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Intrinsic volumes of symmetric cones and applications in convex programming

Dennis Amelunxen · Peter Bürgisser

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Abstract We express the probability distribution of the solution of a random (standard Gaussian) instance of a convex cone program in terms of the intrinsic volumes and curvature measures of the reference cone. We then compute the intrinsic volumes of the cone of positive semidefinite matrices over the real numbers, over the complex numbers, and over the quaternions in terms of integrals related to Mehta's integral. In particular, we obtain a closed formula for the probability that the solution of a random (standard Gaussian) semidefinite program has a certain rank.

Keywords random convex programs · semidefinite programming · intrinsic volumes · symmetric cones · Mehta's integral

Mathematics Subject Classification (2000) 15B48 · 52A55 · 53C65 · 60D05 · 90C22

1 Introduction

In modern convex optimization it is by now a widely accepted standard to formulate problems as *cone programs* [9]. In a cone program the task is to maximize a linear functional over the intersection of an affine subspace, given by a set of equations, with a certain cone, which we call the *reference cone.*

D. Amelunxen
School of Operations Research and Information Engineering, Cornell University
Tel.: 607 255 1525
E-mail: damelunx@gmail.com

P. Bürgisser
Institute of Mathematics, University of Paderborn, Germany
Tel.: +49 5251 60 2643
Fax: +49 5251 60 3516
E-mail: pbuerg@upb.de

This framework generalizes linear, second-order, and semidefinite programming, where the reference cone is chosen as the positive orthant, a product of Lorentz cones, and the cone of symmetric positive semidefinite matrices of a certain format, respectively. In fact, any convex program can be brought into this conic form.

A first step towards understanding the generic behavior of a cone program is to perform *average analyses* of this problem. That is, to analyze the probabilities for certain outcomes of a random cone program like infeasibility, unboundedness, or that the solution lies in a predefined region. Arguably, one of the most elementary probabilistic models for a cone program is to assume that the functional to be maximized and the equations given are i.i.d. standard Gaussian vectors. Assuming this probabilistic model, we will give concrete and simple formulas for the probability distribution of the solution of a random cone program in terms of certain invariants of the cone, called the *intrinsic volumes* and the *curvature measures*. In the case of linear programming (LP) this has been repeatedly done by various authors [39,10,1,27,37,12], but no extensions beyond the LP-case has been achieved so far. It should be noted that although the probabilistic model is rather restricted, the fact that our result holds for *any* reference cone makes this result applicable to *any* convex program. This is our first main result, cf. Theorem 3.1 and Theorem 3.2.

Our second main result concerns a particularly important class of reference cones, the *symmetric cones*, also known as self-scaled cones, i.e., closed convex cones which are self-dual and whose automorphism group acts transitively on the interior. Recall the well-known classification of these cones. It says that every symmetric cone is a direct product of the following basic families of symmetric cones:

- the Lorentz cones $\mathcal{L}^n := \{x \in \mathbb{R}^n \mid x_n \geq (x_1^2 + \ldots + x_{n-1}^2)^{1/2}\}$,
- the cones of positive semidefinite matrices over the real numbers, the complex numbers, or the quaternions,
- the single (exceptional, 27-dimensional) cone of 3×3 positive semidefinite matrices over the octonions.

This result follows from the theory of Jordan algebras, which is intimately related to the theory of symmetric cones, cf. [14]. Self-scaled cones form the basis of interior-point methods in convex optimization. This has been observed in the mid '90s, cf. [29,30,20,15], cf. also the book [32] and the survey article [21].

The intrinsic volumes and the curvature measures of the Lorentz cones are well understood, see for example [7, Ex. 2.15]. We give in this paper, apparently for the first time, an explicit formula for the intrinsic volumes of the cone of positive semidefinite matrices over the real numbers, over the complex numbers, and over the quaternions, cf. Theorem 4.1. The resulting formulas involve integrals that are related to Mehta's integral [28]. Moreover, we also give formulas for the curvature measures evaluated at the rank r-strata, so that we obtain a *closed formula for the probability that the solution of a random SDP has a certain rank.* To the best of our knowledge, this is the first result advancing with this question [3] dating from 1997.

Another interesting aspect, which deserves further investigation, is the observation that there seems to be a connection between the curvature measures of the cone of positive semidefinite matrices and the algebraic degree of semidefinite programming, cf. [31, 11].

The organization of the paper is as follows. In Section 2 we explain the notions of intrinsic volumes and curvature measures in the special case of polyhedral cones. We also state the spherical kinematic formula, which is the main integral geometric tool that we need for our first result. Section 3 states the applications to the probabilistic analysis of random convex programs with the corresponding proofs deferred to Section 5. Section 4 states explicit formulas for the curvature measures of SDP cones evaluated at the rank r-strata. The derivation of these formulas is the topic of Section 6.

This paper is an abridged version of [6] to which we will occasionally refer for integral geometric background or overly technical details of proofs.

2 Background from spherical convex geometry

2.1 Intrinsic volumes of polyhedral cones

Although intrinsic volumes and curvature measures are defined for general closed convex cones[1], we provide in this section characterizations of these quantities only for *polyhedral cones*. These are cones that arise as the intersection of finitely many closed half-spaces. The *polar* of a cone $C \subseteq \mathbb{R}^d$ is defined as $\breve{C} := \text{polar}(C) := \{x \in \mathbb{R}^d \mid \forall y \in C : \langle x, y\rangle \leq 0\}$. A cone is called *self-dual* if $\breve{C} = -C$.

If H is a supporting hyperplane of C, then we call $F = H \cap C$ a *face*[2] of C. Thus the faces are of the form $C \cap v^\perp$ for $v \in \breve{C}$, where $v^\perp := \{x \in \mathbb{R}^d \mid \langle x, v\rangle = 0\}$. The boundary of the cone C decomposes in the disjoint union of the relative interiors of its faces. More precisely, we have $C = \dot{\bigcup}_{F\in\mathcal{F}} F$, where $\mathcal{F} := \{\text{relint}(C \cap v^\perp) \mid v \in \breve{C}\}$. Let $\mathcal{F}_j := \{F \in \mathcal{F} \mid \dim(\text{span}\, F) = j\}$ denote the set of the (relative interiors of) j-dimensional faces of C for $j = 0, 1, \dots, d$.

Denoting by $\Pi_C \colon \mathbb{R}^d \to C$, $x \mapsto \text{argmin}\{\|x - y\| \mid y \in C\}$ the canonical projection on a polyhedral cone $C \subseteq \mathbb{R}^d$, the *intrinsic volumes* of C can be defined by

$$V_j(C) := \sum_{F\in\mathcal{F}_j} \operatorname*{Prob}_{x\in\mathcal{N}(0,I_d)} \bigl\{\Pi_C(x) \in F\bigr\}, \quad j = 0, 1, \dots, d, \tag{2.1}$$

where $\mathcal{N}(0, I_d)$ stands for the standard Gaussian distribution on $\mathbb{R}^d$. Note that $V_d(C) = \text{rvol}(C \cap S^{d-1})$ and $V_0(C) = \text{rvol}(\breve{C} \cap S^{d-1})$, rvol denoting the normalized volume, where $\text{rvol}(S^{d-1}) = 1$.

[1] In fact, intrinsic volumes are usually defined for intersections of convex cones with the unit sphere. We adopt the conical viewpoint for technical reasons, and also adopt a convenient shift in the indices of the intrinsic volumes compared to [18, 19, 35, 5].

[2] Some authors differentiate between faces and exposed faces, cf. for example [34]. We do not make this distinction as for the cones we are interested in both notions coincide.

In order to localize the intrinsic volumes, we denote by $\mathscr{B}(\mathbb{R}^d)$ the σ-algebra of Borel measurable sets in $\mathbb{R}^d$, and we define the subalgebra $\hat{\mathscr{B}}(\mathbb{R}^d) := \{M \in \mathscr{B}(\mathbb{R}^d) \mid \forall \lambda > 0 : \lambda M = M\}$ of conic Borel measurable sets. We define the jth *curvature measure* of a polyhedral cone $C \subseteq \mathbb{R}^d$ localized at $M \in \hat{\mathscr{B}}(\mathbb{R}^d)$ by

$$\Phi_j(C, M) := \sum_{F \in \mathcal{F}_j} \operatorname*{Prob}_{x \in \mathcal{N}(0, I_d)} \big\{ \Pi_C(x) \in F \cap M \big\}, \quad j = 0, 1, \ldots, d. \tag{2.2}$$

Note that $\Phi_d(C, M) = \operatorname{rvol}(C \cap M \cap S^{d-1})$ and

$$\Phi_0(C, M) = V_0(C) = \operatorname{rvol}(\breve{C} \cap S^{d-1}) \quad \text{if } 0 \in M, \tag{2.3}$$

and $\Phi_0(C, M) = 0$ otherwise. Moreover, we set $V_j(C) := 0$ and $\Phi_j(C, M) := 0$ for $j > d$.

These definitions could be extended to any closed convex cones by using an approximation procedure. A more convenient way is to use a spherical version of Steiner's formula for the volume of the tube around a convex set, cf. Proposition 6.1.

The following well-known facts about the intrinsic volumes and the curvature measures hold for any closed convex cone. They are easily verified for polyhedral cones.

Proposition 2.1 *Let $C \subseteq \mathbb{R}^d$ be a closed convex cone.*

1. *Interpreting C as a cone in $\mathbb{R}^{d'}$ with $d' \geq d$ does not change the intrinsic volumes nor the curvature measures. We have $V_j(\mathbb{R}^i) = \delta_{ij}$.*
2. *The intrinsic volumes and the curvature measures are nonnegative and satisfy $\sum_{j=0}^d V_j(C) = 1$.*
3. *$V_j(QC) = V_j(C)$ and $\Phi_j(QC, QM) = \Phi_j(C, M)$ for $Q \in O(d)$ (orthogonal invariance).*
4. *We have $V_j(C) = V_{d-j}(\breve{C})$.*
5. *$V_j(C_1 \times C_2) = \sum_{i=0}^j V_i(C_1) \cdot V_{j-i}(C_2)$ for closed convex cones C_1, C_2.*
6. *For $M \in \hat{\mathscr{B}}(\mathbb{R}^d)$ we have $\operatorname*{Prob}_{x \in \mathcal{N}(0, I_d)} \{\Pi_C(x) \in M\} = \sum_{j=0}^d \Phi_j(C, M)$.*
7. *For a linear subspace $W \subseteq \mathbb{R}^d$ of codimension m with orthogonal projection $\Pi_W \colon \mathbb{R}^d \to W$ we have $\Phi_j(\Pi_W(C), \Pi_W(M)) = \Phi_{j+m}(C + W^\perp, M + W^\perp)$ for $M \in \hat{\mathscr{B}}(\mathbb{R}^d)$.* □

Example 2.1 We have $V_0(\mathbb{R}_+) = V_1(\mathbb{R}_+) = \frac{1}{2}$. From (5) we get $V_j(\mathbb{R}_+^d) = \binom{d}{j}/2^d$ (d-fold convolution of the symmetric Bernoulli distribution).

Another important property of the intrinsic volumes states that for a closed convex cone $C \subseteq \mathbb{R}^d$:

$$V_1(C) + V_3(C) + V_5(C) + \ldots = \tfrac{1}{2} \cdot \chi(C \cap S^{d-1}), \tag{2.4}$$

where χ denotes the *Euler characteristic*, cf. [18, Sec. 4.3] or [35, Thm. 6.5.5]. Note that $\chi(C \cap S^{d-1}) = 1$ if C is not a linear subspace.

2.2 The kinematic formula

Kinematic formulas for Euclidean space are well documented, cf. the survey [25] and the references given therein. For our purposes we need the less known spherical kinematic formulas [35, §6.5].

The *Grassmann manifold* $\mathrm{Gr}^c_m(\mathbb{R}^d)$ consists of the linear subspaces of $\mathbb{R}^d$ with *co*dimension m. The uniform probability distribution on $\mathrm{Gr}^c_m(\mathbb{R}^d)$ is characterized as the unique probability distribution that is invariant under the action of the orthogonal group of $O(d)$. (The kernel of a $m \times d$ standard Gaussian matrix is uniform on $\mathrm{Gr}^c_m(\mathbb{R}^d)$.)

The following result is a consequence of a kinematic formula for spheres due to Glasauer [18], cf. [19] or [25, §2.4].

