SPECTRA: solving exactly linear matrix inequalities

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Outline

1. Spectrahedra and linear matrix inequalities
2. Semidefinite programming
3. Exact algorithms for spectrahedra

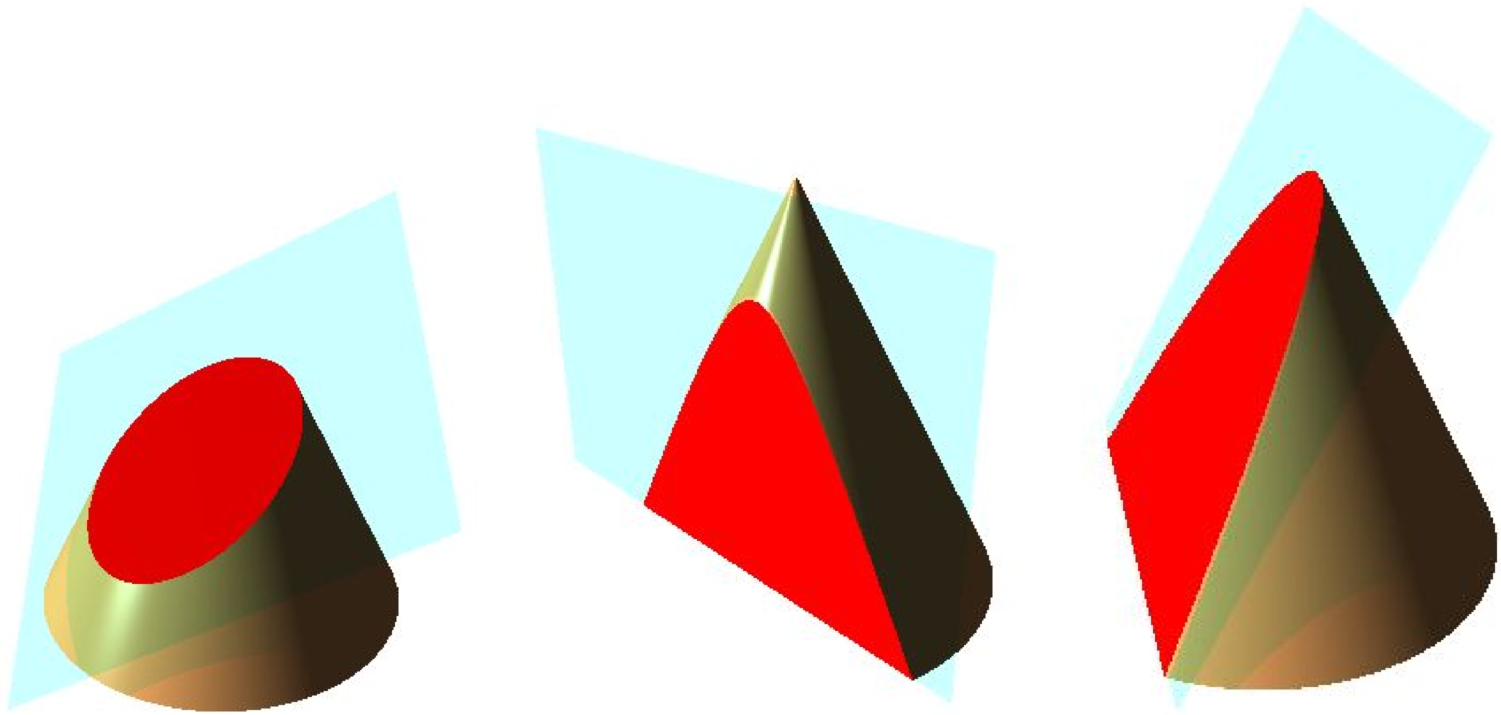
What are spectrahedra ?

Affine sections (i.e. slices) of the semidefinite cone
(i.e. the set of positive semidefinite quadratic forms)

Generalizations of polyhedra

Convex basic semialgebraic sets

Slicing cones



Slices of the Minkowksi (aka Lorentz or second order) cone

Spectrahedra and LMIs

Given symmetric real matrices $A_0, A_1, \ldots, A_n \in \mathbb{SR}^m$
a spectrahedron is described by a **linear matrix inequality** (LMI)

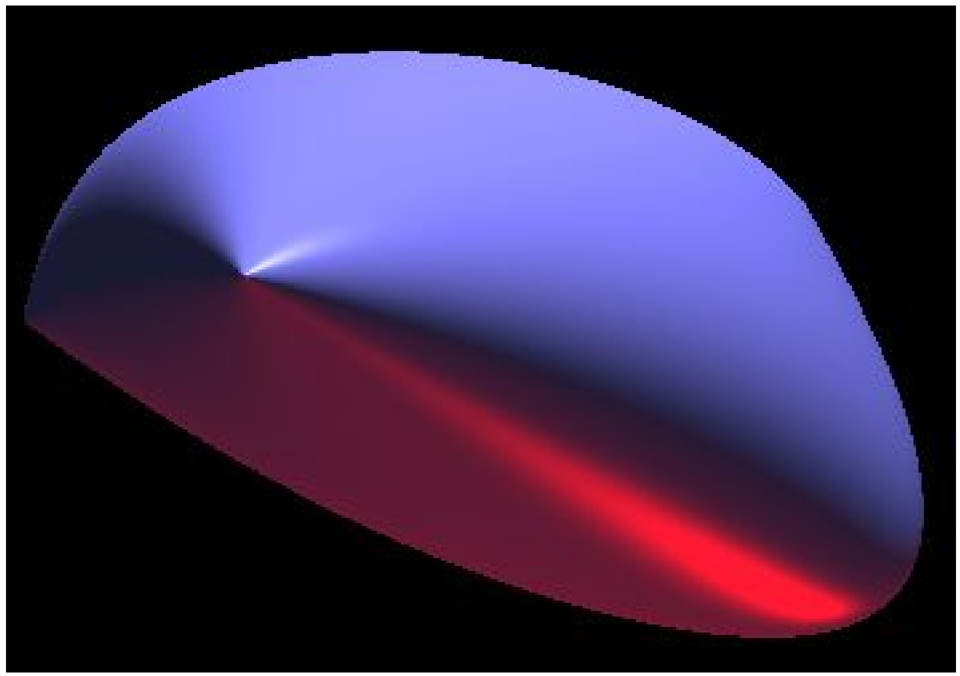
$$\mathscr{S} := \{x \in \mathbb{R}^n \ : \ A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\}$$

