1$ and $t = c - 1 \leq m^2 - 1$. We have

$$\frac{(r + 2t)^2}{2t} = \frac{(b + 2c - 3)^2}{2(c - 1)} \leq \max\left(\frac{(b + 1)^2}{2}, \frac{(b + 2m^2 - 3)^2}{2(m^2 - 1)}\right) \leq m^4,$$

where we use the fact that $(b + 2c - 3)^2/2(c - 1)$ is a convex function of c and so attains its maximum at the end points of the interval $[2, m^2]$. We can then find an even integer i in the interval $[\frac{(r+2t)^2}{2t}, \frac{(r+2t)^2}{2t} + r + t + 1] \subseteq [1, m^4 + 2m^2]$. By Proposition 6.5, there exists a nonzero highest weight vector f of weight $\nu = b \times 1 + c \times i + 1 \times j'$ in $\text{Sym}^d \text{Sym}^N V$ for

$$d := 3m^4 > 3m^2 + 2m^2 + m^4 \geq r + 2t + i, \quad N := 8m^2 > 6t + 2r.$$

By Corollary 2.7, the polynomial $F \in \text{Sym}^d(\text{Sym}^N V)^*$ corresponding to f does not vanish on $p := S_{N,n}^{-1}(v_1^N + \dots + v_d^N)$ for generic $v_i \in V$. Moreover, by Theorem 2.8, $q := X_1^{n-N}(v_1^N + \dots + v_d^N)$ is contained in Ω_n for all $n \geq d \cdot N$, in particular for $n \geq 24m^6$. Consider the lifting $h \in \text{Sym}^d \text{Sym}^n V$ of f ; it has the weight $\lambda = \nu + 1 \times d(n - N)$ with $dn = 3m^4 n$. Let $H \in \text{Sym}^d(\text{Sym}^n V)^*$ denote the polynomial corresponding to h . By Theorem 4.14 we have $H(q) = f(p) = F(p) \neq 0$. Therefore, λ occurs in $\mathbb{C}[\Omega_n]_{3m^4}$. \square

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