Theorem 2.1 (Kinematic formula) *Let $C \subseteq \mathbb{R}^d$ be a closed convex cone and $M \in \hat{\mathscr{B}}(\mathbb{R}^d)$. Fix $1 \le m \le d-1$ and let $W \subseteq \mathbb{R}^d$ be a uniformly random subspace of codimension m. Then the* random intersection $C \cap W$ *satisfies*

$$\mathbb{E}\big[\Phi_j(C \cap W, M \cap W)\big] = \Phi_{m+j}(C, M), \quad \text{for } j = 1, 2, \ldots, d-m, \tag{2.5}$$

$$\mathbb{E}\big[V_0(C \cap W)\big] = V_0(C) + V_1(C) + \ldots + V_m(C), \tag{2.6}$$

and for the random projection $\Pi_W(C)$ *we have*

$$\mathbb{E}\big[\Phi_j(\Pi_W(C), \Pi_W(M))\big] = \Phi_j(C, M), \quad \text{for } j = 0, 1, \ldots, d-m-1, \tag{2.7}$$

$$\mathbb{E}\big[V_{d-m}(\Pi_W(C))\big] = V_{d-m}(C) + V_{d-m+1}(C) + \ldots + V_d(C). \quad \square \tag{2.8}$$

Remark 2.1 A proof for (2.5) is contained in [35, §6.5], whereas the projection formula (2.7) is harder to trace in the literature. See [6, Appendix] for a detailed derivation of (2.7) from Glasauer's formula.

Corollary 2.1 *Let $C \subset \mathbb{R}^d$ be a closed convex cone, which is not a linear subspace. Then for $W \subseteq \mathbb{R}^d$ a uniformly random subspace of codimension m*

$$\operatorname{Prob}\big\{C \cap W = \{0\}\big\} = 2 \cdot \big(V_{m-1}(C) + V_{m-3}(C) + V_{m-5}(C) + \ldots\big).$$

Proof The Euler characteristic $\chi(C \cap W \cap S^{d-1})$ vanishes if $C \cap W = \{0\}$ and equals 1 otherwise, provided $C \cap W$ is not a linear subspace. Moreover, the intersection $C \cap W$ is almost surely not a linear subspace. Therefore,

$$\operatorname{Prob}\big\{C \cap W \ne \{0\}\big\} = \mathbb{E}\big[\chi(C \cap W \cap S^{d-1})\big] \overset{(2.4)}{=} 2 \cdot \sum_{j \text{ odd}} \mathbb{E}\big[V_j(C \cap W)\big].$$

Moreover, $\mathbb{E}\big[V_j(C \cap W)\big] = V_{m+j}(C)$ by (2.5). Taking into account that the intrinsic volumes with even/odd indices add up to $\frac{1}{2}$, the assertion follows. $\square$

3 Probability distributions of solutions of random convex programs

We consider the following forms of convex programming. Let $\mathcal{E}$ be a finite-dimensional Euclidean space with inner product $\langle .,. \rangle \colon \mathcal{E} \times \mathcal{E} \to \mathbb{R}$. Furthermore, let $C \subseteq \mathcal{E}$ be a closed convex cone. The classical convex programming problem (with *reference cone* C) has the inputs $a_1, \ldots, a_m, z \in \mathcal{E}$ and $b_1, \ldots, b_m \in \mathbb{R}$, and consists of the task

$$\begin{aligned} \text{maximize} \quad & \langle z, x \rangle \\ \text{subject to} \quad & \langle a_i, x \rangle = b_i, \quad i = 1, \ldots, m, \\ & x \in C, \end{aligned} \tag{CP}$$

which is to be solved in $x \in \mathcal{E}$. We also consider a homogeneous version, which is easier to analyze. It has only the inputs $a_1, \ldots, a_m, z \in \mathcal{E}$, and again is to be solved in $x \in \mathcal{E}$:

$$\begin{aligned} \text{maximize} \quad & \langle z, x \rangle \\ \text{subject to} \quad & \langle a_i, x \rangle = 0, \quad i = 1, \ldots, m, \\ & x \in C\,, \ \|x\| \le 1. \end{aligned} \tag{hCP}$$

The *(standard) normal distribution* $\mathcal{N}(\mathcal{E})$ is defined by requiring that the components of $x \in \mathcal{E}$ with respect to an orthonormal basis are i.i.d. standard normal.

Definition 3.1 We say that an instance of (hCP) is *standard Gaussian* if $a_1, \ldots, a_m, z$ are i.i.d. in $\mathcal{N}(\mathcal{E})$. An instance of (CP) is called *standard Gaussian* if, additionally, the random vector $(b_1, \ldots, b_m)$ is almost surely nonzero.

We call $\mathcal{F}(\mathrm{CP}) := \{x \in C \mid \forall i : \langle a_i, x \rangle = b_i\}$ the *feasible set* of (CP). The *value* of (CP) is $\mathrm{val}(\mathrm{CP}) := \sup\{\langle z, x \rangle \mid x \in \mathcal{F}(\mathrm{CP})\}$ and its *solution set* is defined as $\mathrm{Sol}(\mathrm{CP}) := \{x \in \mathcal{F}(\mathrm{CP}) \mid \langle z, x \rangle = \mathrm{val}(\mathrm{CP})\}$. Similar definitions apply to (hCP).

Note that val(hCP) is a maximum, as the set $\mathcal{F}(\mathrm{hCP})$ is compact and contains the origin. For the affine version (CP) this need not be the case. The feasible set $\mathcal{F}(\mathrm{CP})$ may be unbounded, and the value val(CP) may be ∞, in which case we say that (CP) is *unbounded*. Also, the feasible set $\mathcal{F}(\mathrm{CP})$ may be empty, so that $\mathrm{val}(\mathrm{CP}) = \sup \emptyset := -\infty$. In this case we say that (CP) is *infeasible*. If Sol(CP) consists of a single element x_0 only, then we write $x_0 = \mathrm{sol}(\mathrm{CP})$ (and we use a similar convention for Sol(hCP)). Well-known results from convex geometry, e.g. [34, Thm. 2.2.9], imply that almost surely Sol(hCP) and Sol(CP) are either empty or consist of single elements.

The first results of our paper describe the distribution of the solutions of (hCP) and (CP) in terms of curvature measures.

Theorem 3.1 *The probability distribution of the solution of a standard Gaussian instance of* (hCP) *is given by* $\mathrm{Prob}\left\{ \mathrm{sol}(\mathrm{hCP}) = 0 \right\} = \sum_{j=0}^{m} V_j(C)$ *and*

$$\mathrm{Prob}\left\{ \mathrm{sol}(\mathrm{hCP}) \in M \right\} = \textstyle\sum_{j=m+1}^{d} \Phi_j(C, M),$$

where $M \in \hat{\mathscr{B}}(\mathcal{E})$ *with* $0 \notin M$*. Furthermore, if* C *is not a linear subspace, then* $\mathrm{Prob}\{\mathcal{F}(\mathrm{hCP}) = \{0\}\} = 2\sum_j V_j(C)$ *where the sum is over all* $0 \le j \le m-1$ *such that* $j \equiv m-1 \bmod 2$*.*

Theorem 3.2 *The probability distribution of the solution of a standard Gaussian instance of* (CP) *is given by*

$$\mathrm{Prob}\{\mathrm{CP}\ \textit{infeasible}\} = \sum_{j=0}^{m-1} V_j(C), \quad \mathrm{Prob}\{\mathrm{CP}\ \textit{unbounded}\} = \sum_{j=m+1}^{d} V_j(C).$$

Furthermore, for $M \in \hat{\mathscr{B}}(\mathcal{E})$ *we have*

$$\mathrm{Prob}\{\mathrm{sol}(\mathrm{CP}) \in M\} = \Phi_m(C, M), \tag{3.1}$$

and $\mathrm{Prob}\{\mathrm{sol}(\mathrm{CP}) \in M \wedge \mathrm{val}(\mathrm{CP}) > 0\} = \mathrm{Prob}\{\mathrm{sol}(\mathrm{CP}) \in M \wedge \mathrm{val}(\mathrm{CP}) < 0\}$*.*

Example 3.1 The intrinsic volumes of the positive orthant $\mathbb{R}^d_+$ are given by the symmetric binomial distribution $V_j(\mathbb{R}^d_+) = \binom{d}{j}/2^d$, cf. Remark 2.1. Plugging this in Theorem 3.2 yields the corresponding probabilities for linear programming, which have already been computed in various places, cf. [39,10,1,27,37,12].

Remark 3.1 The random model (standard Gaussian) in Theorems 3.1 and 3.2 can be relaxed. In fact, the proofs only use the weaker assumptions that $(a_1^\perp \cap \ldots \cap a_m^\perp, z/\|z\|)$ induce the uniform distribution on the product $\mathrm{Gr}^c_m(\mathcal{E}) \times S(\mathcal{E})$ of the Grassmann manifold $\mathrm{Gr}^c_m(\mathcal{E})$ with the unit sphere $S(\mathcal{E})$.

3.1 Semidefinite programming

Throughout the paper we use the parameter $\beta \in \{1, 2, 4\}$ to indicate whether we are working over the real numbers $\mathbb{R}$, over the complex numbers $\mathbb{C}$, or over the quaternions $\mathbb{H}$. We denote the ground (skew) field by $\mathbb{F}_\beta$, i.e., $\mathbb{F}_1 := \mathbb{R}$, $\mathbb{F}_2 := \mathbb{C}$, and $\mathbb{F}_4 := \mathbb{H}$. In particular, $\mathbb{H}$ has the $\mathbb{R}$-basis $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfying the well-known quaternion multiplication rules. The real part of $z \in \mathbb{F}_\beta$ is given by $\Re(z) := (z + \bar{z})/2$, where $\bar{z}$ denotes the conjugation of z.

The space $\mathrm{Her}_{\beta,n} := \{A \in \mathbb{F}_\beta^{n\times n} \mid A^\dagger = A\}$ of $n \times n$-Hermitian matrices over $\mathbb{F}_\beta$ is a real vector space of dimension $d_{\beta,n} := n + \beta\binom{n}{2}$. Here $A^\dagger = (\bar{a}_{ji})$ for $A = (a_{ij})$. We regard $\mathrm{Her}_{\beta,n}$ as a Euclidean vector space with the inner product given by $A \cdot B := \Re(\mathrm{tr}(A^\dagger B))$, where $A, B \in \mathrm{Her}_{\beta,n}$, and $\mathrm{tr}(A)$ denotes the trace. The standard normal distribution in $\mathrm{Her}_{\beta,n}$ with respect to this inner product is called the *Gaussian Orthogonal/Unitary/Symplectic Ensemble* (GOE/GUE/GSE), briefly denoted GβE for $\beta = 1, 2, 4$.

The *cone of positive semidefinite matrices over* $\mathbb{F}_\beta$ defined as

$$\mathcal{C}_{\beta,n} = \{A \in \mathrm{Her}_{\beta,n} \mid \forall x \in \mathbb{F}_\beta^n : x^\dagger A x \ge 0\} \tag{3.2}$$

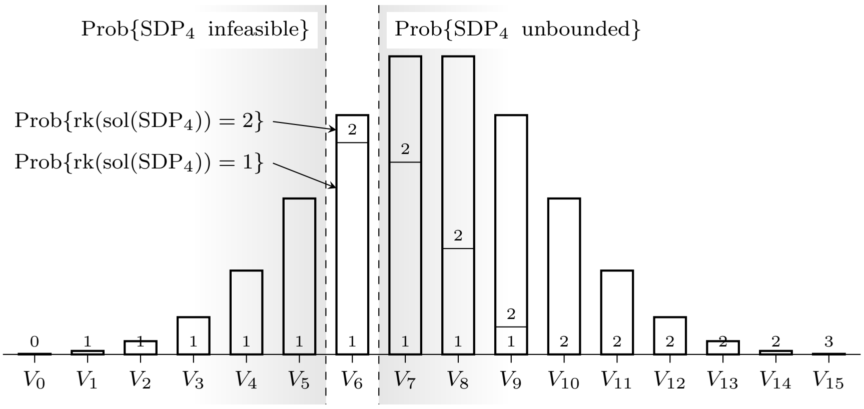


Fig. 3.1: The intrinsic volumes of $\mathcal{C}_{4,3}$ and their decompositions in curvature measures. The small numbers indicate the contributions of the ranks. The probabilities from Corollary 3.1 are indicated for $m = 6$.

is self-dual, i.e., $\mathrm{polar}(\mathcal{C}_{\beta,n}) = -\mathcal{C}_{\beta,n}$, cf. [8, §II.12]. The cone $\mathcal{C}_{\beta,n}$ has a natural decomposition according to the rank of the matrices: $\mathcal{C}_{\beta,n} = \bigcup_{r=0}^{n} \mathcal{W}_{\beta,n,r}$, with $\mathcal{W}_{\beta,n,r} := \{A \in \mathcal{C}_{\beta,n} \mid \mathrm{rk}\, A = r\}$, cf. [41] for the quaternion case. For the jth curvature measure of $\mathcal{C}_{\beta,n}$ evaluated at the set of its rank r-matrices we write $\Phi_j(\beta,n,r) := \Phi_j(\mathcal{C}_{\beta,n}, \mathcal{W}_{\beta,n,r})$. The decomposition of the cone $\mathcal{C}_{\beta,n}$ into the rank r-strata yields $V_j(\mathcal{C}_{\beta,n}) = \sum_{r=0}^{n} \Phi_j(\beta,n,r)$ for $j = 0,\dots,d_{\beta,n}$.