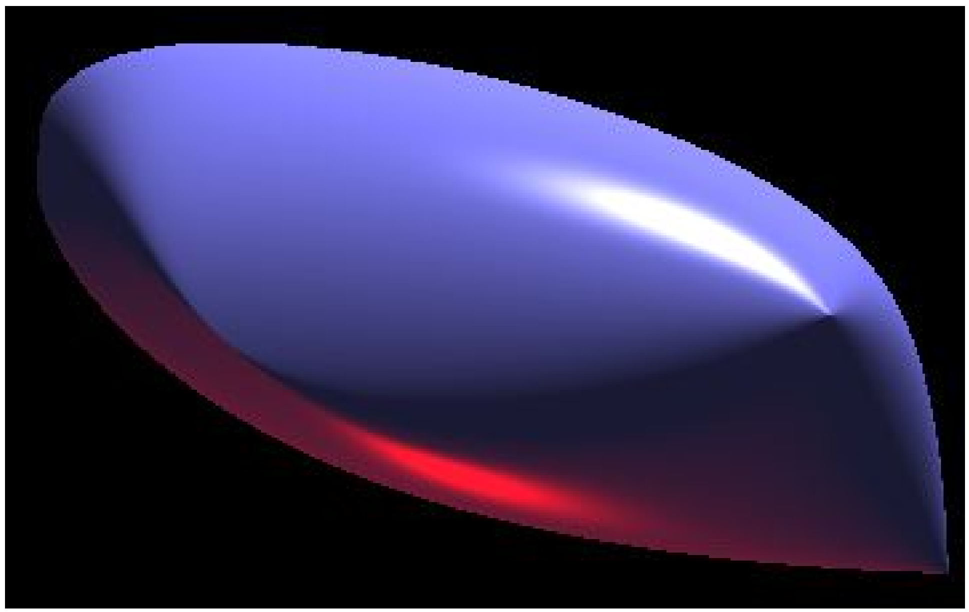
where $\succeq 0$ means **positive semidefinite**, i.e.

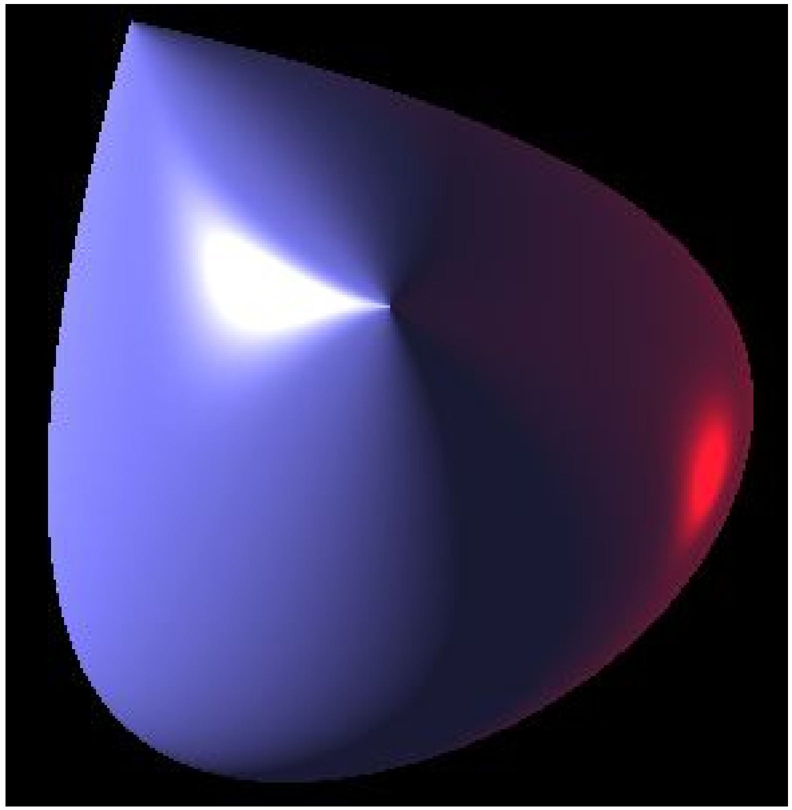
$$\begin{aligned}
\mathscr{S} &= \{x \in \mathbb{R}^n \ : \ \mathrm{eig}(A_0 + A_1 x_1 + \cdots + A_n x_n) \geq 0\} \\
&= \{x \in \mathbb{R}^n : y'(A_0 + A_1 x_1 + \cdots + A_n x_n) y \geq 0, \forall y \in \mathbb{R}^m\} \\
&= \{x \in \mathbb{R}^n : (y' A_0 y) + (y' A_1 y) x_1 + \cdots + (y' A_n y) x_n \geq 0, \forall y \in \mathbb{R}^m\}
\end{aligned}$$