The semidefinite programming task (SDP_β) stands for the task (CP) of convex programming for the cone $C = \mathcal{C}_{\beta,n}$ in $\mathcal{E} = \mathrm{Her}_{\beta,n}$.

Specializing Theorem 3.2 immediately implies the following result.

Corollary 3.1 *The probability distribution of the solution of a standard Gaussian instance of* (SDP_β) *is given by* $\mathrm{Prob}\{\mathrm{rk}(\mathrm{sol}(\mathrm{SDP}_\beta)) = r\} = \Phi_m(\beta,n,r)$ *for* $0 \le r \le n$.

See Theorem 4.1 for explicit formulas for $\Phi_j(\beta,n,r)$. Figure 3.1 illustrates the case $\beta = 4$, $n = 3$, $m = 6$. We note that the self-duality of $\mathcal{C}_{\beta,n}$ and Proposition 2.1(5) imply $V_j(\mathcal{C}_{\beta,n}) = V_{d_{\beta,n}-j}(\mathcal{C}_{\beta,n})$.

4 Curvature Measures of SDP cones

We state the curvature measures $\Phi_j(\beta,n,r)$ in terms of certain integrals for which we need to introduce some notation first. Consider the Vandermonde determinant $\Delta(z) := \prod_{1\le i<j\le n}(z_i - z_j)$ for $z = (z_1,\dots,z_n)$. For $0 \le r \le n$ let $x := (z_1,\dots,z_r)$ and $y := (z_{r+1},\dots,z_n)$, so that $z = (x,y)$. This yields the decomposition

$$\Delta(z)^\beta = \Delta(x)^\beta \cdot \Delta(y)^\beta \cdot \prod_{i=1}^{r}\prod_{j=1}^{n-r}(x_i - y_j)^\beta. \tag{4.1}$$

We regard the rightmost factor in (4.1) as a polynomial in x and decompose it into its homogeneous parts. For convenience, we change the sign, and define

$$f_{\beta,k}(x;y) := \left(\text{the } x\text{-homog. part of } \prod_{i=1}^{r}\prod_{j=1}^{n-r}(x_i+y_j)^\beta \text{ of degree } k\right). \quad (4.2)$$

See (6.19) for a more explicit formula for $f_{\beta,k}(x;y)$. The Vandermonde determinant thus decomposes as $\Delta(z)^\beta = \Delta(x)^\beta \cdot \Delta(y)^\beta \cdot \sum_{k=0}^{\beta r(n-r)} f_{\beta,k}(x;-y)$.

Definition 4.1 We define for $0 \le r \le n$ and $0 \le k \le \beta r(n-r)$ the integrals

$$J_\beta(n,r,k) := \frac{1}{(2\pi)^{n/2}} \cdot \int_{z\in\mathbb{R}^n_+} e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(x)|^\beta \cdot |\Delta(y)|^\beta \cdot f_{\beta,k}(x;y)\, dz, \quad (4.3)$$

where $z = (x,y)$ with $x \in \mathbb{R}^r$, $y \in \mathbb{R}^{n-r}$, and $\mathbb{R}^n_+$ denotes the positive orthant in $\mathbb{R}^n$. We set $J_\beta(n,r,k) := 0$, if $k < 0$ or $k > \beta r(n-r)$.

Exchanging the roles of x and y yields the following symmetry relation

$$J_\beta(n,r,k) = J_\beta(n,n-r,\beta r(n-r)-k). \quad (4.4)$$

For $r \in \{0,n\}$ and $k = 0$ we obtain the integrand $e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(z)|^\beta$, which also appears in *Mehta's integral* (cf. [16] and the references therein)

$$F_n(\beta/2) := \frac{1}{(2\pi)^{n/2}} \cdot \int_{z\in\mathbb{R}^n} e^{-\frac{\|z\|^2}{2}}\, |\Delta(z)|^\beta\, dz = \prod_{j=1}^{n} \frac{\Gamma(1+\frac{j\beta}{2})}{\Gamma(1+\frac{\beta}{2})}. \quad (4.5)$$

It is well-known that the distribution of the joint probability density function for the eigenvalues of matrices from GβE is given by (cf. [16])

$$\frac{1}{(2\pi)^{n/2} F_n(\beta/2)} \cdot e^{-\frac{\|z\|^2}{2}} \cdot |\Delta(z)|^\beta\,. \quad (4.6)$$

Using this, we see that one can write the integrals $J_\beta(n,r,k)$ succinctly as expected values: choosing $A \in$ GβE(r) and $B \in$ GβE$(n-r)$,

$$J_\beta(n,r,k) = F_r(\beta/2) \cdot F_{n-r}(\beta/2) \cdot \mathbb{E}\big[\mathbf{1}_+(A) \cdot \mathbf{1}_+(B) \cdot f_{\beta,k}(A;B)\big],$$

where $\mathbf{1}_+(A) = 1$ if A is positive semidefinite and 0 otherwise, and $f_{\beta,k}(A;B)$ denotes the evaluation of $f_{\beta,k}$ at the eigenvalues of A and B.

Recall that $\Phi_j(\beta,n,r) = \Phi_j(\mathcal{C}_{\beta,n}, \mathcal{W}_{\beta,n,r})$ denotes the jth curvature measures of $\mathcal{C}_{\beta,n}$ evaluated at the set $\mathcal{W}_{\beta,n,r}$ of its rank r matrices. Recall also that $d_{\beta,n} = n + \beta\binom{n}{2}$. The following theorem is another main result of this paper.

Theorem 4.1 *Let $\beta \in \{1,2,4\}$ and $0 \le r \le n$. We have for $0 \le j \le d_{\beta,n}$*

$$\Phi_j(\beta,n,r) = \binom{n}{r} \cdot \frac{J_\beta(n,r,j-d_{\beta,r})}{F_n(\beta/2)}, \quad (4.7)$$

where $J_\beta(n,r,k)$ and $F_n(\beta/2)$ are defined in (4.4) *and* (4.5)*, respectively.*

$\beta \backslash j$	1	2	3	4	5	6	7	8	9
1	$\frac{\sqrt{2}}{4} - \frac{1}{4}$	$\frac{\sqrt{2}}{2\pi}$	$\frac{1}{2} - \frac{\sqrt{2}}{4}$	0	0	0	0	0	0
2	$\frac{3}{16} - \frac{1}{2\pi}$	$\frac{1}{4\pi}$	$\frac{1}{2\pi}$	$\frac{1}{2\pi}$	$\frac{3}{16} - \frac{3}{8\pi}$	0	0	0	0
4	$\frac{11}{64} - \frac{8}{15\pi}$	$\frac{1}{40\pi}$	$\frac{4}{15\pi} - \frac{1}{16}$	$\frac{19}{120\pi}$	$\frac{3}{32}$	$\frac{2}{5\pi}$	$\frac{1}{16} + \frac{1}{6\pi}$	$\frac{1}{5\pi}$	$\frac{7}{64} - \frac{7}{24\pi}$

Table 4.1: The values of $\Phi_j(\beta, 3, 1)$.

See Table 4.1 for some values of $\Phi_j(\beta, n, r)$; the computation of these values is explained in [6, §3.4].

Remark 4.1 1. We have $\Phi_j(\beta, n, r) > 0$ iff $d_{\beta,r} \leq j \leq d_{\beta,r} + \beta r(n-r)$. This is closely related to *Pataki's inequalities* (for $\beta = 1$), cf. [3,31], which state that the rank r of the solution of a generic instance of (SDP$_\beta$) almost surely satisfies $d_{\beta,r} \leq m \leq d_{\beta,r} + \beta r(n-r)$.
2. The relation (4.4) implies $\Phi_j(\beta, n, r) = \Phi_{d_{\beta,n}-j}(\beta, n, n-r)$, which is a refinement of the duality relation $V_j(\mathcal{C}_{\beta,n}) = V_{d_{\beta,n}-j}(\mathcal{C}_{\beta,n})$.
3. Both of the above properties of $\Phi_j(\beta, n, r)$ also hold for the *algebraic degree of semidefinite programming*, cf. [31, Prop. 9]. There should be a deeper reason for this coincidence that would be interesting to explore.

5 Proof of Theorems 3.1 and 3.2

In this section we adopt the following convention. Suppose we want to maximize a function f over a set M. Putting $m := \sup\{f(y) \mid y \in M\}$, we write $\operatorname{Argmax}\{f(x) \mid x \in M\} := \{x \in M \mid f(x) = m\}$. If this set consists of a single element only, then we denote it by $\operatorname{argmax}\{f(x) \mid x \in M\}$. Similarly for Argmin and argmin.

5.1 The homogeneous problem (hCP)

We will see that the homogeneous case (hCP) is easily reformulated in such a way that the kinematic formula yields the proof of Theorem 3.1. The key observation is made in the following simple lemma, which is verified easily.

Lemma 5.1 *Let $C \subseteq \mathbb{R}^d$ be a closed convex cone and let $B \subset \mathbb{R}^d$ denote the closed unit ball. Then for $v \in \mathbb{R}^d \setminus \{0\}$*

$$\operatorname{argmax}\{\langle v, x\rangle \mid x \in C \cap B\} = \begin{cases} \|\Pi_C(v)\|^{-1} \cdot \Pi_C(v) & \text{if } v \notin \breve{C} \\ 0 & \text{if } v \in \operatorname{int}(\breve{C}). \end{cases} \qquad \square$$

The problem (hCP) can now be phrased in the following form: We have the closed convex cone C in d-dimensional Euclidean space $\mathcal{E}$. This cone is intersected with the closed unit ball $B := \{x \in \mathcal{E} \mid \|x\| \leq 1\}$ and with the linear subspace $W := \{x \in \mathcal{E} \mid \langle a_1, x\rangle = \ldots = \langle a_m, x\rangle = 0\}$. In other words, we have

$$\mathcal{F}(\text{hCP}) = C \cap W \cap B.$$

If the a_i are from the standard normal distribution $\mathcal{N}(\mathcal{E})$ then W has almost surely codimension m, and W is uniformly distributed among all $(d-m)$-dimensional subspaces of $\mathcal{E}$. So we may assume w.l.o.g. that W is a uniformly random $(d-m)$-dimensional subspace of $\mathcal{E}$.