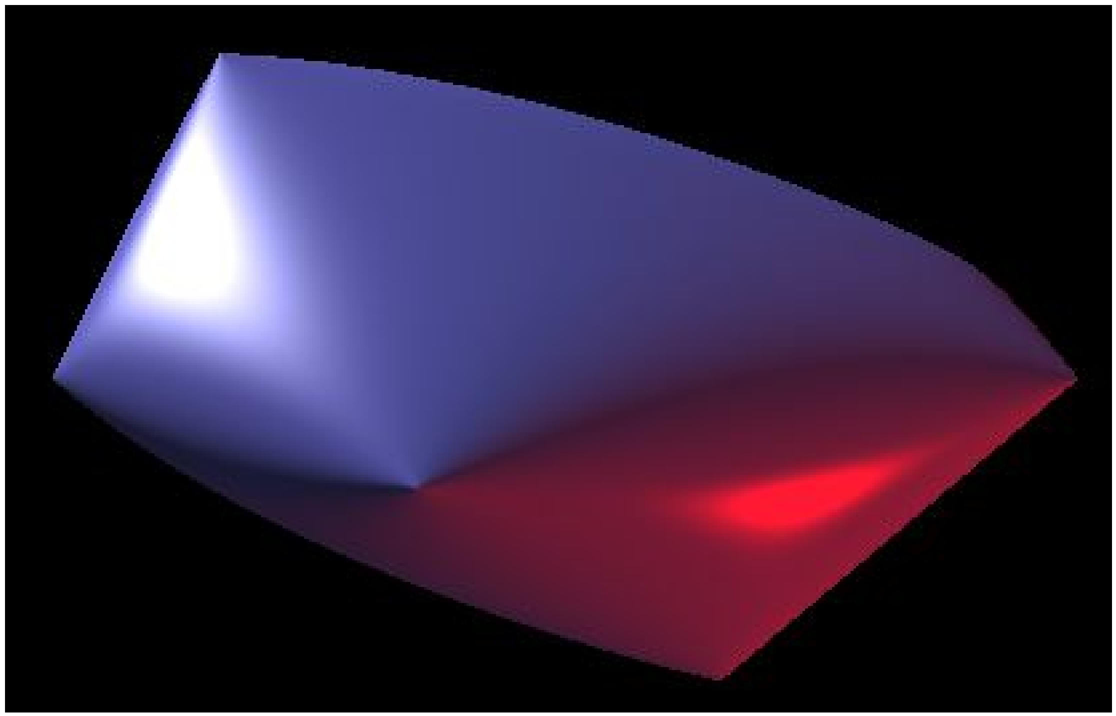
Convexity of $\mathscr{S}$ follows readily from the latter expression,
an intersection of infinitely many half spaces

Polyhedra correspond to commutating matrices A_i, e.g. diagonal





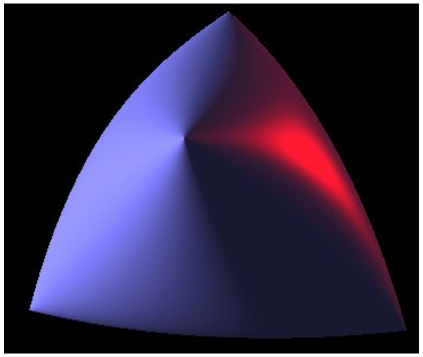




Cayley spectrahedron

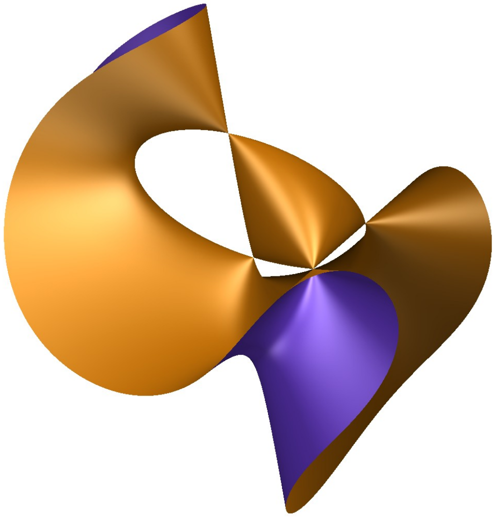
Boundary of spectrahedron

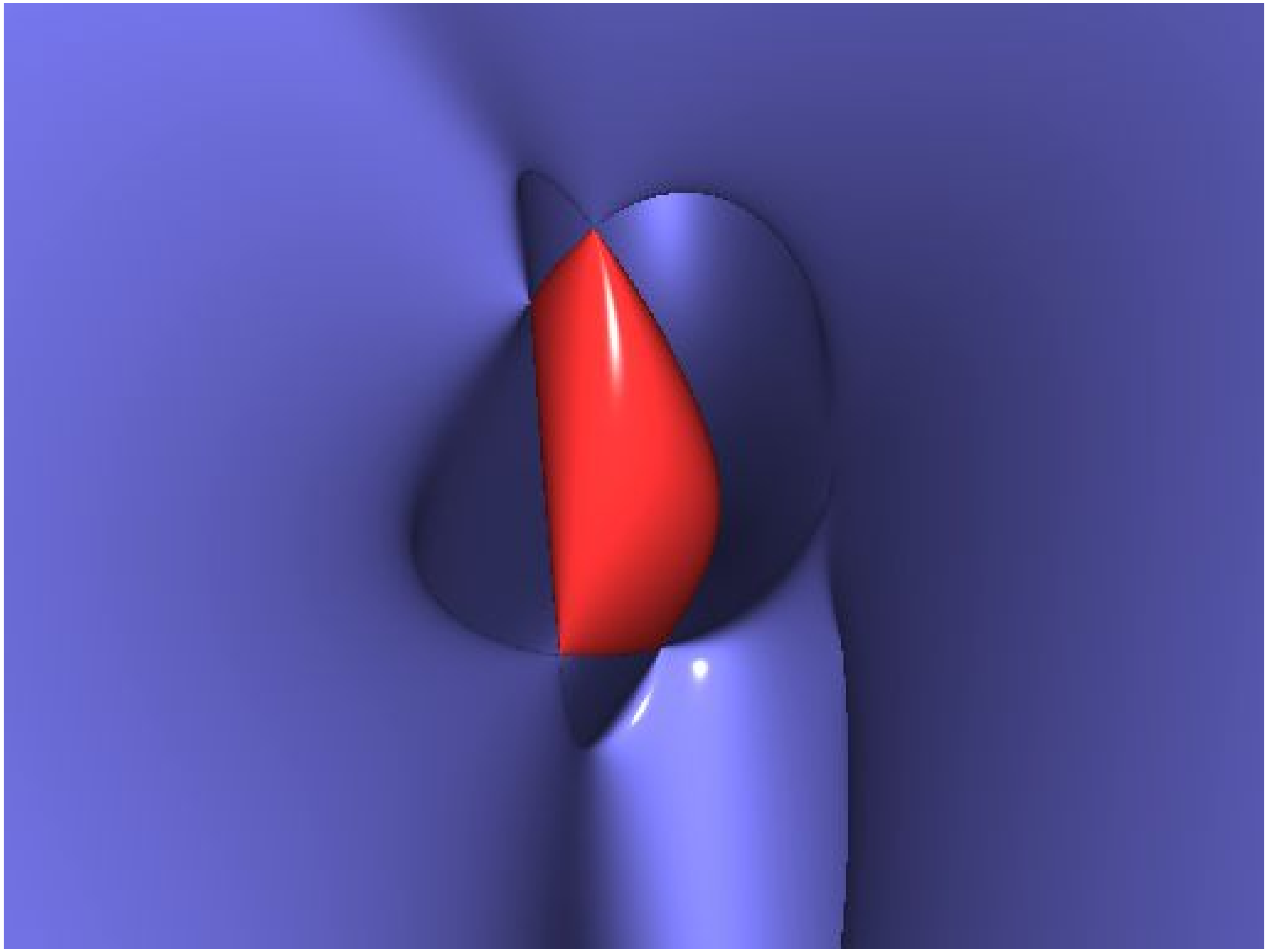
$$\left\{ x \in \mathbb{R}^3 \;:\; \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} \succeq 0 \right\}$$

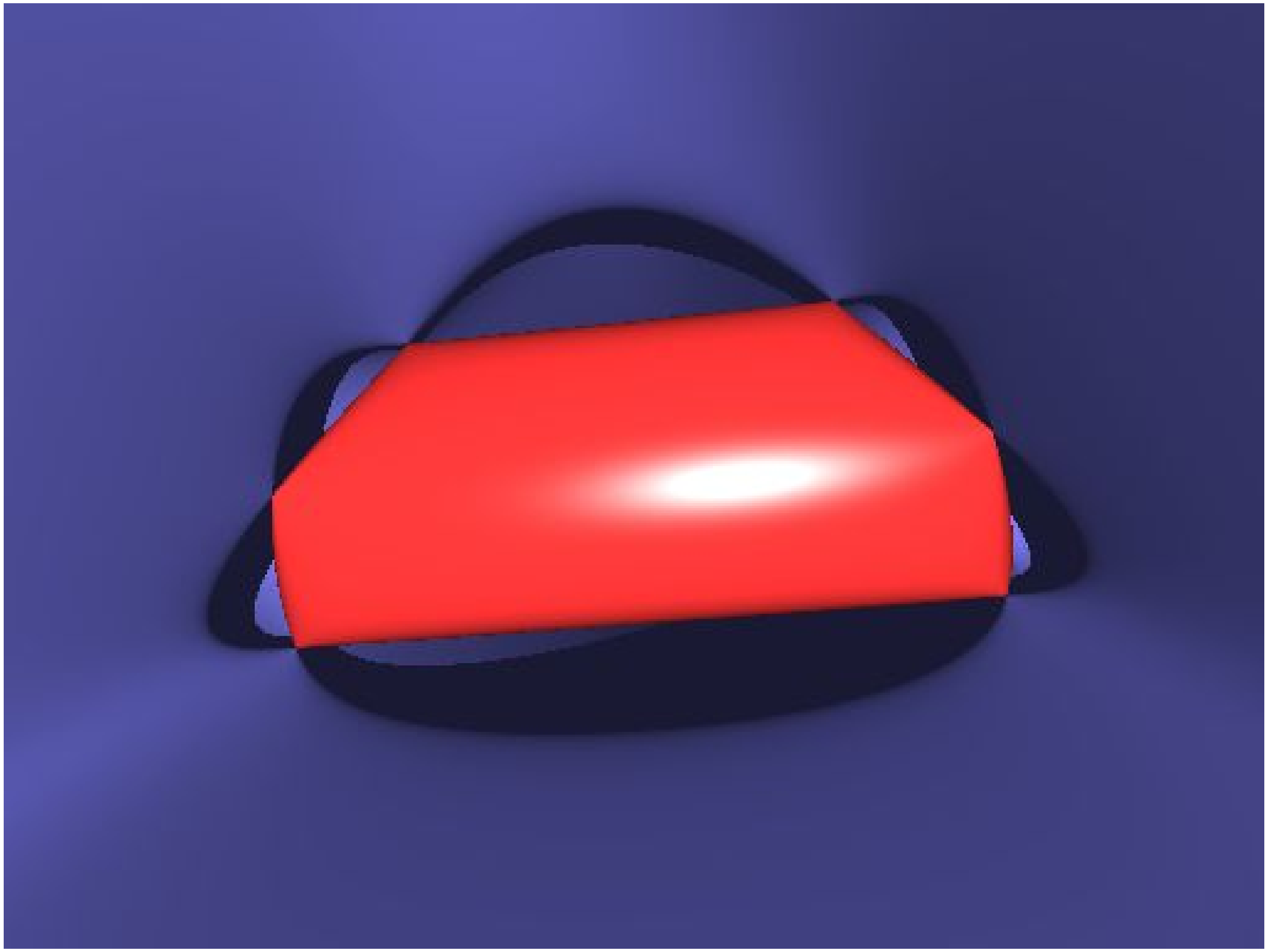


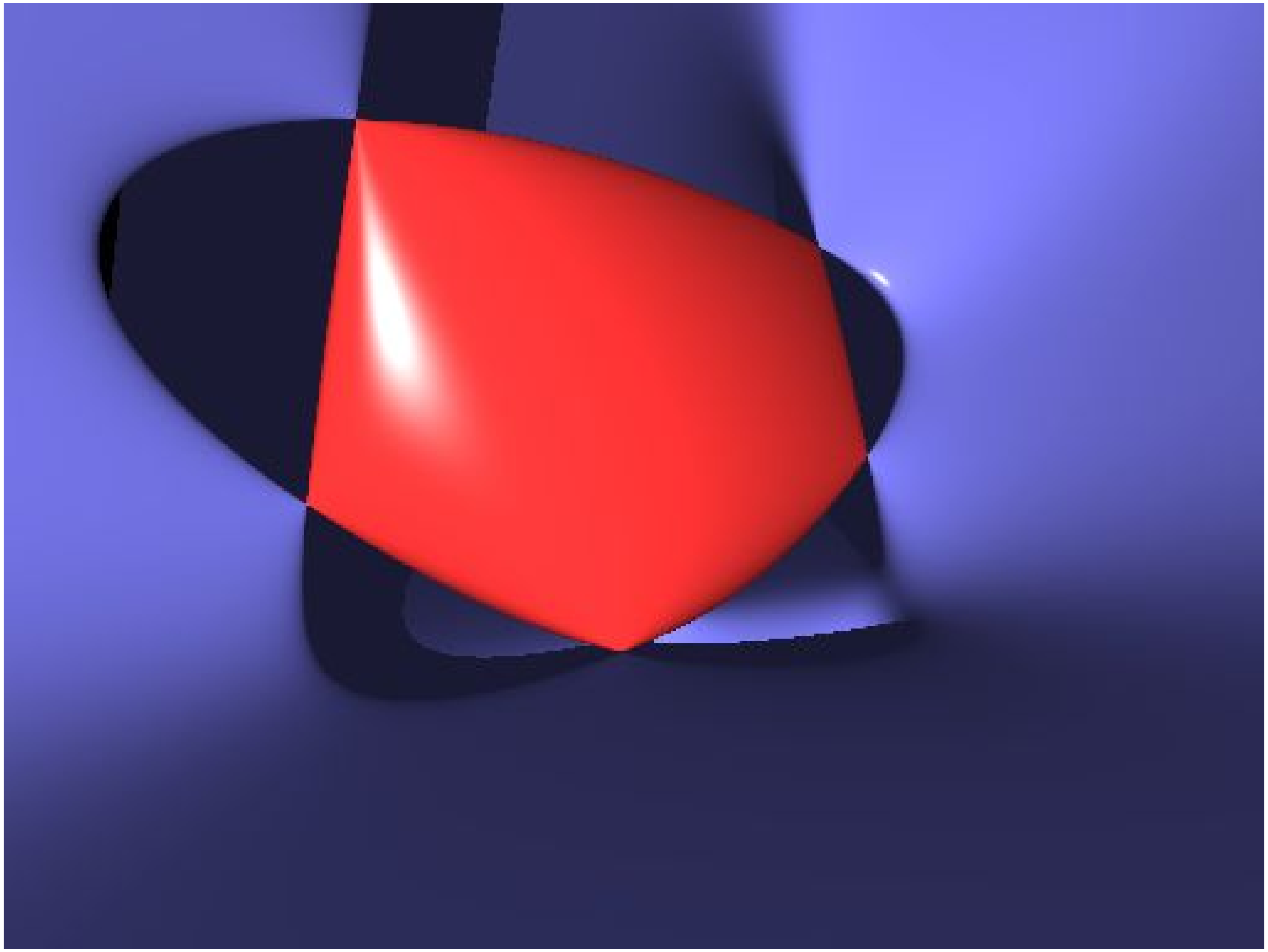
belongs to **determinantal variety**

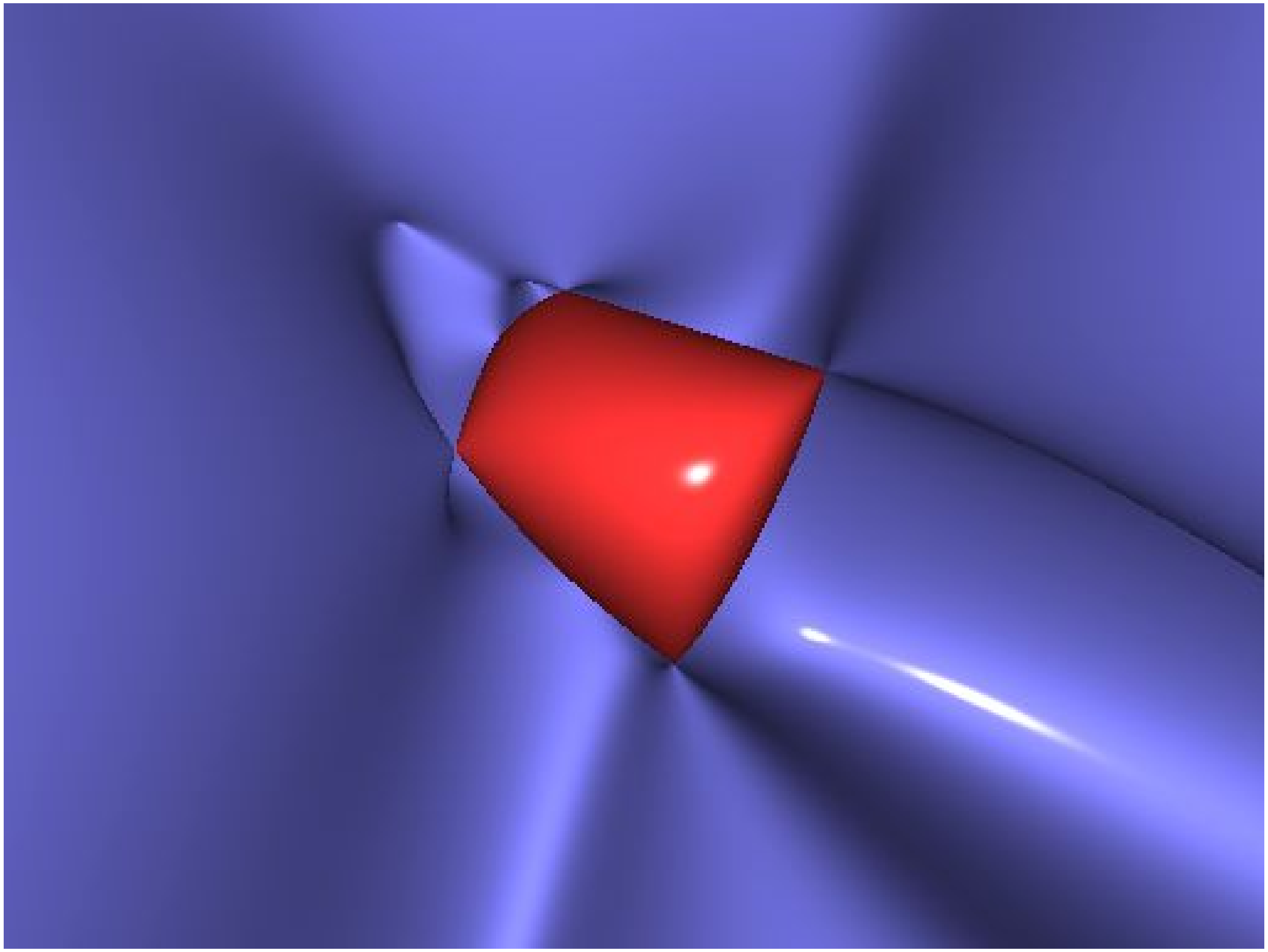
$$\left\{ x \in \mathbb{R}^3 \;:\; \det \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & x_3 \\ x_2 & x_3 & 1 \end{pmatrix} = 0 \right\}$$