Proof (Theorem 3.1) In (hCP) we may replace z by its orthogonal projection $\bar{z}$ to W, as this does not change the value of the functional $\langle z, \cdot\rangle$ on W. For fixed W we thus obtain a conditional distribution for $\bar{z}$, which, by the well-known properties of the normal distribution, is again the standard normal distribution (on W). Hence the probability that the origin is the solution of (hCP) is given in the following way

$$\begin{aligned}\operatorname*{Prob}_{a_1,\ldots,a_m,z}\{\text{sol(hCP)} = 0\} &= \mathop{\mathbb{E}}_{W}\Big[\operatorname*{Prob}_{\bar{z}}\big\{\operatorname{argmax}\{\langle \bar{z}, x\rangle \mid x \in W \cap C \cap B\} = 0\big\}\Big] \\ &\overset{\text{Lem. 5.1}}{=} \mathop{\mathbb{E}}_{W}\Big[\operatorname*{Prob}_{\bar{z}}\big\{\bar{z} \in \operatorname{polar}(W \cap C)\big\}\Big] \overset{(2.3)}{=} \mathop{\mathbb{E}}_{W}\big[V_0(W \cap C)\big] \overset{(2.6)}{=} \sum_{j=0}^{m} V_j(C),\end{aligned}$$

which shows the first claim in Theorem 3.1. As for the second claim in Theorem 3.1, let Π_{C_W} denote the projection onto $C_W := C \cap W$. Then we obtain for $M \in \hat{\mathscr{B}}(\mathcal{E})$ such that $0 \notin M$,

$$\begin{aligned}\operatorname*{Prob}_{a_1,\ldots,a_m,z}\big\{\text{sol(hCP)} \in M\big\} &= \mathop{\mathbb{E}}_{W}\Big[\operatorname*{Prob}_{\bar{z}}\big\{\operatorname{argmax}\{\langle \bar{z}, x\rangle \mid x \in W \cap C \cap B\} \in M\big\}\Big] \\ &\overset{\text{Lem. 5.1}}{=} \mathop{\mathbb{E}}_{W}\Big[\operatorname*{Prob}_{\bar{z}}\big\{\Pi_{C_W}(\bar{z}) \in M\big\}\Big].\end{aligned}$$

For fixed W we have by Proposition 2.1(6)

$$\operatorname*{Prob}_{\bar{z}}\big\{\Pi_{C_W}(\bar{z}) \in M\big\} = \sum_{j=1}^{d-m} \Phi_j(C \cap W, M).$$

For random W we may apply the kinematic formula and continue with

$$\operatorname*{Prob}_{a_1,\ldots,a_m,z}\big\{\text{sol(hCP)} \in M\big\} = \sum_{j=1}^{d-m} \mathop{\mathbb{E}}_{W}\left[\Phi_j(C \cap W, M)\right] \overset{(2.5)}{=} \sum_{j=1}^{d-m} \Phi_{j+m}(C, M),$$

which shows the second claim in Theorem 3.1.

Finally, if the cone C is not a linear subspace, we note that $\mathcal{F}(\text{hCP}) = \{0\}$ iff $C \cap W = \{0\}$ and we conclude with Corollary 2.1. $\square$

5.2 The inhomogeneous problem (CP)

The geometric interpretation of (CP) is slightly more complicated than in the homogeneous case (hCP). The key observation is in the following lemma, which reduces the d-dimensional to the 2-dimensional case.

Lemma 5.2 *Let $v, w \in \mathbb{R}^d \setminus \{0\}$ be such that $\langle v, w\rangle = 0$, let $L := \operatorname{span}\{v, w\}$ denote the plane spanned by v and w, and consider the affine hyperplane $W_{\text{aff}} := \{x \in \mathbb{R}^d \mid \langle w, x\rangle = 1\}$. Further, let $\Pi_L \colon \mathbb{R}^d \to L$ denote the orthogonal projection onto L. For a closed convex cone $C \subseteq \mathbb{R}^d$ we have*

$$\sup\{\langle v, x\rangle \mid x \in C \cap W_{\text{aff}}\} = \sup\{\langle v, x\rangle \mid x \in \Pi_L(C) \cap W_{\text{aff}}\}\ .$$

Moreover, $M := \operatorname{Argmax}\{\langle v, x\rangle \mid x \in C \cap W_{\text{aff}}\}$ and $M_L := \operatorname{Argmax}\{\langle v, x\rangle \mid x \in \Pi_L(C) \cap W_{\text{aff}}\}$ are related by $M = C \cap \Pi_L^{-1}(M_L)$.

Proof Let $x \in \mathbb{R}^d$ be decomposed in $x = x_1 + x_2$ with $x_1 \in L$ and $x_2 \in L^\perp$, i.e., $x_1 = \Pi_L(x)$. Then we have $\langle v, x\rangle = \langle v, x_1\rangle + \langle v, x_2\rangle = \langle v, x_1\rangle$, and similarly $\langle w, x\rangle = \langle w, x_1\rangle$. This implies $\sup\{\langle v, x\rangle \mid x \in C\ ,\ \langle w, x\rangle = 1\} = \sup\{\langle v, x_1\rangle \mid x_1 \in \Pi_L(C)\ ,\ \langle w, x_1\rangle = 1\}$. Analogously, we obtain the second claim. □

We now discuss the 2-dimensional case. For convenience we assume that v, w are normalized. So let $v, w \in S^1$ with $\langle v, w\rangle = 0$, i.e., the matrix with columns v, w lies in $O(2)$. The orthogonal group $O(2)$ is isometric to the disjoint union $S^1 \,\dot\cup\, S^1$ via the map $\varphi \colon O(2) \to S^1 \times \{\pm 1\}$ defined by $\varphi(v, w) = (v, 1)$ iff (v, w) has positive orientation and $\varphi(v, w) = (v, -1)$ otherwise.

In the following let $\bar{C} \subset \mathbb{R}^2$ be a fixed closed convex cone that is not a linear subspace, i.e., $\bar{C}$ is a wedge with an angle between 0 and π. Denote by R_1 and R_2 the two rays forming the boundary of $\bar{C}$. Furthermore, depending on v, w, we write $\bar{\mathcal{F}} := \{x \in \bar{C} \mid \langle w, x\rangle = 1\}$, $\overline{\text{val}} := \sup\{\langle v, x\rangle \mid x \in \bar{\mathcal{F}}\}$, and $\overline{\text{Sol}} := \operatorname{Argmax}\{\langle v, x\rangle \mid x \in \bar{\mathcal{F}}\}$. Assuming that v, w are random vectors with $(v, w) \in O(2)$ uniformly at random, it is easily seen that only four cases appear with positive probability: The intersection $\bar{\mathcal{F}}$ may be empty, the functional v may be unbounded on $\bar{\mathcal{F}}$, or the solution set $\overline{\text{Sol}}$ consists of a single point, which either lies in R_1 or in R_2. In the latter case we again adopt the convention to denote the single point by $\overline{\text{sol}}$.

The following lemma is easily checked.

Lemma 5.3 *For uniformly random $(v, w) \in O(2)$, we have*

$$\operatorname{Prob}\left\{\bar{\mathcal{F}} = \emptyset\right\} = V_0(\bar{C}), \quad \operatorname{Prob}\left\{\overline{\text{val}} = \infty\right\} = V_2(\bar{C}).$$

Furthermore, for $M \in \hat{\mathscr{B}}(\mathbb{R}^2)$, we have

$$\operatorname{Prob}\left\{\overline{\text{sol}} \in M \ \textit{and}\ \overline{\text{val}} > 0\right\} = \operatorname{Prob}\left\{\overline{\text{sol}} \in M \ \textit{and}\ \overline{\text{val}} < 0\right\} = \tfrac{1}{2}\Phi_1(\bar{C}, M). \quad \square$$

As in the homogeneous case, we will now transform the problem (CP) into a geometric form to which we can apply the kinematic formula.

The affine linear subspace

$$W_{\text{aff}} := \{x \in \mathcal{E} \mid \langle a_1, x\rangle = b_1, \dots, \langle a_m, x\rangle = b_m\}.$$

is a shift of the linear space $W := \{x \in \mathcal{E} \mid \langle a_1, x\rangle = \ldots = \langle a_m, x\rangle = 0\}$. If $W_{\text{aff}} \neq W$, then there is a unique vector $w \in S(W^\perp)$ and a unique $\lambda > 0$ such that $W_{\text{aff}} = W + \lambda w$. We write $\tilde{W} := W + \mathbb{R}w$.

Lemma 5.4 *Suppose that $a_1, \dots, a_m$ are i.i.d. standard Gaussian in $\mathcal{E}$ and $b \in \mathbb{R}^m$ is a random vector such that $b \neq 0$ almost surely. Then, almost surely, W is uniformly distributed in $\mathrm{Gr}^c_m(\mathcal{E})$. Further, conditional on W, the vector w is uniformly distributed in $S(W^\perp)$. Finally, $\tilde{W}$ is uniformly distributed in $\mathrm{Gr}^c_{m-1}(\mathcal{E})$.*

Proof (Sketch) It suffices to show that, conditional on W, the vector w is uniformly distributed in $S(W^\perp)$. For seeing this, we may assume that $\mathcal{E} = \mathbb{R}^d$, $W = 0 \times \mathbb{R}^{d-m}$, and $a_1, \dots, a_m$ are i.i.d. standard Gaussian in $\mathbb{R}^m \times 0$. Let $\begin{pmatrix} A & 0 \end{pmatrix}$ denote the $m \times d$ matrix with rows a_i, so that $A \in \mathbb{R}^{m\times m}$ is almost surely invertible. Denote $b := (b_1, \dots, b_m)$, and put $x := A^{-1}b$. It is easy to see that conditional on $b \neq 0$, the vector $w = x/\|x\|$ is uniformly distributed in S^{m-1}. Hence the assertion follows.

As we are not interested in the specific value of (CP) (provided it is $< \infty$) but only where the maximum is attained, we may consider $W + w$ instead of $W_{\text{aff}} = W + \lambda w$, i.e., instead of (CP) we consider

$$\text{maximize} \quad \langle z, x\rangle \quad \text{s.t.} \quad x \in C \cap \tilde{W}, \quad \langle w, x\rangle = 1. \tag{5.1}$$

Without loss of generality, we may further replace z by its orthogonal projection $\bar{z}$ on W. For fixed W the induced distribution of $\bar{z}$ is the normal distribution on W. As $\bar{z}$ is almost surely nonzero, we may define the normalization $v := \|\bar{z}\|^{-1} \cdot \bar{z} \in S(W)$. Finally, we denote the plane spanned by v, w by $L := \mathrm{span}\{v, w\}$.

We can generate the distribution of $(\tilde{W}, L, v, w)$ induced by the standard normal distributed $a_1, \dots, a_m$ and by $b_1, \dots, b_m$ in the following way:

1. choose a uniformly random subspace $\tilde{W}$ of $\mathcal{E}$ of codimension $m - 1$,
2. choose a plane $L \subseteq \tilde{W}$ uniformly at random,
3. choose $v \in S(L)$ uniformly at random,
4. choose w as one of the points in $S(L) \cap v^\perp$, each with probability $\frac{1}{2}$.

Proof (Theorem 3.2) Lemma 5.2 tells us that instead of (5.1) we may consider the following problem in the 2-dimensional plane L

$$\text{maximize} \quad \langle v, x\rangle \quad \text{s.t.} \quad x \in \Pi_L(C \cap \tilde{W}), \quad \langle w, x\rangle = 1. \tag{5.2}$$

More precisely, using the notation introduced before for the analysis of the situation in dimension two, we have

$$\bar{C} := \Pi_L(C \cap \tilde{W}), \qquad \bar{\mathcal{F}} := \{x \in \bar{C} \mid \langle w, x\rangle = 1\},$$
$$\overline{\mathrm{val}} := \sup\{\langle v, x\rangle \mid x \in \bar{\mathcal{F}}\}, \qquad \overline{\mathrm{Sol}} := \mathrm{Argmax}\{\langle v, x\rangle \mid x \in \bar{\mathcal{F}}\},$$

and we obtain from Lemma 5.2 that (CP) is infeasible iff $\bar{\mathcal{F}} = \emptyset$, (CP) is unbounded iff $\overline{\mathrm{val}} = \infty$, and $\mathrm{Sol(CP)} = C \cap \Pi_L^{-1}(\overline{\mathrm{Sol}})$. We thus obtain by Lemma 5.3

$$\operatorname*{Prob}_{\substack{a_1,\ldots,a_m\\ b_1,\ldots,b_m}}\{\text{CP infeasible}\} = \mathop{\mathbb{E}}_{\tilde{W},L}\big[\operatorname*{Prob}_{v,w}\{\bar{\mathcal{F}} = \emptyset\}\big] = \mathop{\mathbb{E}}_{\tilde{W},L}\big[V_0(\Pi_L(C \cap \tilde{W}))\big].$$

Applying the kinematic formula twice yields (recall $\operatorname{codim} \tilde{W} = m-1$)

$$\mathop{\mathbb{E}}_{\tilde{W},L}\big[V_0(\Pi_L(C\cap\tilde{W}))\big] \overset{(2.7)}{=} \mathop{\mathbb{E}}_{\tilde{W}}\big[V_0(C\cap\tilde{W})\big] \overset{(2.6)}{=} V_0(C) + V_1(C) + \ldots + V_{m-1}(C),$$

which proves the first assertion of Theorem 3.2. Analogously, we obtain

$$\begin{aligned}\operatorname*{Prob}_{\substack{a_1,\ldots,a_m,z\\ b_1,\ldots,b_m}}\{\text{CP unbounded}\} &= \mathop{\mathbb{E}}_{\tilde{W},L}\big[\operatorname*{Prob}_{v,w}\{\overline{\mathrm{val}} = \infty\}\big] \overset{5.3}{=} \mathop{\mathbb{E}}_{\tilde{W},L}\big[V_2(\Pi_L(C \cap \tilde{W}))\big]\\ &\overset{(2.8)}{=} \mathop{\mathbb{E}}_{\tilde{W}}\big[V_2(C \cap \tilde{W}) + V_3(C \cap \tilde{W}) + \ldots + V_{d-m+1}(C \cap \tilde{W})\big]\\ &\overset{(2.5)}{=} V_{m+1}(C) + V_{m+2}(C) + \ldots + V_d(C),\end{aligned}$$

which proves the second assertion of Theorem 3.2.