Spectrahedra are convex basic semialgebraic

The pencil

$$A(x) := A_0 + A_1 x_1 + \cdots + A_n x_n$$

has characteristic polynomial

$$\det(tI_m + A(x)) = t^m + p_1(x)t^{m-1} + \cdots + p_{m-1}(x)t + \cdots + p_m(x)$$

such that the spectrahedron

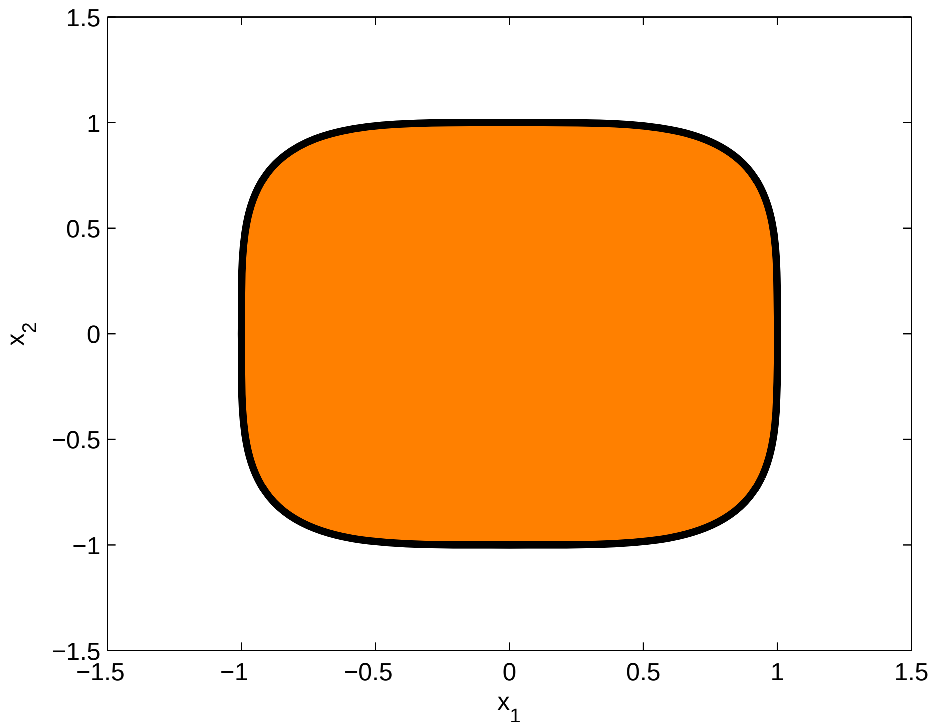
$$\begin{aligned} \mathscr{S} &= \{x \in \mathbb{R}^n \ : \ A(x) \succeq 0\} \\ &= \{x \in \mathbb{R}^n \ : \ p_1(x) \geq 0, \ldots, p_m(x) \geq 0\} \end{aligned}$$

can be expressed as an intersection of polynomial superlevel sets

Are all convex basic semialgebraic sets spectrahedral ?

TV screen is convex basic semialgebraic but **not** spectrahedral

$$\left\{ x \in \mathbb{R}^2 \,:\, 1 - x_1^4 - x_2^4 \geq 0 \right\}$$



Thm [Helton-Vinnikov]: A planar algebraic interior is spectrahedral if and only if its defining polynomial is hyperbolic