As for the claim (3.1), we have for $M \in \hat{\mathscr{B}}(\mathcal{E})$

$$\big(\mathrm{sol(CP)} \in M \cap \tilde{W} \text{ and } \mathrm{val(CP)} > 0\big) \iff \big(\overline{\mathrm{sol}} \in \Pi_L(M \cap \tilde{W}) \text{ and } \overline{\mathrm{val}} > 0\big).$$

Therefore, $\operatorname*{Prob}_{\substack{a_1,\ldots,a_m,z\\ b_1,\ldots,b_m}}\{\mathrm{sol(CP)} \in M \text{ and } \mathrm{val(CP)} > 0\}$ equals by Lemma 5.3

$$\begin{aligned}&\mathop{\mathbb{E}}_{\tilde{W},L}\big[\operatorname*{Prob}_{v,w}\{\overline{\mathrm{sol}} \in \Pi_L(M \cap \tilde{W}) \text{ and } \overline{\mathrm{val}} > 0\}\big]\\ &= \mathop{\mathbb{E}}_{\tilde{W},L}\big[\tfrac{1}{2}\,\Phi_1(\Pi_L(C \cap \tilde{W}), \Pi_L(M \cap \tilde{W}))\big].\end{aligned}$$

Applying the kinematic formula twice finally yields

$$\mathop{\mathbb{E}}_{\tilde{W},L}\big[\tfrac{1}{2}\cdot\Phi_1(\Pi_L(C \cap \tilde{W}), \Pi_L(M \cap \tilde{W}))\big] \overset{(2.7)}{=} \tfrac{1}{2}\cdot\mathop{\mathbb{E}}_{\tilde{W}}\big[\Phi_1(C \cap \tilde{W}, M \cap \tilde{W})\big],$$

which equals $\frac{1}{2}\cdot\Phi_m(C, M)$ by (2.5). An analogous arguments yields the claim with the constraint $\mathrm{val(CP)} < 0$. □

6 Proof of Theorem 4.1

In this section we derive the formulas for the curvature measures of the symmetric cones as stated in Theorem 4.1. For completeness we state in Section 6.1 the formula for the volume of the tube around a spherically convex set, which may serve as a defining formula for the intrinsic volumes of general convex cones. This formula is also needed to justify the generalized version of Weyl's tube formula for cones with stratified smooth boundary, which we state in Section 6.2 without proof. In Section 6.3 we will provide some differential geometric background for the proof of Theorem 4.1, which we give in Section 6.4.

6.1 Intrinsic volumes of general convex cones

In this and in the subsequent sections we adopt the spherical viewpoint by considering intersections of convex cones with the unit sphere. The intrinsic volumes and the curvature measures of a convex cone C can be characterized through the volume of the (local) tube around the spherically convex set $C \cap S^{d-1}$. We introduce the following notation for convex cone $C \subseteq \mathbb{R}^d$, a conic Borel set $M \in \hat{\mathscr{B}}(\mathbb{R}^d)$ and an angle $\alpha \in [0, \pi/2)$

$$\mathcal{T}(C,\alpha) := \{p \in S^{d-1} \mid \|\Pi_C(p)\| \geq \cos(\alpha)\},$$
$$\mathcal{T}(C,\alpha;M) := \{p \in \mathcal{T}(C,\alpha) \mid \Pi_C(p) \in M\},$$

where Π_C denotes the canonical projection map. We suppress the dependence on the ambient sphere S^{d-1} to keep the notation simple.

The following proposition forms the basis for the general definition of the curvature measures and the intrinsic volumes. For a proof see for example [23, 4,33,26,18].

Proposition 6.1 *Let $C \subseteq \mathbb{R}^d$ be a closed convex cone and $M \in \hat{\mathscr{B}}(\mathbb{R}^d)$ be a conic Borel set. Then for $0 \leq \alpha < \pi/2$*

$$\operatorname{rvol} \mathcal{T}(C,\alpha;M) = \Phi_d(C,M) + \sum_{j=1}^{d-1} \Phi_j(C,M) \cdot \operatorname{rvol} \mathcal{T}(W_j,\alpha), \tag{6.1}$$

where $W_j \subseteq \mathbb{R}^d$ denotes a j-dimensional linear subspace. □

6.2 Expressing intrinsic volumes in terms of curvature

The characterizations (2.1) and (2.2) provide formulas for the curvature measures of polyhedral cones. Another class of cones, for which one has closed formulas for the intrinsic volumes, are the *smooth cones*, i.e., cones $C \subseteq \mathbb{R}^d$ such that the intersection of its boundary with the unit sphere $M := \partial C \cap S^{d-1}$ is a smooth (i.e., C^∞) hypersurface of S^{d-1}. In this case the formulas for the intrinsic volumes involve the curvature of M, which we shall describe next.

In general, let $M \subset S^{d-1}$ be a smooth submanifold of the unit sphere. For $p \in M$ we denote the tangent space of M in p by T_pM, and we denote its orthogonal complement in $T_pS^{d-1} = p^\perp$ by $T_p^\perp M$. Let $\zeta \in T_pM$ be a tangent vector, and $\eta \in T_p^\perp M$ a normal vector. It can be shown that if $c\colon \mathbb{R} \to M$ is a (smooth) curve with $c(0) = p$ and $\dot{c}(0) = \zeta$, and if $w\colon \mathbb{R} \to \mathbb{R}^d$ is a normal extension of η along c, i.e., $w(t) \in T_{c(t)}^\perp M$ and $w(0) = \eta$, then the orthogonal projection of $\dot{w}(0)$ onto T_pM neither depends on the choice of the curve c nor on the choice of the normal extension w of η (cf. for example [38, Ch. 14] for the hypersurface case, or [13, Ch. 6] for general Riemannian manifolds). It therefore makes sense to define the map

$$W_{p,\eta}\colon T_pM \to T_pM, \qquad \zeta \mapsto -\Pi_{T_pM}(\dot{w}(0)),$$

where $w\colon \mathbb{R} \to \mathbb{R}^d$ is a normal extension of η along a curve $c\colon \mathbb{R} \to M$ which satisfies $c(0) = p$ and $\dot{c}(0) = \zeta$, and Π_{T_pM} denotes the orthogonal projection onto the tangent space T_pM. The map $W_{p,\eta}$ is called the *Weingarten map*.

It can be shown that $W_{p,\eta}$ is a symmetric linear map (cf. [13, Ch. 6]), so that it has $m := \dim M$ real eigenvalues $\kappa_1(p,\eta), \ldots, \kappa_m(p,\eta)$, which are called the *principal curvatures* of M at p in direction η. The corresponding eigenvectors are called *principal directions*. Furthermore, we denote the elementary symmetric functions in the principal curvatures by

$$\sigma_i(p,\eta) := \sum_{1 \le j_1 < \ldots < j_i \le m} \kappa_{j_1}(p,\eta) \cdots \kappa_{j_i}(p,\eta). \tag{6.2}$$

When we are working with orientable hypersurfaces, i.e., with submanifolds of codimension 1 that are endowed with a global unit normal vector field $\nu\colon M \to T^\perp M$, i.e., $\nu(p) \in T_p^\perp M$, $\|\nu(p)\| = 1$, then we abbreviate $\sigma_i(p) := \sigma_i(p, \nu(p))$. When $M = \partial C \cap S^{d-1}$ is the boundary of a convex cone intersected with the unit sphere as well as a smooth hypersurface of S^{d-1}, then we always consider M to be endowed with the unit normal field pointing *inwards* the cone C (this implies $\kappa_i(p) \ge 0$ for all $i = 1, \ldots, d-2$).

In the context of (spherically) convex sets, Weyl's classical tube formula [40] says the following: Let $C \subseteq \mathbb{R}^d$ be a closed convex cone such that $M = \partial C \cap S^{d-1}$ is a smooth hypersurface of S^{d-1}. Then, for $1 \le j \le d-1$,

$$V_j(C) = \tfrac{1}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{d-j-1}} \cdot \int_{p \in M} \sigma_{d-j-1}(p)\, dM, \tag{6.3}$$

where $\mathcal{O}_{d-1} := \mathrm{vol}_{d-1} S^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, and dM denotes the volume element induced from the Riemannian metric on M.

The problem is that the cones $\mathcal{C}_{\beta,n}$, whose intrinsic volumes we want to compute, are neither polyhedral nor smooth (for $n \ge 3$). But the rank decomposition $\mathcal{C}_{\beta,n} = \cup_{r=0}^n \mathcal{W}_{\beta,n,r}$ yields a decomposition of $\mathcal{C}_{\beta,n}$ into smooth pieces, which is the basic idea behind the proof of Theorem 4.1. In the remainder of this section we define the notion of a *stratifiable convex set*, which is a generalization of both polyhedral and smooth convex sets, and we state a suitable generalization of (6.3).

In the following let $M \subset S^{d-1}$ be a smooth submanifold of the unit sphere. We may consider the *tangent* resp. *normal bundle of* M (cf. [36, Ch. 3]) as submanifolds of $\mathbb{R}^d \times \mathbb{R}^d$ via

$$TM = \bigcup\nolimits_{p\in M} \{p\} \times T_pM, \qquad T^\perp M = \bigcup\nolimits_{p\in M} \{p\} \times T_p^\perp M.$$

Furthermore, we also consider the *spherical normal bundle*

$$T^S M := \bigcup\nolimits_{p\in M} \{p\} \times T_p^S M, \qquad T_p^S M := T_p^\perp M \cap S^{d-1}. \tag{6.4}$$

The tangent and the normal bundle are both so-called *vector bundles*, as all fibers of the canonical projection maps $(x, v) \mapsto x$ are vector spaces. The spherical normal bundle is a *sphere bundle*, as all fibers are subspheres of the unit sphere. For the generalization of Weyl's tube formula we need to consider another class of fiber bundles, where each fiber is given by (the relative interior of) a spherically convex set.

Let $C \subseteq \mathbb{R}^d$ be a closed convex cone. For $p \in C$ we define the *normal cone* of C in p by

$$N_p(C) := \{v \in \mathbb{R}^d \mid \Pi_C(v + p) = p\},$$

which is easily seen to be a closed convex cone with $N_p(C) \subseteq p^\perp$. For a subset $M \subseteq C$, we define the *spherical duality bundle* via

$$N^S M := \bigcup_{p\in M} \{p\} \times N_p^S M, \qquad N_p^S M := \operatorname{relint}(N_p(C)) \cap S^{d-1}. \tag{6.5}$$

Note that we have not imposed any smoothness assumption yet, but if $M \subseteq C \cap S^{d-1}$ is smooth, then we have $N^S M \subseteq T^S M$. Note also that $N^S M$ in fact depends on M and C.

Definition 6.1 Let $C \subseteq \mathbb{R}^d$ be a closed convex cone. We call the spherically convex set $K := C \cap S^{d-1}$ *stratifiable* if it decomposes into a disjoint union $K = \dot{\bigcup}_{i=0}^t M_i$, such that:

1. For all $0 \le i \le t$, M_i is a smooth connected submanifold of S^{d-1}.
2. For all $0 \le i \le t$ the spherical duality bundle $N^S M_i$ is a smooth manifold.

If (1) and (2) are satisfied, then we call $K = \dot{\bigcup}_{i=0}^t M_i$ a *valid decomposition*. Furthermore, we call a stratum M_i *essential* if $\dim N^S M_i = d - 2$, otherwise we call it *negligible*.

The following theorem is the announced generalization of Weyl's tube formula (6.3) to stratified sets. A proof may be found in [5, §4.3]. Similar formulas may also be found in [2].