However, the TV screen is the **projection** of a spectrahedron

$$\left\{x \in \mathbb{R}^2 \ : \ 1 - x_1^4 - x_2^4 \geq 0\right\} =$$

$$\left\{ x \in \mathbb{R}^2 \ : \ \exists u \in \mathbb{R}^2 : \begin{pmatrix} 1+u_1 & u_2 & & & & \\ u_2 & 1-u_1 & & & & \\ & & 1 & x_1 & & \\ & & x_1 & u_1 & & \\ & & & & 1 & x_2 \\ & & & & x_2 & u_2 \end{pmatrix} \succeq 0 \right\}$$

u_1 and u_2 are called **liftings**

Spectrahedral shadows

A spectrahedral shadow is described by a lifted LMI

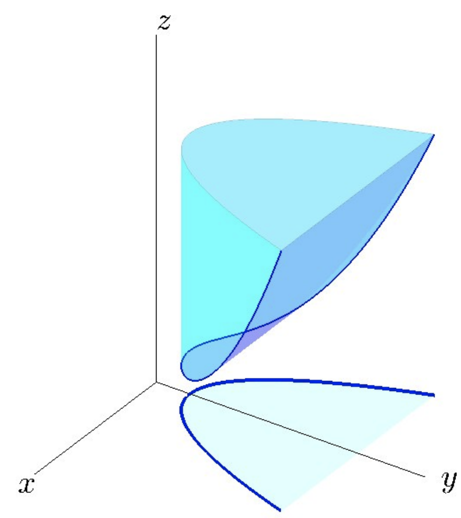
$$\{x \in \mathbb{R}^n : \exists u \in \mathbb{R}^p : A_0 + A_1 x_1 + \cdots + A_n x_n + B_1 u_1 + \cdots + B_p u_p \succeq 0\}$$

and it is a convex semialgebraic set

Conjecture [Nemirovski, Helton-Nie]: Every convex semialgebraic set is a spectrahedral shadow

Thm [Scheiderer]: True in the plane

.. but the proof is not constructive



Outline

1. Spectrahedra and linear matrix inequalities
2. **Semidefinite programming**
3. Exact algorithms for spectrahedra

Semidefinite programming (SDP)

Optimization of a linear form on a spectrahedron

SDP is linear programming (LP) in the semidefinite cone

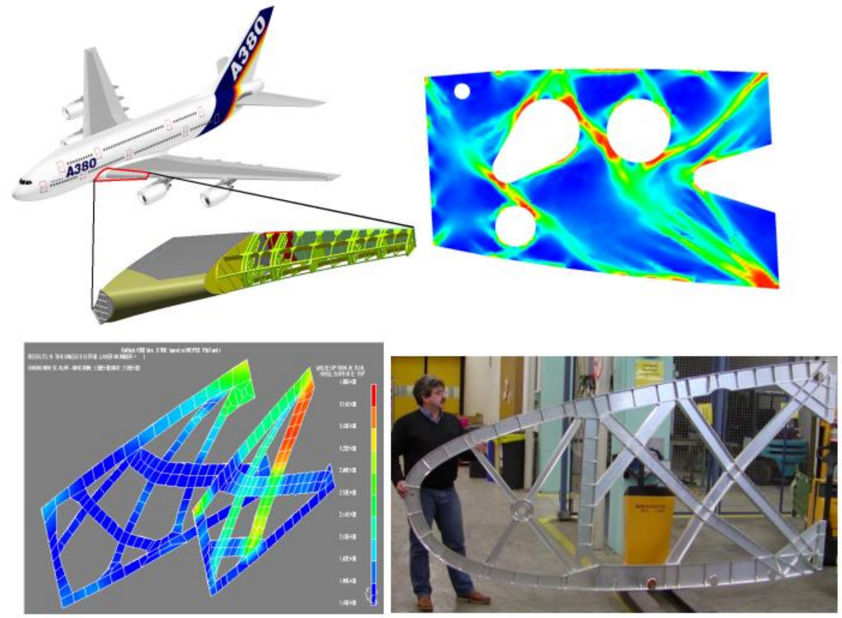
$$
\begin{array}{ll}
\inf\limits_{x \in \mathbb{R}^n} & c_1 x_1 + \cdots + c_n x_n \\
\text{s.t.} & A(x) = A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0
\end{array}
$$

with dual SDP

$$
\begin{array}{ll}
\sup\limits_{Y \in \mathbb{SR}^m} & -\text{trace}\, A_0 Y \\
\text{s.t.} & \text{trace}\, A_1 Y = c_1 \\
 & \quad \cdots \\
 & \text{trace}\, A_n Y = c_n \\
 & Y \succeq 0
\end{array}
$$

Success stories of SDP

- eigenvalue optimization
- systems control
- combinatorics
- structural mechanics
- computer vision ?



Since 2000: polynomial optimization, Lasserre hierarchies, sums of squares certificates

Duality and complementarity

If $x \in \mathbb{R}^n$ and $Y \in \mathbb{SR}^m$ are admissible, i.e. $A(x) \succeq 0$, $Y \succeq 0$, and

$$\text{trace}\, A(x) Y = 0$$

or equivalently

$$A(x) Y = 0$$

then x and Y are optimal

Conversely if x and Y are optimal then $A(x)Y = 0$

SDP solvers use **numerical** primal-dual interior points to ensure trace $A(x)Y \leq \varepsilon$ in time $\mathcal{O}(\sqrt{m}(m^3 n + m^2 n^2) \log \varepsilon^{-1})$

Outline

1. Spectrahedra and linear matrix inequalities
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3. **Exact algorithms for spectrahedra**

Finding a point in a spectrahedron

Given symmetric **rational** matrices $A_0, A_1, \ldots, A_n \in \mathbb{SQ}^m$ compute **exactly** at least one point in the spectrahedron

$$\mathscr{S} = \{x \in \mathbb{R}^n \,:\, A_0 + A_1 x_1 + \cdots + A_n x_n \succeq 0\}$$

or conclude that $\mathscr{S}$ is empty

Numerical SDP solvers assume that $\mathscr{S}$ has an interior point and return an approximate solution in floating-point arithmetic

We do **not** assume existence of an interior point

We do **not** want an approximate solution, but an **exact** solution

Exact samples

Samples are provided via a **rational univariate parametrization**

$$\left\{\left(\frac{q_1(t)}{q_0(t)}, \ldots, \frac{q_n(t)}{q_0(t)}\right) \;:\; q(t) = 0\right\}, \; (q, q_0, q_1, \ldots, q_n) \subset \mathbb{Q}[t]$$

with the smallest possible degree of q

The **input** of our algorithm are the matrices $A_0, A_1, \ldots, A_n \in \mathbb{S}\mathbb{Q}^m$