Theorem 6.1 *Let $C \subseteq \mathbb{R}^d$ such that $K := C \cap S^{d-1}$ is stratifiable and decomposes into the valid decomposition $K = \dot{\bigcup}_{i=0}^t M_i$. Let $M_0 = \operatorname{int}(K)$ and*

$M_1, \dots, M_k$ be the essential and $M_{k+1}, \dots, M_t$, $k \le t$ the negligible pieces. Then, for $1 \le i \le k$ and $1 \le j \le d-1$,

$$\begin{aligned} V_j(C) &= \sum_{i=1}^{k} \Phi_j(C, M_i), \\ \Phi_j(C, M_i) &= \frac{1}{\mathcal{O}_{j-1} \cdot \mathcal{O}_{d-j-1}} \cdot \int_{p\in M_i} \int_{\eta \in N_p^S(C)} \sigma^{(i)}_{d_i-j-1}(p, -\eta)\, dN_p^S(C)\, dM_i\,, \end{aligned}$$

where $d_i := \dim M_i + 2$ and $\sigma_\ell^{(i)}(p, -\eta)$ denotes the ℓth elementary symmetric function in the principal curvatures of M_i at p in direction $-\eta$ (and $\sigma_\ell := 0$ if $\ell < 0$). □

6.3 Orthogonal, unitary, and (compact) symplectic groups

In this section we discuss the compact Lie groups

$$G(n) := G_\beta(n) := \{U \in \mathbb{F}_\beta^{n\times n} \mid U^\dagger U = I_n\},$$

of linear isomorphisms $\mathbb{F}_\beta \to \mathbb{F}_\beta$ preserving the standard scalar product on $\mathbb{F}_\beta^n$ given by $\langle x, y\rangle = x^\dagger y = \sum_{i=1}^n \bar{x}_i y_i$ for $x, y \in \mathbb{F}_\beta^n$. The groups $G_\beta(n)$ are called the *orthogonal groups*, *unitary groups*, and *(compact) symplectic groups* depending on the value of $\beta = 1, 2, 4$, cf. for example [17, §7.2]. Note that an element $U \in G_\beta(n)$ may be identified with an orthonormal basis of $\mathbb{F}_\beta^n$ by interpreting the matrix U as the n-tuple of its columns. We drop the index β to simplify the notation.

The Lie algebra of $G(n)$, i.e., the tangent space of $G(n)$ at the identity matrix I_n, is given by the real vector space of skew-Hermitian matrices

$$\mathrm{Skew}_n := \mathrm{Skew}_{\beta,n} := T_{I_n} G(n) = \{A \in \mathbb{F}_\beta^{n\times n} \mid A^\dagger = -A\}.$$

To specify a left-invariant Riemannian metric on $G(n)$ it suffices to declare an $\mathbb{R}$-basis of the Lie algebra Skew_n to be orthonormal (and then extend the metric to $G(n)$ by pushing it forward via the left-multiplication). For $\beta = 4$ we declare the following basis of Skew_n to be orthonormal:

$$\begin{aligned} &\{\iota E_{ii} \mid 1 \le i \le n, \iota \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}\} \cup \{E_{ij} - E_{ji} \mid 1 \le j < i \le n\} \\ &\quad \cup \{\iota(E_{ij} + E_{ji}) \mid 1 \le j < i \le n, \iota \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}\}\,, \end{aligned} \tag{6.6}$$

and for $\beta = 1, 2$ we use its intersections with $\mathbb{R}^{n\times n}$ and $\mathbb{C}^{n\times n}$, respectively. It is readily checked that this yields a bi-invariant metric on $G(n)$ (the bi-invariance in fact determines the Riemannian metric up to scaling).

Applying the coarea formula [24, Appendix] to the Riemannian submersion $\varphi\colon G(n) \to S(\mathbb{F}_\beta^n) = \{x \in \mathbb{F}_\beta^n \mid \|x\| = 1\}$, $U \mapsto U \cdot e_1$, implies $\mathrm{vol}\, G(n) = \mathrm{vol}\, S(\mathbb{F}_\beta^n) \cdot \mathrm{vol}\, G(n-1)$ and hence

$$\mathrm{vol}\, G(n) = \prod_{i=1}^n \mathcal{O}_{\beta i - 1} = 2^n \cdot \pi^{n(n+1)\beta/4} \cdot \prod_{i=1}^n \frac{1}{\Gamma(\frac{\beta i}{2})}. \tag{6.7}$$

By a *distribution* of $r \in \mathbb{Z}_{>0}$ we understand a tuple $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{Z}_{>0}^m$ such that $|\rho| := \rho_1 + \ldots + \rho_m = r$. For such ρ with $|\rho| \le n$ we define the closed subgroup $G(n,\rho)$ of $G(n)$ consisting of the matrices having a block-diagonal form prescribed by ρ, namely:

$$G(n,\rho) := \{\operatorname{diag}(U_1, \dots, U_m, U') \mid U_i \in G(\rho_i), U' \in G(n-r)\}\,. \tag{6.8}$$

Note that $G(n,\rho)$ with its induced Riemannian metric is isometric to the direct product $G(\rho_1) \times \ldots \times G(\rho_m) \times G(n-r)$. Furthermore, the homogeneous space $G(n)/G(n,\rho)$ is a smooth manifold. The case $\rho = 1^{(r)} = (1, \dots, 1)$ (r-times) will be of particular importance. Note that $G(1) = S(\mathbb{F}_\beta) = \{a \in \mathbb{F}_\beta \mid \|a\| = 1\}$, so that $G(n, 1^{(r)}) \cong S(\mathbb{F}_\beta) \times \ldots \times S(\mathbb{F}_\beta) \times G(n-r)$. We use the notation

$$G_{n,r} := G(n)/G(n, 1^{(r)}). \tag{6.9}$$

Furthermore, we denote by $G(n) \to G_{n,r}$, $U \mapsto [U] := U \cdot G(n, 1^{(r)})$ the canonical map, which is a Riemannian submersion. An application of the coarea formula [24, Appendix] yields

$$\operatorname{vol} G_{n,r} = \frac{\operatorname{vol} G(n)}{\operatorname{vol} G(n, 1^{(r)})} = \frac{\operatorname{vol} G(n)}{\mathcal{O}_{\beta-1}^r \operatorname{vol} G(n-r)}. \tag{6.10}$$

Note that $G(n)$ has a natural action on $G_{n,r}$ given by $(U_1, [U_2]) \mapsto [U_1 U_2]$ for $U_1, U_2 \in G(n)$. Moreover, as $G(n)$ acts transitively on $G_{n,r}$, there exists up to scaling at most one Riemannian metric on $G_{n,r}$, which is $G(n)$-invariant.

In the following paragraphs we will give a concrete description of the tangent space $T_{[I_n]} G_{n,r}$, and specify on it a $G(n)$-invariant Riemannian metric. We have

$$G(n, 1^{(r)}) = \left\{ \begin{pmatrix} \Lambda & 0 \\ 0 & U' \end{pmatrix} \middle| \, \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_r), \lambda_i \in S(\mathbb{F}_\beta), U' \in G(n-r) \right\},$$

hence the tangent space of $G_{n,r}$ at $[I_n]$ equals

$$T_{I_n} G(n, 1^{(r)}) = \left\{ \begin{pmatrix} D & 0 \\ 0 & S \end{pmatrix} \middle| \, D = \operatorname{diag}(a_1, \dots, a_r)\,,\ \Re(a_i) = 0\,,\ S^\dagger = -S \right\}.$$

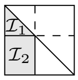
The orthogonal complement of $T_{I_n} G(n, 1^{(r)})$ in $T_{I_n} G(n) = \mathrm{Skew}_n$, the space of skew-Hermitian matrices, is given by ($X \in \mathbb{F}_\beta^{r \times r}, Y \in \mathbb{F}_\beta^{(n-r) \times r}$)

$$\overline{\mathrm{Skew}_n} := (T_{I_n} G(n, 1^{(r)}))^\perp = \left\{ \begin{pmatrix} X & -Y^\dagger \\ Y & 0 \end{pmatrix} \middle| \, X^\dagger = -X \right\}. \tag{6.11}$$

It can be shown (cf. [22, Lemma II.4.1]) that there exists an open ball B around the origin in $T_{I_n} G(n) = \mathrm{Skew}_n$ such that the intersection $B \cap \overline{\mathrm{Skew}_n}$ is diffeomorphic to an open neighborhood of $[I_n]$ in $G_{n,r}$. Moreover, the tangent space of $G_{n,r}$ in $[I_n]$ may be identified with $\overline{\mathrm{Skew}_n}$, and the restriction of the inner product on Skew_n to $\overline{\mathrm{Skew}_n}$ yields a well-defined Riemannian metric on $G_{n,r}$, which is $G(n)$-invariant. (See [7, §5.2] for a more detailed description of the induced Riemannian metric on a homogeneous space in a similar situation.)

For $\beta = 4$ we have the following orthonormal basis of $\overline{\mathrm{Skew}_n}$, cf. (6.6),

$$\{E_{ij} - E_{ji} \mid (i,j) \in \mathcal{I}\} \cup \{\iota(E_{ij} + E_{ji}) \mid \iota \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}, (i,j) \in \mathcal{I}\} \ ,$$

where $\mathcal{I} := \mathcal{I}_1 \cup \mathcal{I}_2 =$ 

with

$$\mathcal{I}_1 := \{(i,j) \mid 1 \le j < i \le r\}, \ \mathcal{I}_2 := \{(i,j) \mid r{+}1 \le i \le n\,,\ 1 \le j \le r\}. \quad (6.12)$$

For further use in Section 6.4, we denote this orthonormal basis of $\overline{\mathrm{Skew}_n} \cong T_{[I_n]}G_{n,r}$ (for $\beta = 4$) by

$$\eta_{ij}^1 := E_{ij} - E_{ji}, \qquad \eta_{ij}^\iota := \iota(E_{ij} + E_{ji}), \qquad (i,j) \in \mathcal{I},\ \iota \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}. \quad (6.13)$$

For $\beta = 2$ we have the orthonormal basis $\{\eta_{ij}^\iota \mid \iota \in \{1,\mathbf{i}\}, (i,j) \in \mathcal{I}\}$, and for $\beta = 1$ we have the orthonormal basis $\{\eta_{ij}^1 \mid (i,j) \in \mathcal{I}\}$.

6.4 Deducing the formulas for $\Phi_j(\beta, n, r)$

We first note that the face structure of $\mathcal{C}_{\beta,n}$ described in [8, §II.12] for the real case extends to the complex and the quaternion case in a straightforward way. We shall see that $\mathcal{C}_{\beta,n}$ is a stratified cone and determine its essential and the negligible pieces.

We change to the spherical viewpoint and write $K_n := \mathcal{C}_{\beta,n} \cap S(\mathrm{Her}_{\beta,n}) = \{A \in \mathrm{Her}_{\beta,n} \mid A \succeq 0\,,\ \|A\| = 1\}$. In order to see that K_n is stratifiable and to exhibit a valid decomposition of K_n (cf. Definition 6.1), we define the *eigenvalue pattern* of an element $A \in K_n$ via

$$\mathrm{patt}(A) := (\rho_1, \ldots, \rho_m), \quad \text{iff } \lambda_1 = \ldots = \lambda_{\rho_1} > \lambda_{\rho_1+1} = \ldots = \lambda_{\rho_1+\rho_2} > \ldots,$$

where $\lambda_1 \ge \ldots \ge \lambda_r > 0$ are the positive eigenvalues of A. Note that $\mathrm{patt}(A)$ is a distribution of $r = \mathrm{rk}(A)$. The spherical cap K_n thus decomposes into

$$K_n = \dot{\bigcup_{r=1}^{n}} \dot{\bigcup_{|\rho|=r}} M_{n,\rho}, \qquad M_{n,\rho} := \{A \in K_n \mid \mathrm{patt}(A) = \rho\}. \quad (6.14)$$

Note that $\mathrm{int}(K_n) = \dot{\bigcup}_{|\rho|=n} M_{n,\rho}$ and $\partial K_n = \dot{\bigcup}_{r=1}^{n-1} \dot{\bigcup}_{|\rho|=r} M_{n,\rho}$. Put

$$P_r := \left\{\lambda \in S^{n-1} \mid \lambda_1 > \lambda_2 > \ldots > \lambda_r > 0 = \lambda_{r+1} = \ldots = \lambda_n\right\}. \quad (6.15)$$

For the proof of the following result we refer to [6].

Proposition 6.2 *The set $M_{n,\rho}$, $|\rho| \le n$, defined in* (6.14) *is a smooth submanifold of the unit sphere $S(\mathrm{Her}_{\beta,n})$. Moreover, the duality bundle $N^S M_{n,\rho}$ defined in* (6.5) *is a smooth manifold for all $|\rho| \le n$. Hence* (6.14) *is a valid decomposition.*

The strata $\{M_{n,1^{(r)}} \mid 1 \le r \le n\}$, *where* $1^{(r)} := (1,1,\dots,1)$, *are essential and all the other strata* $M_{n,\rho}$ *are negligible. Moreover,*

$$\varphi_r \colon P_r \times G_{n,r} \to M_{n,1^{(r)}}, \qquad (\lambda,[U]) \mapsto U \cdot \mathrm{diag}(\lambda) \cdot U^\dagger. \tag{6.16}$$

is a well-defined diffeomorphism and its Jacobian determinant satisfies

$$|\det(D_{(\lambda,[U])}\varphi_r)| = 2^{r(2n-r-1)\beta/4} \cdot \Delta(\lambda)^\beta \cdot \prod_{i=1}^r \lambda_i^{\beta(n-r)}, \tag{6.17}$$

where $\Delta(\lambda) = \prod_{1\le i<j\le r}(\lambda_i - \lambda_j)$ *denotes the Vandermonde determinant.* □

Note that $\dim M_{n,1^{(r)}} = \dim P_r + \dim G_{n,r} = \beta r(n-r) + r - 1 + \beta\binom{r}{2}$. We next compute the principal curvatures of the essential strata $M_{n,1^{(r)}}$.

Proposition 6.3 *Let* $A = U\cdot\mathrm{diag}(\lambda)\cdot U^\dagger \in M_{n,1^{(r)}}$ *with* $\lambda \in P_r$. *Furthermore, let* $A'' \in \mathcal{C}_{\beta,n-r}$, *so that* $B := U \cdot \begin{pmatrix} 0 & 0 \\ 0 & -A'' \end{pmatrix} \cdot U^\dagger \in N_A(K_n)$ *is a vector in the normal cone of* K_n *at* A. *If* $\mu_1 \ge \ldots \ge \mu_{n-r} \ge 0$ *denote the eigenvalues of* A'', *then the principal curvatures of* $M_{n,1^{(r)}}$ *at* A *in direction* $-B$ *are given by*

$$\frac{\mu_1}{\lambda_1},\dots,\frac{\mu_{n-r}}{\lambda_1}, \frac{\mu_1}{\lambda_2},\dots,\frac{\mu_{n-r}}{\lambda_2}, \dots, \frac{\mu_1}{\lambda_r},\dots,\frac{\mu_{n-r}}{\lambda_r} \quad \text{(each value } \beta\text{-times)}$$

and $r - 1 + \beta\binom{r}{2}$ *times the value* 0.

Proof By orthogonal invariance we may assume w.l.o.g. that $U = I_n$, so that $A = \mathrm{diag}(\lambda)$ and $A'' = \mathrm{diag}(\mu)$. From (6.16) we get that the tangent space of $M_{n,1^{(r)}}$ at A is given by (omitting the argument $(\lambda,[I_n])$)

$$T_A M_{n,1^{(r)}} = D\varphi_r\big(T_\lambda P_r \times \overline{\mathrm{Skew}_n}\big).$$

It is easily seen that all the vectors in $D\varphi_r(T_\lambda P_r \times \{0\})$ are principal directions of $M_{n,1^{(r)}}$ at A with principal curvature 0, thus giving $r-1$ of the claimed $r - 1 + \beta\binom{r}{2}$ zero curvatures.

Concerning the second component, we again only consider the quaternion case $\beta = 4$, the other cases being similar. Let $U_{ij}^\iota \colon \mathbb{R} \to G(n)$, with $\iota \in \{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$ and $(i,j) \in \mathcal{I}$ (cf. (6.12)) be curves such that the induced curves $[U_{ij}^\iota]\colon \mathbb{R} \to G_{n,r}$ define the directions η_{ij}^ι, cf. (6.13). We denote the images of $D\varphi_r$ by

$$\zeta_{ij}^\iota := D\varphi_r(0,\eta_{ij}^\iota) \in T_A M_{n,1^{(r)}}, \qquad \iota \in \{1,\mathbf{i},\mathbf{j},\mathbf{k}\}. \tag{6.18}$$

We compute the derivative of φ_r in the second component for $\iota \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}$:

$$\begin{aligned} D_{(\lambda,[I_n])}\varphi_r(0,\eta_{ij}^\iota) &= \tfrac{d}{dt}\big(U_{ij}^\iota(t)\cdot\mathrm{diag}(\lambda)\cdot U_{ij}^\iota(t)^\dagger\big)(0) \\ &= \eta_{ij}^\iota \cdot \mathrm{diag}(\lambda) - \mathrm{diag}(\lambda)\cdot\eta_{ij}^\iota \\ &= \begin{cases} (\lambda_j - \lambda_i)\cdot\iota(E_{ij} - E_{ji}) & \text{if } 1 \le j < i \le r \\ \lambda_j \cdot \iota(E_{ij} - E_{ji}) & \text{if } r+1 \le i \le n,\ 1 \le j \le r. \end{cases} \end{aligned}$$

For $\iota = 1$ one obtains a similar formula (replace $E_{ij} - E_{ji}$ by $E_{ij} + E_{ji}$). We define normal extensions of $-B = \mathrm{diag}(0, \mu)$ along the curves $\varphi_r\big(\lambda, \big[U_{ij}^\iota(t)\big]\big)$ via

$$v_{ij}^\iota(t) := U_{ij}^\iota(t) \cdot \mathrm{diag}(0, \mu) \cdot U_{ij}^\iota(t)^\dagger, \qquad \iota \in \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}.$$

Differentiating these normal extensions $t = 0$ yields for $\iota \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, using $\eta_{ij}^\iota = \iota(E_{ij} + E_{ji})$,

$$\begin{aligned}\tfrac{d}{dt} v^\iota(0) &= \iota(E_{ij} + E_{ji}) \cdot \mathrm{diag}(0, \mu) - \mathrm{diag}(0, \mu) \cdot \iota(E_{ij} + E_{ji}) \\ &= \begin{cases} 0 & \text{if } 1 \le j < i \le r \\ -\mu_{i-r} \cdot \iota(E_{ij} - E_{ji}) & \text{if } r+1 \le i \le n,\ 1 \le j \le r. \end{cases}\end{aligned}$$

Again, the formula for $\iota = 1$ is obtained by replacing $E_{ij} - E_{ji}$ by $E_{ij} + E_{ji}$. Comparing this with the values of ζ_{ij}^ι given above implies for $\iota \in \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$

$$\tfrac{d}{dt} v_{ij}^\iota(0) = \begin{cases} 0 \cdot \zeta_{ij}^\iota & \text{if } 1 \le j < i \le r \\ -\frac{\mu_{i-r}}{\lambda_j} \cdot \zeta_{ij}^\iota & \text{if } r+1 \le i \le n,\ 1 \le j \le r. \end{cases}$$

We conclude that the directions $\zeta_{ij}^1, \zeta_{ij}^{\mathbf{i}}, \zeta_{ij}^{\mathbf{j}}, \zeta_{ij}^{\mathbf{k}}$ are principal directions with curvature 0 and $\frac{\mu_{i-r}}{\lambda_j}$, respectively. □

Before we finally get to the proof of Theorem 4.1 note that we can write the polynomial $f_{\beta,k}(x;y)$, defined in (4.2), in an explicit form if we rearrange

$$\prod_{i=1}^{r} \prod_{j=1}^{n-r} (x_i + y_j)^\beta = \prod_{i=1}^{r} \prod_{j=1}^{n-r} \left(\tfrac{x_i}{y_j} + 1\right)^\beta \cdot \prod_{j=1}^{n-r} y_j^{\beta r}.$$

Denoting by σ_k the kth elementary symmetric function, we obtain

$$f_{\beta,k}(x;y) = \sigma_k\left((x \otimes y^{-1})^{\times\beta}\right) \cdot \prod_{j=1}^{n-r} y_j^{\beta r}\ , \tag{6.19}$$

where $(x \otimes y^{-1})^{\times\beta} = \big(\underbrace{x \otimes y^{-1}, \ldots, x \otimes y^{-1}}_{\beta\text{-times}}\big)$, and

$$x \otimes y^{-1} := \left(\frac{x_1}{y_1}, \ldots, \frac{x_r}{y_1}, \frac{x_1}{y_2}, \ldots, \frac{x_r}{y_2}, \ldots, \frac{x_1}{y_{n-r}}, \ldots, \frac{x_r}{y_{n-r}}\right) \in \mathbb{R}^{r(n-r)}. \tag{6.20}$$

Proof (Theorem 4.1) In the stratification (6.14) of $K_n = \mathcal{C}_{\beta,n} \cap S(\mathrm{Her}_{\beta,n})$ only the strata $M_{n,1^{(r)}}$ are essential, cf. Proposition 6.2. Denoting $\delta(n,r) := \dim M_{n,1^{(r)}} + 2 = \beta r(n-r) + d_{\beta,r} + 1$ and $c := \mathcal{O}_{j-1} \cdot \mathcal{O}_{d_{\beta,n}-j-1}$ we thus obtain from Theorem 6.1

$$\Phi_j(\beta, n, r) = c^{-1} \int\limits_{A \in M_{n,1^{(r)}}} \int\limits_{B \in N_A^S} \sigma_{\delta(n,r)-j-1}^{(r)}(A, -B)\ dN_A^S\ dM_{n,1^{(r)}}\ ,$$

where the superscript in $\sigma^{(r)}_{\delta(n,r)-j-1}$ indicates the dependence on $M_{n,1^{(r)}}$.

In Proposition 6.3 we computed the principal curvatures of $M_{n,1^{(r)}}$. Using the notation $(x \otimes y^{-1})^{\times\beta} = \big(x \otimes y^{-1}, \ldots, x \otimes y^{-1}\big)$ (β-times) and $x \otimes y^{-1}$ defined in (6.20), we obtain

$$\Phi_j(\beta, n, r) = c^{-1} \int\limits_{A \in M_{n,1^{(r)}}} \int\limits_{B \in N^S_A} \sigma_{\delta(n,r)-j-1}\left((\lambda^{-1} \otimes \mu)^{\times\beta}\right) dN^S_A \, dM_{n,1^{(r)}}, \tag{6.21}$$

where λ and μ denote the (positive) eigenvalues of A and $-B$, respectively. Using the relation $\sigma_k(\frac{1}{x_1}, \ldots, \frac{1}{x_N}) = (x_1 \cdots x_N)^{-N} \cdot \sigma_{N-k}(x_1, \ldots, x_N)$ and observing $\delta(n,r) - j - 1 = \beta r(n-r) + d_{\beta,r} - j$, we can rewrite the integrand:

$$\sigma_{\delta(n,r)-j-1}\left((\lambda^{-1} \otimes \mu)^{\times\beta}\right) = \sigma_{j-d_{\beta,r}}\left((\lambda \otimes \mu^{-1})^{\times\beta}\right) \cdot \frac{\prod_{i=1}^{n-r} \mu_i^{\beta r}}{\prod_{i=1}^{r} \lambda_i^{\beta(n-r)}} \, . \tag{6.22}$$

By (6.17), the absolute value of the Jacobian of φ_n equals $2^{n(n-1)\beta/4}\, \Delta(\mu)^\beta$. It is easy to see that the normal cone of $\mathcal{C}_{\beta,n}$ at $A \in M_{n,1^{(r)}}$ is isometric to $\mathcal{C}_{\beta,n-r}$. Further, $M_{n-r,1^{(n-r)}}$ equals $K_{n-r} = \mathcal{C}_{\beta,n-r} \cap S(\mathrm{Her}_{\beta,n-r})$ up to strata of lower dimension. Applying (6.17) to φ_{n-r} (note that n needs to be replaced by $n-r$), we can transform the inner integral of (6.21) via the coarea formula [24, Appendix] to obtain

$$\int\limits_{B \in N^S_A} f(\lambda, \mu)\, dN^S_A = \int\limits_{P_{n-r} \times G_{n-r,n-r}} f(\lambda, \mu) \cdot 2^{(n-r)(n-r-1)\beta/4} \cdot \Delta(\mu)^\beta \, d(\mu, [U_2]) \, ,$$

where we have abbreviated $f(\lambda, \mu)$ for the integrand (6.22).