The **output** are the univariate polynomials $q, q_0, q_1, \ldots, q_n \in \mathbb{Q}[t]$

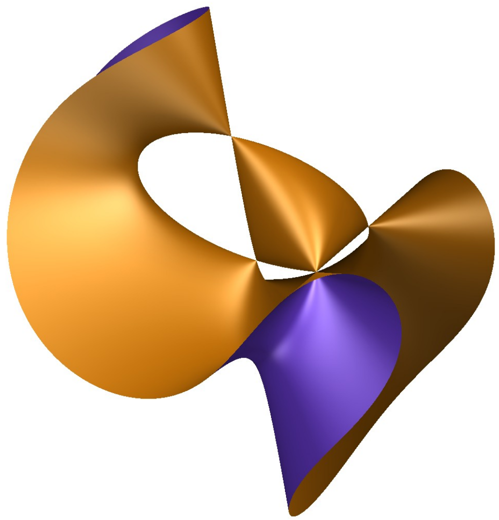
Determinantal varieties

Given the integer $r = 1, 2, \ldots, m-1$, define the variety

$$\mathscr{D}_r := \{x \in \mathbb{C}^n \;:\; \text{rank}\, A(x) \le r\}$$

and the minimum rank on the spectrahedron

$$r(A) := \min\{\, \text{rank}\, A(x) \;:\; x \in \mathscr{S}\}$$



The sequence is nested: $\mathscr{D}_1 \subset \mathscr{D}_2 \subset \cdots \subset \mathscr{D}_{m-1}$

Smallest rank theorem [H.-Naldi-Safey El Din] Let $\mathscr{C}$ be a connected component of $\mathscr{D}_{r(A)} \cap \mathbb{R}^n$ such that $\mathscr{C} \cap \mathscr{S}$ is non-empty. Then $\mathscr{C} \subset \mathscr{S}$.

Polar varieties

Consequence of the smallest rank theorem: we can apply a probabilistic algorithm based on polar varieties that returns at least one point per connected component

Step 1: lift the determinantal variety $\mathscr{D}_r$ to the polar variety

$$\mathscr{V}_r := \{(x, y) \,:\, A(x) \begin{pmatrix} y_{1,1} & \cdots & y_{1,m-r} \\ \vdots & & \vdots \\ y_{m,1} & \cdots & y_{m,m-r} \end{pmatrix} = 0\}$$

which is **smooth** and **equidimensional** if pencil A is generic

Step 2: compute the critical points of the random linear map $\pi(x, y) := a_1 x_1 + \cdots + a_n x_n$ on $\mathscr{V}_r$; they are **finitely many** if A and a are generic

Step 3: iterate Step 2 over fibers of π

Complexity and implementation

We can exploit the **determinantal structure**, and especially the **symmetry**, to derive explicit complexity estimates

$$\mathscr{O}\left(n \sum_{r \leq r(A)} \binom{m}{r} (n + p_r + r(m-r))^7 \binom{p_r + n}{n}^6 \right)$$

with $\mathscr{O}(k) = \mathcal{O}(k \log_k^c)$, $c \in \mathbb{N}$ and $p_r := (m-r)(m+r+1)/2$

Polynomial time in number of variables n if size m is fixed

Implementation in a Maple library called SPECTRA
using Safey El Din's RAGLIB and Faugère's FGb

Genericity assumptions can be checked on the fly

Some experiments on random spectrahedra

Maple 17 on `Intel(R) Xeon(R) CPU E7540@2.00GHz 256 Gb of RAM`

(m, r, n)	RAGLIB	SPECTRA	deg q
$(3, 2, 8)$	109	18	39
$(3, 2, 9)$	230	20	39
$(4, 2, 5)$	12.2	26	100
$(4, 2, 6)$	∞	593	276
$(4, 2, 7)$	∞	6684	532
$(4, 2, 8)$	∞	42868	818
$(4, 2, 9)$	∞	120801	1074
$(4, 3, 10)$	∞	303	284
$(4, 3, 11)$	∞	377	284
$(5, 2, 9)$	∞	903	175
$(6, 5, 4)$	∞	8643	726

m = size of LMI, r = expected rank, n = number of variables
deg q = degree of rational parametrization
times in seconds, with ∞ if more than 2 days

Scheiderer's spectrahedron

Trivariate quartic with no rational sum of squares decomposition

$$f = u_1^4 + u_1u_2^3 + u_2^4 - 3u_1^2u_2u_3 - 4u_1u_2^2u_3 + 2u_1^2u_3^2 + u_1u_3^3 + u_2u_3^3 + u_3^4$$

written as $f = v'A(x)v$ with $v = (u_1^2, u_1u_2, u_2^2, u_1u_3, u_2u_3, u_3^2)$ and

$$A(x) = \begin{pmatrix} 1 & 0 & x_1 & 0 & -3/2 - x_2 & x_3 \\ 0 & -2x_1 & 1/2 & x_2 & -2 - x_4 & -x_5 \\ x_1 & 1/2 & 1 & x_4 & 0 & x_6 \\ 0 & x_2 & x_4 & -2x_3 + 2 & x_5 & 1/2 \\ -3/2 - x_2 & -2 - x_4 & 0 & x_5 & -2x_6 & 1/2 \\ x_3 & -x_5 & x_6 & 1/2 & 1/2 & 1 \end{pmatrix}$$

With SPECTRA we can check that there are

- no rank-one $A(x) \succeq 0$ hence $f \neq g^2$
- two rank-two $A(x) \succeq 0$ hence $f = g_1^2 + g_2^2 = g_3^2 + g_4^2$
- no rank-three $A(x) \succeq 0$ hence $f \neq h_1^2 + h_2^2 + h_3^2$