Similarly, we may transform the outer integral of (6.21) by applying the coarea formula to the map φ_r. As a result we obtain

$$\begin{aligned} &\int\limits_{A \in M_{n,1^{(r)}}} \int\limits_{B \in N^S_A} f(\lambda, \mu)\, dN^S_A \, dM_{n,1^{(r)}} \\ &\quad = \int\limits_{(\lambda,[U_1]) \in P_r \times G_{n,r}} \int\limits_{(\mu,[U_2]) \in P_{n-r} \times G_{n-r,n-r}} f(\lambda, \mu) \cdot 2^{r(2n-r-1)\beta/4} \cdot \Delta(\lambda)^\beta \\ &\qquad \cdot \prod_{i=1}^{r} \lambda_i^{\beta(n-r)} \cdot 2^{(n-r)(n-r-1)\beta/4} \cdot \Delta(\mu)^\beta \, d(\mu, [U_2])\, d(\lambda, [U_1]) \, . \end{aligned} \tag{6.23}$$

Note that we have

$$\mathrm{vol}\, G_{n,r} \cdot \mathrm{vol}\, G_{n-r,n-r} \overset{(6.10)}{=} \frac{\mathrm{vol}\, G(n)}{\mathcal{O}^r_{\beta-1} \cdot \mathrm{vol}\, G(n-r)} \cdot \frac{\mathrm{vol}\, G(n-r)}{\mathcal{O}^{n-r}_{\beta-1}} = \frac{\mathrm{vol}\, G(n)}{\mathcal{O}^n_{\beta-1}} \, .$$

Replacing $f(\lambda,\mu)$ again by (6.22), the integral (6.23) simplifies to

$$2^{n(n-1)\beta/4}\,\frac{\operatorname{vol} G(n)}{\mathcal{O}_{\beta-1}^n}\cdot\int\limits_{P_r}\int\limits_{P_{n-r}}\Delta(\lambda)^\beta\Delta(\mu)^\beta\sigma_{j-d_{\beta,r}}\left((\lambda\otimes\mu^{-1})^{\times\beta}\right)\prod_{i=1}^{n-r}\mu_i^{\beta r}\,d\lambda\,d\mu$$

$$\overset{(*)}{=}\frac{(2\pi)^{n(n-1)\beta/4}\cdot n!}{F_n(\beta/2)}\cdot\int\limits_{P_r}\int\limits_{P_{n-r}}\Delta(\lambda)^\beta\Delta(\mu)^\beta f_{\beta,j-d_{\beta,r}}(\lambda;\mu)\,d\lambda\,d\mu\,, \tag{6.24}$$

where in $(*)$ we have used (6.19) and the small computation

$$\frac{\operatorname{vol} G(n)}{\mathcal{O}_{\beta-1}^n}\overset{(6.7)}{=}\frac{2^n\pi^{n(n+1)\beta/4}\prod_{i=1}^n\frac{1}{\Gamma(\frac{\beta i}{2})}}{(2\pi^{\beta/2}/\Gamma(\frac{\beta}{2}))^n}=\pi^{\frac{n(n-1)\beta}{4}}\prod_{i=1}^n\frac{\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\beta i}{2})}\overset{(4.5)}{=}\frac{\pi^{\frac{n(n-1)\beta}{4}}n!}{F_n(\beta/2)}.$$

The integrand in (6.24) is bihomogeneous in λ and μ. Its degree in λ equals $\beta\binom{r}{2}+j-d_{\beta,r}=j-r$, and its degree in μ is given by $\beta\binom{n-r}{2}+\beta r(n-r)-j+d_{\beta,r}=\beta\binom{n}{2}-j+r$.

The following is easily seen using polar coordinates: let $f\colon\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ be a homogeneous function of degree k, i.e., $f(x)=\|x\|^k\cdot f(\|x\|^{-1}\cdot x)$. Then for a Borel set $U\subseteq S^{n-1}$ and $\hat U=\{s\cdot p\mid s\ge 0\,,\ p\in U\}$

$$\int\limits_{p\in U}f(p)\,dp=\frac{1}{2^{\frac{n+k}{2}-1}\cdot\Gamma(\frac{n+k}{2})}\cdot\int\limits_{x\in\hat U}e^{-\frac{\|x\|^2}{2}}\cdot f(x)\,dx\,.$$

Using this observation twice, we get

$$\begin{aligned}(6.24)&=\frac{(2\pi)^{n(n-1)\beta/4}\,n!}{F_n(\beta/2)}\cdot\frac{2}{2^{j/2}\cdot\Gamma(\frac{j}{2})}\cdot\frac{2}{2^{(\beta\binom{n}{2}-j+n)/2}\cdot\Gamma\big(\frac{\beta\binom{n}{2}+n-j}{2}\big)}\\&\qquad\cdot\int\limits_{\hat P_r}\int\limits_{\hat P_{n-r}}e^{-\frac{\|\lambda\|^2+\|\mu\|^2}{2}}\cdot\Delta(\lambda)^\beta\cdot\Delta(\mu)^\beta\cdot f_{\beta,j-d_{\beta,r}}(\lambda;\mu)\,d\lambda\,d\mu\\&=\frac{c\,n!}{F_n(\beta/2)\,(2\pi)^{n/2}}\cdot\int\limits_{\hat P_r}\int\limits_{\hat P_{n-r}}e^{-\frac{\|\lambda\|^2+\|\mu\|^2}{2}}\Delta(\lambda)^\beta\Delta(\mu)^\beta f_{\beta,j-d_{\beta,r}}(\lambda;\mu)\,d\lambda\,d\mu\,.\end{aligned}$$

The positive orthant $\mathbb{R}_+^r$ decomposes into $r!$ isometric copies of $\hat P_r$, such that their interiors are disjoint. More precisely, the copies of $\hat P_r$ are parametrized by the permutations of $\{1,\dots,r\}$, which indicate the order of the components of a vector in $\mathbb{R}_+^r$. The same applies to $\mathbb{R}_+^{n-r}$ and $\hat P_{n-r}$. As the Vandemonde determinant is antisymmetric, and $f_{\beta,k}(\lambda;\mu)$ is symmetric both in λ and in μ, we finally see that $\Phi_j(\beta, n, r)$ equals

$$\frac{n!}{F_n(\beta/2)\,(2\pi)^{n/2}} \cdot \int\limits_{\hat{P}_r} \int\limits_{\hat{P}_{n-r}} e^{-\frac{\|\lambda\|^2+\|\mu\|^2}{2}} \Delta(\lambda)^\beta \Delta(\mu)^\beta f_{\beta,j-d_{\beta,r}}(\lambda;\mu)\, d\lambda\, d\mu$$

$$= \binom{n}{r} \cdot \frac{1}{F_n(\beta/2)\cdot(2\pi)^{n/2}} \int\limits_{\nu:=(\lambda,\mu)\in\mathbb{R}^n_+} e^{-\frac{\|\nu\|^2}{2}} |\Delta(\lambda)|^\beta |\Delta(\mu)|^\beta f_{\beta,j-d_{\beta,r}}(\lambda;\mu)\, d\nu$$

$$\overset{(4.3)}{=} \binom{n}{r} \cdot \frac{J_\beta(n,r,j-d_{\beta,r})}{F_n(\beta/2)} . \qquad \square$$

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References

1. I. Adler and S. E. Berenguer. Random linear programs. *Operations Research Center Report*, No. 81-4, University of California, Berkeley, CA, 1981.
2. R. J. Adler and J. E. Taylor. *Random fields and geometry.* Springer Monographs in Mathematics. Springer, New York, 2007.
3. F. Alizadeh, J.-P. A. Haeberly, and M. L. Overton. Complementarity and nondegeneracy in semidefinite programming. *Math. Programming*, 77(2, Ser. B):111–128, 1997. Semidefinite programming.
4. C.B. Allendoerfer. Steiner's formulae on a general S^{n+1}. *Bull. Amer. Math. Soc.*, 54:128–135, 1948.
5. D. Amelunxen. Geometric analysis of the condition of the convex feasibility problem. PhD Thesis, Universität Paderborn, 2011.
6. D. Amelunxen and P. Bürgisser. Intrinsic volumes of symmetric cones. arXiv 1205.1863.
7. D. Amelunxen and P. Bürgisser. Probabilistic analysis of the Grassmann condition number. arXiv:1112.2603v1 [math.OC].
8. A. Barvinok. *A course in convexity*, volume 54 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2002.
9. A. Ben-Tal and A. Nemirovski. *Lectures on modern convex optimization.* MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001. Analysis, algorithms, and engineering applications.
10. S. E. Berenguer. Random linear programs. PhD Thesis, University of California, Berkeley, CA, 1978.
11. H.-C. G. v. Bothmer and K. Ranestad. A general formula for the algebraic degree in semidefinite programming. *Bull. Lond. Math. Soc.*, 41(2):193–197, 2009.
12. D. Cheung and F. Cucker. Solving linear programs with finite precision. I. Condition numbers and random programs. *Math. Program.*, 99(1, Ser. A):175–196, 2004.
13. M. P. do Carmo. *Riemannian geometry.* Mathematics: Theory & Applications. Birkhäuser Boston Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.
14. J. Faraut and A. Korányi. *Analysis on symmetric cones.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
15. L. Faybusovich. Euclidean Jordan algebras and interior-point algorithms. *Positivity*, 1(4):331–357, 1997.

16. P. J. Forrester and S. O. Warnaar. The importance of the Selberg integral. *Bull. Amer. Math. Soc. (N.S.)*, 45(4):489–534, 2008.
17. W. Fulton and J. Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.
18. S. Glasauer. Integralgeometrie konvexer Körper im sphärischen Raum. PhD Thesis, Universität Freiburg im Breisgau, 1995.
19. S. Glasauer. Integral geometry of spherically convex bodies. *Diss. Summ. Math.*, 1(1-2):219–226, 1996.
20. O. Güler. Barrier functions in interior point methods. *Math. Oper. Res.*, 21(4):860–885, 1996.
21. R. A. Hauser and O. Güler. Self-scaled barrier functions on symmetric cones and their classification. *Found. Comput. Math.*, 2(2):121–143, 2002.
22. S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 80 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1978.
23. G. Herglotz. Über die Steinersche Formel für Parallelflächen. *Abh. Math. Sem. Univ. Hamburg*, 15:165–177, 1943.
24. R. Howard. The kinematic formula in Riemannian homogeneous spaces. *Mem. Amer. Math. Soc.*, 106(509):vi+69, 1993.
25. D. Hug and R. Schneider. Kinematic and Crofton formulae of integral geometry: recent variants and extensions (survey). pages 51–80, 2002. Homenatge al professor Lluís Santaló i. Sors: 22 de novembre de 2002 / C. Barceló i Vidal (ed.), Girona: Universitat de Girona. Càtedra Lluís Santaló d'Aplicacions de la Matemàtica.
26. P. Kohlmann. Curvature measures and Steiner formulae in space forms. *Geom. Dedicata*, 40(2):191–211, 1991.
27. J.H. May and R.L. Smith. Random polytopes: their definition, generation and aggregate properties. *Math. Programming*, 24(1):39–54, 1982.
28. M. L. Mehta. *Random matrices*, volume 142 of *Pure and Applied Mathematics (Amsterdam)*. Elsevier/Academic Press, Amsterdam, third edition, 2004.
29. Yu. E. Nesterov and M. J. Todd. Self-scaled barriers and interior-point methods for convex programming. *Math. Oper. Res.*, 22(1):1–42, 1997.
30. Yu. E. Nesterov and M.J. Todd. Primal-dual interior-point methods for self-scaled cones. *SIAM J. Optim.*, 8(2):324–364 (electronic), 1998.
31. J. Nie, K. Ranestad, and B. Sturmfels. The algebraic degree of semidefinite programming. *Math. Program.*, 122(2):379–405, 2009.
32. J. Renegar. *A mathematical view of interior-point methods in convex optimization.* MPS/SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2001.
33. L. A. Santaló. On parallel hypersurfaces in the elliptic and hyperbolic n-dimensional space. *Proc. Amer. Math. Soc.*, 1:325–330, 1950.
34. R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.
35. R. Schneider and W. Weil. *Stochastic and integral geometry.* Probability and its Applications (New York). Springer-Verlag, Berlin, 2008.
36. M. Spivak. *A comprehensive introduction to differential geometry. Vol. I.* Publish or Perish Inc., Wilmington, Del., third edition, 2005.
37. P. V. Sporyshev. An application of integral geometry to the theory of linear inequalities. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 123:208–220, 1983. Differential geometry, Lie groups and mechanics, V.
38. J. A. Thorpe. *Elementary topics in differential geometry.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1994.
39. J. G. Wendel. A problem in geometric probability. *Math. Scand.*, 11:109–111, 1962.
40. H. Weyl. On the Volume of Tubes. *Amer. J. Math.*, 61(2):461–472, 1939.
41. F. Zhang. Quaternions and matrices of quaternions. *Linear Algebra Appl.*, 251:21–57, 1997.